Notes on the $D_4$ extension $E = \mathbb{Q}(\sqrt[4]{2}, i)$ of $\mathbb{Q}$

**Elements of $D_4$:** 1, (1234), (1234)$^2 = (13)(24)$, (1234)$^3 = (1432)$; (13), (24), (12)(34), (14)(23).

**Subgroups of $D_4$:** $\{1\}$ (order 1), $D_4$ (order 8). The three subgroups of order 4, all automatically normal:

\[
H_1 = \langle (1234) \rangle \\
H_2 = \{1, (13)(24), (12)(34), (14)(23)\} \\
H_3 = \{1, (13)(24), (13), (24)\}.
\]

The five subgroups of order 2:

\[
\langle (13)(24) \rangle, \langle (13) \rangle, \langle (24) \rangle, \langle (12)(34) \rangle, \langle (14)(23) \rangle.
\]

Of these, only $\langle (13)(24) \rangle$ is normal (it is the center of $D_4$).

**The fixed fields:** Label the roots of $x^4 - 2$ as $\alpha_1 = \sqrt[4]{2}$; $\alpha_2 = i\sqrt[4]{2}$; $\alpha_3 = -\sqrt[4]{2}$; $\alpha_4 = -i\sqrt[4]{2}$, corresponding to the labeling of elements of $D_4$ above. Then the fixed field of $\{1\}$ is $E = \mathbb{Q}(\sqrt[4]{2}, i)$ and the fixed field of $D_4$ is $\mathbb{Q}$. As for the subgroups of order 2, they correspond to subfields $K$ of $E$ such that $[K : \mathbb{Q}] = 4$. For example, it is clear that $\sqrt[4]{2} \in E(\langle (24) \rangle)$ and hence by counting degrees that $E(\langle (24) \rangle) = \mathbb{Q}(\sqrt[4]{2})$.

Likewise $E(\langle (13) \rangle) = \mathbb{Q}(i\sqrt[4]{2})$. As for $E(\langle (13)(24) \rangle)$, note that $\sqrt[4]{2} = (\sqrt[4]{2})^2 = (-\sqrt[4]{2})^2$ is fixed by $(13)(24)$, and also $i$ is fixed by $(13)(24)$ since if $\sigma(\sqrt[4]{2}) = -\sqrt[4]{2}$ and $\sigma(i) = -i\sqrt[4]{2}$, then

\[
\sigma(i) = \sigma(i\sqrt[4]{2}/\sqrt[4]{2}) = \sigma(i\sqrt[4]{2})/\sigma(\sqrt[4]{2}) = (-i\sqrt[4]{2})/(-\sqrt[4]{2}) = i.
\]

Thus $\mathbb{Q}(\sqrt[4]{2}, i) \subseteq E(\langle (13)(24) \rangle)$, so again by counting degrees they are equal. As for $E(\langle (12)(34) \rangle)$, note that $\sqrt[4]{2} + i\sqrt[4]{2} = \alpha_1 + \alpha_2 \in E(\langle (12)(34) \rangle)$. In particular, this forces $\mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2}) \neq F$. While it may not be obvious how to compute the degree $[\mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2}) : \mathbb{Q}]$, note that

\[
(\sqrt[4]{2} + i\sqrt[4]{2})^2 = (1 + i)^2(\sqrt[4]{2})^2 = 2i\sqrt[4]{2}.
\]

Thus $[\mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2}) : \mathbb{Q}(i\sqrt[4]{2})] = 2$ since $\sqrt[4]{2} + i\sqrt[4]{2} \notin \mathbb{Q}(i\sqrt[4]{2})$, and since $[\mathbb{Q}(i\sqrt[4]{2}) : \mathbb{Q}] = 2$ since $i\sqrt[4]{2} = \sqrt{-2}$, it follows that

\[
[\mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2}) : \mathbb{Q}(i\sqrt[4]{2})][\mathbb{Q}(i\sqrt[4]{2}) : \mathbb{Q}] = 4.
\]
Hence, again by counting degrees, \( E^{((12)(34))} = \mathbb{Q}(\sqrt{2} + i\sqrt{2}) \). Similarly, \( E^{((14)(23))} = \mathbb{Q}(\sqrt{2} - i\sqrt{2}) \).

Finally, there are the 3 fields \( E^{H_1}, E^{H_2}, E^{H_3} \). A computation shows that \( i \in E^{H_1} \), hence \( E^{H_1} = \mathbb{Q}(i) \). As for the others, clearly \( E^{H_2} = E^{((13)(24))} \cap E^{((12)(34))} \). Since \( E^{((13)(24))} = \mathbb{Q}(\sqrt{2}, i) \) and \( i\sqrt{2} \in E^{((12)(34))} \), \( i\sqrt{2} \in E^{H_2} \) and hence \( E^{H_2} = \mathbb{Q}(i\sqrt{2}) \). The other equality \( E^{H_3} = \mathbb{Q}(\sqrt{2}) \) is similar.

**Picture of the subgroups of \( D_4 \):**

![Diagram of subgroups of D4]

**Picture of the intermediate subfields between \( E \) and \( \mathbb{Q} \):**

![Diagram of intermediate subfields]

2