Liouville Quantum Gravity on Riemann Surfaces

Guillaume Remy

École Normale Supérieure

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Quantum Field Theory: a general framework in physics

Goal: compute correlations of fields: \( \langle \prod_{i \in I} \phi_i(z_i) \rangle \)

Conformal Field Theory: conformal invariance in 2D

Continuum limit of the Ising model
   Fields: spin operator

Liouville Quantum Gravity:
   Polyakov, ”Quantum Geometry of Bosonic Strings”
Brownian motion seen as a path integral

Space of paths: \( \Sigma = \{ \sigma : [0, 1] \rightarrow \mathbb{R}, \sigma(0) = 0 \} \)

Action functional: \( S_{BM}(\sigma) = \frac{1}{2} \int_0^1 |\sigma'(r)|^2 dr \)

\[ \mathbb{E}[F((B_s)_{0 \leq s \leq 1})] = \frac{1}{Z} \int_\Sigma D\sigma F(\sigma) e^{-S_{BM}(\sigma)} \]

\( D\sigma \): formal uniform measure on \( \Sigma \)

Classical Theory / Quantum Theory

Minimum of \( S_{BM} \) → straight line = classical solution
Path integral → Brownian motion = quantum correction
Some definitions

Let \((M, g)\) be a Riemann surface with a metric \(g\).

**Metric tensor** \(g : M \rightarrow S_2^+(\mathbb{R})\)

- Length of a curve \(z = z^i(t)\): 
  \[
  \int_a^b \sqrt{g_{ij}(z(t))} \frac{dz^i}{dt} \frac{dz^j}{dt} dt
  \]

- Area of \(A\): 
  \[
  \int_A \sqrt{\det g(x)} dx^2 = \int_A \lambda_g(dx)
  \]

- Scalar curvature \(R_g\):
  
  For \(g(x) = \begin{pmatrix} e^{f(x)} & 0 \\ 0 & e^{f(x)} \end{pmatrix}\), \(R_g = -e^{-f} \Delta f\).

Spherical metric on \(\mathbb{R}^2\): 
\[
  g(x) = \frac{4}{(1+|x|^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
\(R_g = 2\).
Classical Liouville Theory

For all maps $X : M \rightarrow \mathbb{R}$, we define:

$$S_L(X, g) = \frac{1}{4\pi} \int_M (|\partial^g X|^2 + QR^g X + 4\pi\mu e^{\gamma X}) \lambda_g$$

- $|\partial^g X|^2 = \sum_{i,j} g_{ij} \partial^i X \partial^j X$
- $Q, \gamma, \mu > 0$ positive constants

Uniformization of $(M, g)$

Assume $X_{\text{min}}$ to be the minimum of $S_L$ and define $g' = e^{\gamma X_{\text{min}}} g$. Then $R_{g'} = -2\pi\mu\gamma^2$ if we choose $Q = \frac{2}{\gamma}$. \(\implies \) The minimum of $S_L$ provides a metric of constant negative curvature.
Defining Liouville Quantum Gravity

Formal definition

Random metric $e^{\gamma\phi}g$ where the law of $\phi$ is given by:

$$\mathbb{E}[F(\phi)] = \frac{1}{Z} \int F(X)e^{-S_L(X,g)}DX$$

First goal: give a meaning to $\phi$ for different $M$.

- $M = \text{Riemann sphere}:$ David-Kupiainen-Rhodes-Vargas
- $M = \text{Torus or higher genus}:$ David-Guillarmou-Rhodes-Vargas
- $M = \text{Unit disk}:$ Huang-Rhodes-Vargas
- $M = \text{Annulus}:$ Remy

$\phi = \text{Liouville field}$
Why the Liouville action?

- $|\partial^g X|^2$: analogue of the $|\sigma'|^2$ for Brownian motion

$$\frac{1}{Z} \int F(X) e^{-\frac{1}{4\pi} \int_M |\partial^g X|^2 \lambda_g} DX : \text{formally defines the law of the Gaussian Free Field (GFF)}$$

GFF: Gaussian process with covariance function given by the Green function of the Laplacian $-\Delta_g$

- $QR^g X$: curvature term

- $\int_M e^{\gamma X} \lambda_g = \text{area of } M \text{ in the metric } g' = e^{\gamma X} g$

  $\Rightarrow$ penalizes large areas

  $\Rightarrow$ required to have a well defined Liouville field
Liouville measure

- $\phi$ is a random distribution $\Rightarrow$ difficult to define $e^{\gamma \phi}$
- Well defined Liouville measure $Z(A) = \int_A e^{\gamma \phi} \lambda_g$

Conjectured limit of uniform planar maps for $\gamma = \sqrt{\frac{8}{3}}$

Conjectured limit of planar maps with an Ising model for $\gamma = \sqrt{3}$

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Discrete model / Continuum limit

Brownian motion = scaling limit of random walks
Liouville quantum gravity = limit of discrete 2D models (like planar maps)
For $M = \mathbb{S}^2$, Gauss-Bonnet: $\int_{\mathbb{S}^2} R_g \lambda_g = 8\pi > 0$

No metric of constant negative curvature
$\implies S_L$ has no minimum
$\implies \frac{1}{Z} \int F(X) e^{-S_L(X,g)} DX$ not defined

Instead we consider:

$$\frac{1}{Z} \int F(X) e^{\sum_{i=1}^{n} \alpha_i X(z_i)} e^{-S_L(X,g)} DX = \langle \prod_{i=1}^{n} e^{\alpha_i X(z_i)} \rangle$$

$= $ correlation function of the fields $e^{\alpha_i X(z_i)}$

$(z_i, \alpha_i)$: insertion points $= $ singularities of the metric

For $\mathbb{S}^2$: at least 3 insertions required