REPRESENTATION THEORY SPRING 2018: TWELFTH PROBLEM SET

1. (i) Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$. Show that the conjugate partition λ^T is given by $(\mu_1, \dots, \mu_m)$, where μ_i is equal to the number of λ_i which are greater than or equal to i. (Thus for example $\mu_1 = \ell$.)

2. We consider the irreducible representations of S_4.

   (a) The five partitions of 4 are (4), $(3,1)$, $(2,2)$, $(2,1,1)$, and $(1,1,1,1)$. Note that $(4)^T = (1,1,1,1)$ and $(3,1)^T = (2,1,1)$. The partition (4) corresponds to the trivial representation, $(3,1)$ to the standard 3-dimensional representation W_0, and $(1,1,1,1)$ to the one-dimensional representation given by ε, the sign homomorphism.

   (b) Show that, for $\lambda = (2,2)$, $S_\lambda \cong S_2 \times S_2$, hence $\#(S_4/S_\lambda) = 6$. Show that the 6 λ-tabloids $[t]$ can be indexed by i, j with $i < j$, by taking for t any of the 4 λ-tableaux whose first row has entries i and j, in either order.

   (c) For $t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$, what is the corresponding polytabloid e_t? Denote this polytabloid by E_{12}. Similarly, if $t = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$, denote the corresponding polytabloid by E_{13}. Compute the polytabloids associated to $\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}$, and finally $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline\end{array}$, in terms of E_{12} and E_{13}. Note that in this last case the associated tabloid is the same as that for $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$.

   (d) By looking at the above, argue that the polytabloids associated to $\begin{array}{|c|c|}\hline i & j \\ \hline k & \ell \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline j & i \\ \hline \ell & k \\ \hline\end{array}$ are the same. What about the polytabloid associated to $\begin{array}{|c|c|}\hline i & j \\ \hline \ell & k \\ \hline\end{array}$?

   (e) Show that $\dim S^{(2,2)} = 2$.

3. Let $\lambda = (n-2, 1, 1) \vdash n$. Then every tableau t is of the form

$$\begin{array}{|c|c|c|c|c|}\hline i & * & * & \cdots & * \\ \hline j \\ \cline{1-1} k \\ \cline{1-1}\end{array}$$

The tabloid $[t]$ only depends on j and k; denote it by $[jk]$.

(a) With t as above, show that the polytabloid $A_t([t]) = E_{ijk}$ is given by

$$E_{ijk} = [jk] - [kj] + [ki] - [ik] + [ij] - [ji].$$

(b) Show that, for 3 distinct elements i, j, k in $\{1, \dots, n\}$, $E_{ijk} = E_{jki} = E_{kij}$ and that $E_{jik} = -E_{ijk}$. Thus the polytabloids of the form E_{ijk} with $i < j < k$ span $S^{(n-2,1,1)}$.

(c) Show that, given 4 distinct elements i, j, k, ℓ in $\{1, \dots, n\}$,

$$E_{jk\ell} = E_{ijk} - E_{ij\ell} + E_{ik\ell}.$$

Conclude that the polytabloids E_{1jk} with $1 < i < j$ span $S^{(n-2,1,1)}$ and in fact are a basis. Hence

$$\dim S^{(n-2,1,1)} = \binom{n-1}{2}.$$

(In fact, it is not hard to show that $S^{(n-2,1,1)} = \bigwedge^2 S^{(n-1,1)}$.)