What is Liouville Quantum Gravity?

First introduced by Polyakov in 1981:
"Quantum Geometry of Bosonic Strings"

Brownian motion:
- Canonical random path
- Scaling limit of random walks

Liouville Quantum Gravity:
- Canonical two-dimensional geometry
- Conjectured limit of planar maps
Goal of field theory: compute correlations of certain observables called fields: $\langle \prod_{i \in I} \phi_i(z_i) \rangle$

Conformal Field Theory: conformal invariance in 2D
$\Rightarrow$ Imposes constraints on correlation functions

Examples of CFT’s:
- Continuum limit of critical Ising model
- Liouville Quantum Gravity

Another point of view: SLE curves (see Miller/Sheffield course)
Brownian motion seen as a path integral

Space of paths: \( \Sigma = \{ \sigma : [0, 1] \rightarrow \mathbb{R}, \sigma(0) = 0 \} \)

Action functional: \( S_{BM}(\sigma) = \frac{1}{2} \int_0^1 |\sigma'(r)|^2 dr \)

\[
\mathbb{E}[F((B_s)_{0 \leq s \leq 1})] = \frac{1}{Z} \int_\Sigma D\sigma F(\sigma) e^{-S_{BM}(\sigma)}
\]

\( D\sigma \): formal uniform measure on \( \Sigma \)

Classical Theory / Quantum Theory

Minimum of \( S_{BM} \) → straight line = classical solution
Path integral → Brownian motion = quantum correction
Some definitions

Let $M$ be a two-dimensional surface (sphere, torus,..).

Metric tensor $g : M \to S^+_2(\mathbb{R})$

Simple case: $g(x) = \begin{pmatrix} e^f(x) & 0 \\ 0 & e^f(x) \end{pmatrix}$

- Area of $A$: $\int_A e^f(x) \, dx^2 = \int_A \lambda_g(dx)$
- Gradient squared: $|\partial g X|^2 = e^{-f} |\partial X|^2$
- Scalar curvature $R_g = -e^{-f} \Delta f$.

Spherical metric on $\mathbb{R}^2$: $g(x) = \frac{4}{(1+|x|^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $R_g = 2$. 
Classical Liouville Theory

For all maps $X : M \rightarrow \mathbb{R}$, we define:

$$S_L(X, g) = \frac{1}{4\pi} \int_M (|\partial^g X|^2 + QR_g X + 4\pi \mu e^{\gamma X}) \lambda_g$$

$Q, \gamma, \mu > 0$ positive constants

Uniformization of $(M, g)$

Assume $X_{\text{min}}$ to be the minimum of $S_L$ and define $g' = e^{\gamma X_{\text{min}}} g$. Then $R_{g'} = -2\pi \mu \gamma^2$ if we choose $Q = \frac{2}{\gamma}$.

$\implies$ The minimum of $S_L$ provides a metric of constant negative curvature.
Defining Liouville Quantum Gravity

Formal definition

Random metric $e^{\gamma \phi} g$ where the law of $\phi$ is given by:

$$\mathbb{E}[F(\phi)] = \frac{1}{Z} \int F(X) e^{-S_L(X,g)} DX$$

First goal: give a meaning to $\phi$ for different $M$.

- $M = \text{Riemann sphere}:$ David-Kupiainen-Rhodes-Vargas
- $M = \text{Torus or higher genus}:$ David-Guillarmou-Rhodes-Vargas
- $M = \text{Unit disk}:$ Huang-Rhodes-Vargas
- $M = \text{Annulus}:$ Remy

$\phi = \text{Liouville field}$
Why the Liouville action?

- $|\partial^g X|^2$: analogue of the $|\sigma'|^2$ for Brownian motion

\[
\frac{1}{Z} \int F(X) e^{-\frac{1}{4\pi} \int_M |\partial^g X|^2 \lambda_g} DX : \text{formally defines the law of the Gaussian Free Field (GFF)}
\]

- $QR_g X$: curvature term

- $\int_M e^{\gamma X} \lambda_g = \text{area of } M \text{ in the metric } g' = e^{\gamma X} g$
  $\Rightarrow$ penalizes large areas
  $\Rightarrow$ required to have a well defined Liouville field
Insertion points

- For $M = S^2$, Gauss-Bonnet: $\int_{S^2} R_g \lambda_g = 8\pi > 0$

- No metric of constant negative curvature
  $\Rightarrow$ $S_L$ has no minimum
  $\Rightarrow \frac{1}{Z} \int F(X) e^{-S_L(X,g)} DX$ not defined

- Instead we consider:
  \[
  \frac{1}{Z} \int F(X) e^{\sum_{i=1}^n \alpha_i X(z_i)} e^{-S_L(X,g)} DX = \langle \prod_{i=1}^n e^{\alpha_i X(z_i)} \rangle
  \]
  $= \text{correlation function of the fields } e^{\alpha_i \phi(z_i)}$

- $(z_i, \alpha_i)$: insertion points $= \text{singularities of the metric}$

- For $S^2$: at least 3 insertions required
Computing the partition function

Consider $M = S^2$

Main objective: give a mathematical meaning to

$$\Pi_{\mu, \gamma}^{(z_i, \alpha_i)}(F, g) = \int DX \prod_{i} e^{\alpha_i X(z_i)} e^{-S_L(X, g)} F(X)$$

where again:

$$S_L(X, g) = \frac{1}{4\pi} \int_M (|\partial^g X|^2 + QR g X + 4\pi \mu e^{\gamma X}) \lambda_g$$

Remark: $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$
Goal: give meaning to \( \frac{1}{2} \int \tilde{F}(X)e^{-\frac{1}{4\pi} \int_M |\partial^g X|^2 \lambda_g} DX \)

\[
e^{-\frac{1}{4\pi} \int_M |\partial^g X|^2 d\lambda_g} = e^{-\frac{1}{2} \int_M X(-\frac{\Delta}{2\pi})X d\lambda_g}
\]

