REPRESENTATION THEORY SPRING 2018: FIFTH PROBLEM SET

1. Let D_4 act on $\{1,2,3,4\}$, viewed as the vertices of a square, in the usual way: as a subgroup of S_4,

$$D_4 = \langle (1234) \rangle \cup \{(12)(34), (14)(23), (13), (24)\}.$$

   (a) Show that the action of D_4 on $\{1,2,3,4\}$ is transitive, but not doubly transitive.

   (b) If $\mathbb{C}^4 = W_1 \oplus W_2$ in the usual way, where W_1 is the span of $(1,1,1,1)$ and $W_2 = \{(t_1,t_2,t_3,t_4) : \sum_1^4 t_i = 0\}$, compute the character $\chi_{\mathbb{C}^4}$ for the D_4 action on $\mathbb{C}^4$ and hence the character $\chi_{W_2} = \chi_{\mathbb{C}^4} - 1$ for the D_4-action on W_2.

   (c) What are: $\langle \chi_{\mathbb{C}^4}, \chi_{\mathbb{C}^4} \rangle$? $\langle \chi_{W_2}, 1 \rangle$? $\langle \chi_{W_2}, \chi_{W_2} \rangle$? Is W_2 irreducible as a D_4-representation?

   (d) In the handout on some aspects of group theory, we have constructed a two-dimensional representation V of D_4, whose character χ_V satisfies: $\chi_V(1) = 2$, $\chi_V((13)(24)) = -2$, and $\chi_V(g) = 0$ otherwise. Show that V is a direct summand of W_2, and in fact that $W_2 \cong V \oplus \mathbb{C}(\lambda)$ for a homomorphism $\lambda\colon D_4 \to \mathbb{C}^*$. Compute λ explicitly. Thus $\mathbb{C}^4 \cong V \oplus \mathbb{C}(1) \oplus \mathbb{C}(\lambda)$.

2. Let V be a G-representation. Show that $V \cong V^* \iff \overline{\chi_V} = \chi_V \iff \chi_V(g) \in \mathbb{R}$ for all $g \in G$.

3. Let $\lambda\colon G \to \mathbb{C}^*$ be a homomorphism, corresponding to a one dimensional representation $\mathbb{C}(\lambda)$. General theory tells us that $\mathbb{C}(\lambda)$ occurs exactly once in $\mathbb{C}[G]$, i.e. that, up to scalars, there is a unique vector $v \in \mathbb{C}[G]$ such that $\rho_{\text{reg}}(g)v = \lambda(g)v$ for all $G \in G$. Show in fact that any such v is a multiple of $\sum_{g\in G} \lambda(g)^{-1} \cdot g$. (This is especially clear if we identify $\mathbb{C}[G]$ with the vector space of all functions from G to $\mathbb{C}$. What function does the vector v correspond to?)

4. Let $\rho_V\colon G \to \operatorname{Aut} V$ be a representation and let $\lambda\colon G \to \mathbb{C}^*$ be a homomorphism, corresponding to a one dimensional representation of G.

   (a) Show that the function $\lambda \otimes \rho_V\colon G \to \operatorname{Aut} V$ defined by

$$(\lambda \otimes \rho_V)(g)(v) = \lambda(g)\rho_V(g)(v)$$

   is a representation (i.e. a homomorphism from G to $\operatorname{Aut} V$.)

(b) Show that V is irreducible as a G-representation for the homomorphism $\rho_V \iff V$ is irreducible as a G-representation for the homomorphism $\lambda \otimes \rho_V$.

(c) What is the character of $\lambda \otimes \rho_V$? Show that ρ_V and $\lambda \otimes \rho$ are isomorphic $\iff$ for all $g \in G$, if $\chi_V(g) \neq 0$, then $g \in \operatorname{Ker} \lambda$.

(d) Consider $G = S_3$, ρ_W the irreducible two-dimensional representation of S_3, and ε the sign homomorphism. Show that ρ and $\varepsilon \otimes \rho$ have the same character, hence are isomorphic—of course, this is already clear because, up to isomorphism, there is a unique irreducible representation of S_3. Show however that, if $n \geq 4$ and ρ is the representation of dimension $n - 1$ which we have been denoting W_2, then ρ and $\varepsilon \otimes \rho$ are not isomorphic.

5. Let N be a normal subgroup of the group G and let $\pi \colon G \to G/N$ be the quotient homomorphism. For each representation $\rho \colon G/N \to \operatorname{Aut} V$, show that $\rho \circ \pi \colon G \to \operatorname{Aut} V$ is also a representation. Show that a vector subspace W of V is G/N-invariant (for the representation ρ $\iff W$ is G-invariant for the representation $\rho \circ \pi$, and hence that V is irreducible as a G/N-representation $\iff V$ is irreducible as a G-representation.

6. In class, we discussed the normal subgroup H of S_4 given by

$$H = \{1, (12)(34), (13)(24), (14)(23)\}.$$

As H is clearly a subgroup of A_4, H is a normal subgroup of A_4.

(a) Show that A_4/H is cyclic and generated by $(1, 2, 3)$.

(b) Using the previous problem, show that there are three one-dimensional representations of A_4 and compute their characters.

(c) Show that $(1, 2, 3)$ and $(1, 3, 2)$ are not conjugate in A_4 (although they are conjugate in S_4).

(d) Let W_2 be the three-dimensional representation of A_4 defined as in Problem 2(b). Compute the character χ_{W_2} for this representation of A_4 and conclude that W_2 is an irreducible A_4-representation.

(e) Show that, up to isomorphism, there are exactly 4 irreducible representations of A_4, and they are given by (b) and (c) above.