1. Let $D_4$ act on $\{1, 2, 3, 4\}$, viewed as the vertices of a square, in the usual way: as a subgroup of $S_4$, 

$$D_4 = \langle (1234) \rangle \cup \{(12)(34), (14)(23), (13), (24)\}.$$ 

(a) Show that the action of $D_4$ on $\{1, 2, 3, 4\}$ is transitive, but not doubly transitive.

(b) If $C_4 = W_1 \oplus W_2$ in the usual way, where $W_1$ is the span of $(1, 1, 1, 1)$ and $W_2 = \{(t_1, t_2, t_3, t_4) : \sum_1^4 t_i = 0\}$, compute the character $\chi_{C_4}$ for the $D_4$ action on $C_4$ and hence the character $\chi_{W_2} = \chi_{C_4} - 1$ for the $D_4$-action on $W_2$.

(c) What are: $\langle \chi_{C_4}, \chi_{C_4} \rangle$, $\langle \chi_{W_2}, 1 \rangle$, $\langle \chi_{W_2}, \chi_{W_2} \rangle$? Is $W_2$ irreducible as a $D_4$-representation?

(d) In the handout on some aspects of group theory, we have constructed a two-dimensional representation $V$ of $D_4$, whose character $\chi_V$ satisfies: $\chi_V(1) = 2$, $\chi_V((13)(24)) = -2$, and $\chi_V(g) = 0$ otherwise. Show that $V$ is a direct summand of $W_2$, and in fact that $W_2 \cong V \oplus C(1) \oplus C(\lambda)$. Compute $\lambda$ explicitly. Thus $C_4 \cong V \oplus C(1) \oplus C(\lambda)$.

2. Let $V$ be a $G$-representation. Show that $V \cong V^* \iff \chi_V(g) \in \mathbb{R}$ for all $g \in G$.

3. Let $\lambda: G \to \mathbb{C}^*$ be a homomorphism, corresponding to a one dimensional representation $\mathbb{C}(\lambda)$. General theory tells us that $\mathbb{C}(\lambda)$ occurs exactly once in $\mathbb{C}[G]$, i.e. that, up to scalars, there is a unique vector $v \in \mathbb{C}[G]$ such that $\rho_{\text{reg}}(g)v = \lambda(g)v$ for all $G \in G$. Show in fact that any such $v$ is a multiple of $\sum_{g \in G} \lambda(g)^{-1} \cdot g$. (This is especially clear if we identify $\mathbb{C}[G]$ with the vector space of all functions from $G$ to $\mathbb{C}$. What function does the vector $v$ correspond to?)

4. Let $\rho_V: G \to \text{Aut } V$ be a representation and let $\lambda: G \to \mathbb{C}^*$ be a homomorphism, corresponding to a one dimensional representation of $G$.

(a) Show that the function $\lambda \otimes \rho_V: G \to \text{Aut } V$ defined by 

$$\lambda \otimes \rho_V)(g)(v) = \lambda(g)\rho_V(g)(v)$$

is a representation (i.e. a homomorphism from $G$ to $\text{Aut } V$.)
(b) Show that $V$ is irreducible as a $G$-representation for the homomorphism $\rho_V \iff V$ is irreducible as a $G$-representation for the homomorphism $\lambda \otimes \rho_V$.

(c) What is the character of $\lambda \otimes \rho_V$? Show that $\rho_V$ and $\lambda \otimes \rho$ are isomorphic $\iff$ for all $g \in G$, if $\chi_V(g) \neq 0$, then $g \in \text{Ker } \lambda$.

(d) Consider $G = S_3$, $\rho_W$ the irreducible two-dimensional representation of $S_3$, and $\varepsilon$ the sign homomorphism. Show that $\rho$ and $\varepsilon \otimes \rho$ have the same character, hence are isomorphic—of course, this is already clear because, up to isomorphism, there is a unique irreducible representation of $S_3$. Show however that, if $n \geq 4$ and $\rho$ is the representation of dimension $n - 1$ which we have been denoting $W_2$, then $\rho$ and $\varepsilon \otimes \rho$ are not isomorphic.

5. Let $N$ be a normal subgroup of the group $G$ and let $\pi : G \to G/N$ be the quotient homomorphism. For each representation $\rho : G/N \to \text{Aut } V$, show that $\rho \circ \pi : G \to \text{Aut } V$ is also a representation. Show that a vector subspace $W$ of $V$ is $G/N$-invariant (for the representation $\rho$ $\iff$ $W$ is $G$-invariant for the representation $\rho \circ \pi$, and hence that $V$ is irreducible as a $G/N$-representation $\iff V$ is irreducible as a $G$-representation.

6. In class, we discussed the normal subgroup $H$ of $S_4$ given by

$$H = \{1, (12)(34), (13)(24), (14)(23)\}.$$ 

As $H$ is clearly a subgroup of $A_4$, $H$ is a normal subgroup of $A_4$.

(a) Show that $A_4/H$ is cyclic and generated by $(1,2,3)$.

(b) Using the previous problem, show that there are three one-dimensional representations of $A_4$ and compute their characters.

(c) Show that $(1,2,3)$ and $(1,3,2)$ are not conjugate in $A_4$ (although they are conjugate in $S_4$).

(d) Let $W_2$ be the three-dimensional representation of $A_4$ defined as in Problem 2(b). Compute the character $\chi_{W_2}$ for this representation of $A_4$ and conclude that $W_2$ is an irreducible $A_4$-representation.

(e) Show that, up to isomorphism, there are exactly 4 irreducible representations of $A_4$, and they are given by (b) and (c) above.