REPRESENTATION THEORY SPRING 2018: TENTH PROBLEM SET

1. Let G be a nonabelian group of order pq, where p and q are primes $p < q$, and $q \equiv 1 \bmod p$. Suppose that x is an element of order q and that y is an element of order p, with $yxy^{-1} = x^t$, where $t \in (\mathbb{Z}/q\mathbb{Z})^*$ has order p. Finally, let $H = \langle x \rangle$ be the q-Sylow subgroup, so that H is normal. In class, we have described $\operatorname{Res}_H^G \operatorname{Ind}_H^G W$, for an H-representation W, and used this to give a necessary and sufficient condition for $\operatorname{Ind}_H^G W$ to be irreducible. Finally, we have seen that, if V is an irreducible G-representation of dimension p and $\operatorname{Res}_H^G V = \bigoplus_i W_i$, where each W_i is one-dimensional, then, for all i, W_i is not the trivial representation, $V \cong \operatorname{Ind}_H^G W_i$ for every i, and, for $i \neq j$, W_i and W_j are not isomorphic. Give a more precise description of the irreducible p-dimensional representations of G as follows:

   (a) Let $\lambda \colon H \to \mathbb{C}^*$ be a homomorphism. Given the generator x of H, we can write $\lambda(x) = e^{2\pi i a/q}$ for a unique a such that $0 \leq a \leq q-1$; this identifies $\widehat{H}$ with $\mathbb{Z}/q\mathbb{Z}$. What is $\lambda \circ i_y$? What is $\lambda \circ i_{y^k}$? Show that, if $a \neq 0$ (i.e. if λ is not the trivial character), then, for every $z \in G$, $\lambda \circ i_z = \lambda \iff z \in H$.

   (b) Using (a), the discussion above, and the identification of the one dimensional representations of H (i.e. $\widehat{H}$) with $\mathbb{Z}/q\mathbb{Z}$ above, show that the set of p-dimensional irreducible representations of G up to isomorphism can be identified with $(\mathbb{Z}/q\mathbb{Z})^*/\langle t \rangle$ (where we consider t as an element of the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^*$ and hence $\langle t \rangle = \{1, t, \dots, t^{p-1}\}$). In particular we see that the number of non-isomorphic such representations is $(q-1)/p$.

2. The symmetric group S_3 has three irreducible representations up to isomorphism: the two 1-dimensional representations $\mathbb{C}$ and $\mathbb{C}(\varepsilon)$ and the 2-dimensional representation W_2 which is the nontrivial summand of the standard representation of S_3 acting on $\mathbb{C}^3$. The symmetric group S_4 has three irreducible representations up to isomorphism: the two 1-dimensional representations $\mathbb{C}$ and $\mathbb{C}(\varepsilon)$, the 3-dimensional representation V_3 which is the nontrivial summand of the standard representation of S_4 acting on $\mathbb{C}^4$, as well as $V_3 \otimes \varepsilon$ and an irreducible 2-dimensional representation V_2 which is obtained by using the surjective homomorphism $S_4 \to S_4/H \cong S_3$, where H is the order 4 normal subgroup of S_4 described in Problem 6 of HW5. (Here, we somewhat

carelessly use the notation $\mathbb{C}$ and $\mathbb{C}(\varepsilon)$ for representations both of S_3 and of S_4.)

(a) Show that $\operatorname{Res}_{S_3}^{S_4}\mathbb{C} = \mathbb{C}$ and that $\operatorname{Res}_{S_3}^{S_4}\mathbb{C}(\varepsilon) = \mathbb{C}(\varepsilon)$.

(b) Show that $\operatorname{Res}_{S_3}^{S_4} V_3 \cong W_2 \oplus \mathbb{C}$ and that

$$\operatorname{Res}_{S_3}^{S_4}(V_3 \otimes \varepsilon) \cong (\operatorname{Res}_{S_3}^{S_4} V_3) \otimes \varepsilon \cong W_2 \oplus \mathbb{C}(\varepsilon).$$

(c) Show that $\operatorname{Ind}_{S_3}^{S_4}\mathbb{C} \cong V_3 \oplus \mathbb{C}$ and that $\operatorname{Ind}_{S_3}^{S_4}\mathbb{C}(\varepsilon) \cong V_3(\varepsilon) \oplus \mathbb{C}(\varepsilon)$.

(d) Show that V_3 and $V_3 \otimes \varepsilon$ are isomorphic to irreducible summands of $\operatorname{Ind}_{S_3}^{S_4} W_2$ and that $\mathbb{C}$ and $\mathbb{C}(\varepsilon)$ are not isomorphic to irreducible summands of $\operatorname{Ind}_{S_3}^{S_4} W_2$. Conclude that

$$\operatorname{Ind}_{S_3}^{S_4} W_2 \cong V_3 \oplus (V_3 \otimes \varepsilon) \oplus V_2.$$

3. Let G be a finite group and H a subgroup of G of index two, so that H is a normal subgroup of G. Let W be an irreducible representation of H. For $x \in G$, let W_x be the representation corresponding to $\rho_W \circ i_x$ as in class. Finally set $V = \operatorname{Ind}_H^G W$. Show that exactly one of the following holds:

(i) V is irreducible. In this case, W is not isomorphic to W_x, and $\operatorname{Res}_H^G V \cong W \oplus W_x$.

(ii) $V = \operatorname{Ind}_H^G W$ is reducible, and in fact $V \cong V' \oplus (V' \otimes \varepsilon)$, where V' is an irreducible G-representation and V' is not isomorphic to $V' \otimes \varepsilon$. In this case, $W \cong W_x$ and $W \cong \operatorname{Res}_H^G V' \cong \operatorname{Res}_H^G (V' \otimes \varepsilon)$.

(Hints: we have seen in class that $\operatorname{Res}_H^G V = W \oplus W_x$ and that V is irreducible $\iff$ W is not isomorphic to W_x. If V is irreducible, we can apply the theorem proved in class to study $\operatorname{Res}_H^G V$. If $W \cong W_x$, show that $\langle \chi_V, \chi_V \rangle_G = 2$ and hence that $V \cong V_1 \oplus V_2$ where V_1 and V_2 are irreducible and not isomorphic, and then show that, for $i = 1, 2$, $\operatorname{Res}_H^G V_i \cong W$. Then $V = \operatorname{Ind}_H^G \operatorname{Res}_H^G V_1 \cong V_1 \oplus (V_1 \otimes \varepsilon)$, and so $V_2 \cong V_1 \otimes \varepsilon$.)