REPRESENTATION THEORY SPRING 2018: FOURTH PROBLEM SET

1. (a) Let V_1 and V_2 be two G-representations. Show that $(V_1 \oplus V_2)^G = V_1^G \oplus V_2^G$. Hence, if W is also a G-representation, then
$$\mathrm{Hom}^G(W, V_1 \oplus V_2) \cong \mathrm{Hom}^G(W, V_1) \oplus \mathrm{Hom}^G(W, V_2).$$
   (b) Suppose that G is finite and that W is an irreducible G-representation and that V is a G-representation. Thus we can write $V \cong V_1 \oplus \cdots \oplus V_k$, where each V_i is irreducible. Show that $\dim \mathrm{Hom}^G(W, V)$ is equal to the number of summands V_i such that $V_i \cong W$. Hence this number is independent of the choice of isomorphism of V with the direct sum.

2. (i) For $G = D_3$ and the two dimensional representation of G described for example in the handout, calculate the character χ_2 and verify that $\frac{1}{6}\sum_{g\in D_3} \chi_2(g)\overline{\chi_2(g)} = \frac{1}{6}\sum_{g\in D_3} |\chi_2(g)|^2 = 1$.

   (ii) Compute the character χ_{st} of the standard representation of S_3 on $\mathbb{C}^3$, and verify that (under the usual identification of S_3 with D_3) $\chi_{\mathrm{st}} = \chi_2 + 1$ (where 1 denotes the constant function 1, the character of the trivial representation).

3. Let Q be the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ and let $\rho\colon Q \to GL(2, \mathbb{C})$ be the representation determined by:
$$\rho(\pm 1) = \pm\,\mathrm{Id}; \qquad \rho(\pm i) = \pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix};$$
$$\rho(\pm j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \qquad \rho(\pm k) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$
   Compute the character χ of ρ. Verify that
$$\frac{1}{8}\sum_{g\in Q} \chi(g)\overline{\chi(g)} = \frac{1}{8}\sum_{g\in Q} |\chi(g)|^2 = 1.$$

4. (i) Let V_1 and V_2 be two one-dimensional representations of the group G corresponding to homomorphisms $\lambda_1, \lambda_2\colon G \to \mathbb{C}^*$. Show that V_1 and V_2 are isomorphic $\iff$ $\lambda_1 = \lambda_2$.

   (ii) Let V be a two one-dimensional representation of the group G corresponding to the homomorphisms $\lambda\colon G \to \mathbb{C}^*$. Show that the dual representation V^* corresponds to the homomorphism λ^{-1}. (Here λ^{-1} does not denote the inverse function, but rather the homomorphism $G \to \mathbb{C}^*$ defined by $\lambda^{-1}(g) = (\lambda(g))^{-1} = 1/\lambda(g)$.)

(iii) Let V_1 denote the one-dimensional representation of $\mathbb{Z}/n\mathbb{Z}$ corresponding to the homomorphism $\lambda_1(k) = e^{2\pi ik/n}$. Show that, if $n > 2$, then V_1 and $(V_1)^*$ are not isomorphic. What happens when $n = 2$?

5. If $V = \mathbb{C}^n$ is the standard representation of S_n, which we denote by ρ_{st}, we have defined an S_n-morphism $p\colon V \to \mathbb{C}$ by $p(t_1, \ldots, t_n) = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} t_i$. We have also defined a projection $p'\colon V \to V$ by:

$$p'(v) = \frac{1}{\#(S_n)} \sum_{\sigma \in S_n} \rho_{\text{st}}(v) = \frac{1}{n!} \sum_{\sigma \in S_n} \rho_{\text{st}}(v).$$

What is the relationship between these two maps?

6. Let V be a representation of the finite group G. For $v \in V$, define $G \cdot v$, the orbit of v, to be

$$\{\rho(g)v : g \in G\}$$

and define $\mathbb{C}[G] \cdot v$ to be the span of $G \cdot v$.

(a) Show that $\mathbb{C}[G] \cdot v$ is a G-invariant subspace of V.

(b) Show that, if V is irreducible and $v \neq 0$, then $\mathbb{C}[G] \cdot v = V$.

(c) Conclude that, if V is irreducible and $v \neq 0$, then

$$\#(G \cdot v) \geq \dim V.$$

(d) Suppose that V is irreducible and that $H \leq G$ is an abelian subgroup (not necessarily normal). Show that

$$\dim V \leq (G : H) = \#(G/H).$$

(As we have seen, by viewing V as a not necessarily irreducible representation of the group H, there exists a common eigenvector v for the action of H on V, and by (b) $\mathbb{C}[G] \cdot v = V$. Now argue that $\dim \operatorname{span} G \cdot v \leq \#(G/H)$.)

For example, every irreducible representation of the dihedral group D_n has dimension ≤ 2.