REPRESENTATION THEORY SPRING 2018: SIXTH PROBLEM SET

1. Show that, if V is an irreducible G-representation, then V^* is irreducible as well.

2. Suppose that G acts doubly transitively on a finite set X with $\#(X) = n$. Show that $\#(G) \geq n^2 - n$. (In particular, the group D_n, with $\#(D_n) = 2n$, acts doubly transitively on the set $\{1, \dots, n\}$ only for $n \leq 3$.)

3. Suppose that G is a nonabelian group and that $\#(G) = 6$. Show that G has exactly 3 conjugacy classes. Similarly, if G is a nonabelian group and $\#(G) = 8$, show that G has exactly 5 conjugacy classes.

4. Using last week's HW, show that A_4 has exactly 4 conjugacy classes. Write down the character table for A_4 and verify the orthogonality relations for the columns.

5. We have described the irreducible representations of S_4 in class. In particular, the standard representation $\mathbb{C}^4$ has an irreducible subspace W_2 of dimension 3, with character χ_{W_2}, and there is also a second irreducible representation of dimension 3, $\varepsilon \otimes W_2$ with character $\varepsilon\chi_{W_2}$ and an irreducible 2-dimensional representation W with character χ_W. In this problem, we describe how to write $\mathrm{Hom}(W_2, W_2)$ as a direct sum of irreducible representations. (Note: since χ_{W_2} takes on only real values, $W_2^* \cong W_2$ and hence $\mathrm{Hom}(W_2, W_2) \cong W_2 \otimes W_2$.)

   (a) Show that $\chi_{\mathrm{Hom}(W_2,W_2)} = \chi_{W_2}^2$ and compute its values. Using these, show that $\langle \chi_{W_2}^2, 1\rangle = 1$. Why does this also follow from Schur's lemma?

   (b) Show that $\langle \chi_{W_2}^2, \chi_{W_2}^2\rangle = 4$. Using (a) and the fact that every irreducible representation of S_4 has degree at most 3, conclude that $\mathrm{Hom}(W_2, W_2)$ is isomorphic to the direct sum of the trivial representation, the unique irreducible two dimensional representation W of S_4, and two three dimensional representations.

   (c) By computing characters, show that the two irreducible summands are W_2 and $\varepsilon \otimes W_2$ (as opposed to either $W_2 \oplus W_2$ or $(\varepsilon \otimes W_2) \oplus (\varepsilon \otimes W_2)$. Thus

$$\mathrm{Hom}(W_2, W_2) \cong \mathbb{C}(1) \oplus W \oplus W_2 \oplus (\varepsilon \otimes W_2),$$

where $\mathbb{C}(1)$ denotes the trivial representation of S_4 and W, as above, is the unique irreducible two dimensional representation of S_4.

6. (Tensors) Let V be a $\mathbb{C}$-vector space of dimension d and consider the n-fold tensor product $V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$. An element α of $V^{\otimes n}$ is also caled an *n-tensor*. We will use a few facts about $V^{\otimes n}$:

   (1) If $e_1, \ldots, e_d$ is a basis for V, then a basis for $V^{\otimes n}$ is given by $e_{i_1} \otimes \cdots \otimes e_{i_n}$, where $1 \leq i_k \leq d$ for every k, $1 \leq k \leq n$. In particular, $\dim V^{\otimes n} = d^n$.

   (2) There is a representation of the symmetric group S_n on $V^{\otimes n}$ as follows: given $\sigma \in S_n$ and $v_1 \otimes v_2 \otimes \cdots \otimes v_n \in V^{\otimes n}$, then

$$\rho_{V^{\otimes n}}(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}.$$

   Note that there is **no** action specified of S_n on the vector space V, and the natural numbers n and d are unrelated. In particular, an element $\sigma \in S_n$ acts on a basis vector $e_{i_1} \otimes \cdots \otimes e_{i_n}$ by permuting the **relative positions** of the e_{i_k} in the tensor product, **not** by acting on the indices i_k.

   An n-tensor α is *symmetric* if $\rho_{V^{\otimes n}}(\sigma)(\alpha) = \alpha$ for all $\sigma \in S_n$ and *alternating* if $\rho_{V^{\otimes n}}(\sigma)(\alpha) = \varepsilon(\sigma)\alpha$ for all $\sigma \in S_n$, where $\varepsilon\colon S_n \to \{\pm 1\}$ is the sign homomorphism. (These definitions are perhaps more familiar for the representation on $(V^{\otimes n})^* = (V^*)^{\otimes n}$, which is the vector space of multilinear functions from $V \times \cdots \times V$ to $\mathbb{C}$.) We denote by $\operatorname{Sym}^n V$ the vector space of symmetric elements of $V^{\otimes n}$ and by $\bigwedge^n V$ the vector space of alternating elements of $V^{\otimes n}$.

   (a) For S_2 acting on $V^{\otimes 2}$, show that the character is given as follows:

σ	1	(i,j)
$\chi_{V^{\otimes 2}}(\sigma)$	d^2	d

   Writing $\chi_{V^{\otimes 2}} = A \cdot 1 + B \cdot \varepsilon$, solve for A and B and hence find the multiplicities of the trivial representation $\mathbb{C}(1)$ and of $\mathbb{C}(\varepsilon)$ in $V^{\otimes 2}$.

   (b) For S_3 acting on $V^{\otimes 3}$, show that the character is given as follows:

σ	1	(i,j)	(i,j,k)
$\chi_{V^{\otimes 3}}(\sigma)$	d^3	d^2	d

Writing $\chi_{V^{\otimes 3}} = A\cdot 1 + B\cdot\varepsilon + C\cdot\chi_{W_2}$, where W_2 is the irreducible representation of S_3 of dimension 2, solve for A, B, C and hence find the multiplicities of the trivial representation $\mathbb{C}(1)$, of $\mathbb{C}(\varepsilon)$, and of W_2 in $V^{\otimes 3}$. (Note: this shows that "most" 3-tensors **cannot** be written as a sum of a symmetric and an alternating 3-tensor.)

(c) Generalizing (a) and (b), show that, if $\sigma \in S_n$ is a product of t disjoint cycles $\gamma_1, \dots, \gamma_t$, where γ_i has length $\ell_i \geq 2$, then

$$\chi_{V^{\otimes n}}(\sigma) = d^{n+t-\sum_{i=1}^{t}\ell_i}.$$

Note: The symmetric tensors $\operatorname{Sym}^n V$ can be identified (after a choice of a basis, or more accurately a dual basis) with the space of degree n monomials in variables $x_1, \dots, x_d$, and thus

$$\dim \operatorname{Sym}^n V = \binom{n+d-1}{n}.$$

As for the alternating tensors $\bigwedge^n V$,

$$\dim \bigwedge^n V = \binom{d}{n},$$

with the understanding that this is zero for $d < n$. Again, we see that most tensors are not a sum of a symmetric and an alternating tensor. For $d \geq n$, it is easy to see that $V^{\otimes n}$ contains an S_n-invariant subspace isomorphic to the regular representation of S_n, and thus that every irreducible representation of S_n appears as a direct summand of $V^{\otimes n}$ for $d \geq n$.