Density of an infinite dimensional Gaussian vector of covariance function \((-\frac{\Delta}{2\pi})^{-1} = \text{Green function}
\Rightarrow \text{defines a GFF}

Gradient term: defines \( X \) up to a constant \( c \)
\( c = \text{average value of the field} \)
We integrate over \( c \) with the Lebesgue measure.
Step 1: the squared gradient term

\[ \frac{1}{Z} \int \tilde{F}(X) e^{-\frac{1}{4\pi} \int_M |\partial^g X|^2 \lambda_g} DX = \int_{\mathbb{R}} dc \mathbb{E}[\tilde{F}(X + c)] \]

- On the l.h.s: formal functional integral
- On the r.h.s: \( X \) has the law of a GFF

The partition function \( \Pi_{\mu,\gamma}^{(z_i,\alpha_i)}(F, g) \) becomes:

\[ \int_{\mathbb{R}} dc \mathbb{E}[F(X+c) \prod_i e^{\alpha_i(X(z_i)+c)} e^{-\frac{1}{4\pi} \int_M (QR_g(X+c) + 4\pi \mu e^{\gamma(X+c)}) \lambda_g]} \]
Step 2: Gaussian multiplicative chaos

$X$ is a random distribution, $e^{\gamma X}$ is ill defined.

Regularization procedure: circle average $X_\epsilon$

$$X_\epsilon(z) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} X(z + \epsilon e^{i\theta}) d\theta$$

Gaussian multiplicative chaos

The following limit exists in probability in the sense of weak convergence of measures for $\gamma \in [0, 2)$:

$$\lim_{\epsilon \to 0} e^{\gamma X_\epsilon(z) - \frac{\gamma^2}{2} \mathbb{E}[X_\epsilon(z)^2]} d\lambda_g(z) = e^{\gamma X(z) - \frac{\gamma^2}{2} \mathbb{E}[X(z)^2]} d\lambda_g(z)$$

Remark: $\mathbb{E}[X_\epsilon(z)^2] + \ln \epsilon$ remains bounded as $\epsilon \to 0$. 

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Step 3 : regularization of the partition function

Define \( \Pi_{\mu, \gamma}^{(z_i, \alpha_i)}(F, g, \epsilon) = \)

\[
\int_{\mathbb{R}} dc \mathbb{E}[F(X_\epsilon + c) \prod_i \epsilon \frac{\alpha_i^2}{2} e^{\alpha_i(X_\epsilon(z_i) + c)} e^{-\frac{1}{4\pi} \int_M (QRg(X_\epsilon+c) + 4\pi \mu \epsilon \frac{\gamma^2}{2} e^{\gamma(X_\epsilon+c)}) \lambda_g}]
\]

When does the limit \( \lim_{\epsilon \to 0} \Pi_{\mu, \gamma}^{(z_i, \alpha_i)}(F, g, \epsilon) \) exist?
Main result

Consider $M = \mathbb{S}^2$

Non-triviality of the partition function

Assume $\gamma \in [0, 2)$ and $\mu > 0$, then

$$\prod_{\mu, \gamma}^{(z_i, \alpha_i)}(F, g) = \lim_{\epsilon \to 0} \prod_{\mu, \gamma}^{(z_i, \alpha_i)}(F, g, \epsilon)$$

exists and is finite and non zero

$\iff \sum_i \alpha_i > 2Q$ and $\forall i, \alpha_i < Q$

Definition of the law of the Liouville field $\phi$:

$$\mathbb{E}[F(\phi)] = \frac{\prod_{\mu, \gamma}^{(z_i, \alpha_i)}(F, g)}{\prod_{\mu, \gamma}^{(z_i, \alpha_i)}(1, g)}$$
Liouville measure

- $\phi$ is a random distribution $\Rightarrow$ difficult to define $e^{\gamma \phi}$
- Well defined Liouville measure $Z(A) = \int_A e^{\gamma \phi} \lambda_g$
  
  Conjectured limit of uniform planar maps for
  $\gamma = \sqrt{\frac{8}{3}}$

  Conjectured limit of planar maps with an Ising model for $\gamma = \sqrt{3}$
Surfaces with boundary

Must add boundary terms to $S_L$:

\[
\tilde{S}_L(X, g) = S_L(X, g) + \frac{1}{2\pi} \int_{\partial M} (QK_g X + 2\pi \mu_\partial e^{\frac{\gamma}{2} X}) \lambda_\partial g
\]

Example: $M =$ unit disk

- Bulk insertion points $(z_i, \alpha_i)$
- Boundary insertion points $(s_j, \beta_j)$

Non-triviality of the partition function

Assume $\gamma \in [0, 2)$, $\mu_\partial > 0$, and $\mu > 0$, then

\[
\Pi_{\mu, \gamma}^{(z_i, \alpha_i)}(F, g) = \lim_{\epsilon \to 0} \Pi_{\mu, \gamma}^{(z_i, \alpha_i)}(F, g, \epsilon)
\]

exists and is finite and non zero

\[\iff \sum_i \alpha_i + \sum_j \frac{\beta_j}{2} > Q, \forall i, \alpha_i < Q \text{ and } \forall j, \beta_j < Q\]
Higher genus: non trivial moduli space

- Torus
- Annulus

\[ A(a, b) = \text{annulus in the plane with radii } a < b \]

Then \[ A(a, b) \sim A(a', b') \iff \frac{a}{b} = \frac{a'}{b'} \]

\[ \tau = \frac{a}{b} = \text{modular parameter, important in physics.} \]
Thank you for listening!

References: