PROOF OF THE PONSOT-TESCHNER FORMULA FOR BOUNDARY LIOUVILLE CFT

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Abstract. We prove that the probabilistic definition of the boundary three-point structure constant in Liouville conformal field theory (LCFT) agrees with the formula proposed by Ponsot and Techsner (2002). This formula also describes the fusion kernel of the Virasoro conformal blocks, an important function in various contexts of mathematical physics. As an intermediate step, we obtain the formula for the boundary reflection coefficient of LCFT proposed by Fateev, Zamolodchikov and Zamolodchikov (2000). Earlier work by two of the authors has determined these quantities when there is no bulk Liouville potential. To overcome the essential difficulty coming from the presence of both bulk and boundary Liouville potentials, we prove a Belavin-Polyakov-Zamolodchikov differential equation for Liouville correlation functions on the disk, which only holds if the bulk and boundary cosmological constants are coupled in an intriguing fashion. We also rely on an integrable input from the mating-of-trees theory for Liouville quantum gravity (LQG). Our method can be used to derive the one-bulk-one-boundary structure constant, which completes the determination of all structure constants needed for the conformal bootstrap of boundary LCFT. Our results also give the joint law of the area and boundary lengths of quantum triangles and two-pointed quantum disks in LQG, which are quantum surfaces appeared naturally in LQG coupled with Schramm-Loewner evolution.

1. Introduction

Liouville conformal field theory (LCFT) describes the law of the conformal factor of random surfaces in Liouville quantum gravity. Introduced by Polyakov [Pol81] in theoretical physics, LCFT was recently made rigorous in probability theory, first in the case of the Riemann sphere in [DKRV16], and then in the case of a simply connected domain with boundary in [HRV18]; see also [DRV16, Rem18, GRV19] for the case of other typologies. Following the framework of [BPZ84], the correlation functions of LCFT can be solved by the conformal bootstrap program. In the probabilistic framework, this was recently carried out on surfaces without boundary [KRV20], [GKRV20, GKRV21]. The initial input of the conformal bootstrap is the *structure constant*. For surfaces without boundary, it is the three-point correlation function on the sphere. It has an exact expression called the DOZZ formula which was proposed in [DO94, ZZ96] and proved in [KRV20].

For LCFT on surfaces with boundary, the structure constants are the correlation functions on the disk with three points on the boundary, or one point in the bulk and one on the boundary. The theory involves both the bulk and the boundary Liouville potentials. When the bulk Liouville potential is absent, the structure constants were obtained by Remy and Zhu [RZ21]. When there is one bulk insertion and no boundary ones, the structure constant was obtained by Ang, Remy, and Sun in [ARS21], confirming an earlier proposal of Fateev, Zamolodchikov and Zamolodchikov (FZZ) in [FZZ00]. The conformal bootstrap is also applicable in the boundary case and Wu [Wu22] recently proved a bootstrap formula for the annulus with one insertion at each boundary, where the FZZ formula gives the needed structure constant.

In this paper we obtain the exact formula for the boundary three-point structure constant proposed by Ponsot and Techsner [PT02], where both the boundary and bulk Liouville potential are

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non-zero. As an intermediate step, we obtain the formula for the boundary reflection coefficient of LCFT proposed in [FZZ00]; see Section 1.1. Our results together with those in [ARS21] can determine the bulk-boundary structure constant via known techniques. This completes the determination of all structure constants required for the conformal bootstrap of boundary LCFT. To overcome the essential difficulty coming from the bulk-boundary coupling, we prove a novel Belavin-Polyakov-Zamolodchikov (BPZ) differential equation for LCFT on the disk, which is of independent interest. We also rely on the exact law of the quantum disk from the mating-of-trees theory for Liouville quantum gravity [DMS14, AG21]. See Section 1.2 for a summary of our method.

Besides its relevance to the conformal bootstrap, the boundary three-point structure constant agrees modulo a prefactor with the fusion kernel of the Virasoro conformal blocks [PT02], an important special function with various interpretations in mathematical physics. Moreover, it gives the joint area and boundary length distribution of natural random surfaces called quantum triangles in Liouville quantum gravity coupled with the Schramm-Loewner evolution [She16, DMS14, AHS20, ASY22].

1.1. **Main results.**

We start by recalling the probabilistic construction of LCFT on the disk from [HRV18], which is adapted from the construction on the Riemann sphere performed in [DKRV16]. By conformal invariance we will use the upper half plane $\mathbb{H}$ as the base domain. Our presentation is brief with more details provided in Sections 2 and 3. Fix the global constants

$$\gamma \in (0,2) \quad \text{and} \quad Q = \frac{\gamma}{2} + \frac{2}{\gamma}. \tag{1.1}$$

In physics LCFT is defined using a formal path integral. Fix N points $z_i \in \mathbb{H}$ with associated weights $\alpha_i \in \mathbb{R}$ and similarly M points $s_j \in \mathbb{R}$ with associated weights $\beta_j \in \mathbb{R}$. The correlation function associated to these points is given by the formal integral

$$\left\langle \prod_{i=1}^N V_{\alpha_i}(z_i) \prod_{j=1}^M B_{\beta_j}^{\sigma_j,\sigma_{j+1}}(s_j) \right\rangle = \int_{\phi:\mathbb{H}\mapsto\mathbb{R}} D\phi \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \prod_{j=1}^M e^{\frac{\beta_j}{2}\phi(s_j)} e^{-S_L(\phi)}, \tag{1.2}$$

where here S_L is the Liouville action given by:

$$S_L(\phi) = \frac{1}{4\pi}\int_{\mathbb{H}} \left(|\partial^g \phi|^2 + QR_g\phi + 4\pi\mu e^{\gamma\phi}\right) d\lambda_g + \frac{1}{2\pi}\int_{\mathbb{R}} \left(QK_g\phi + 2\pi\mu_B e^{\frac{\gamma}{2}\phi}\right) d\lambda_{\partial g}. \tag{1.3}$$

Here g is a background metric with R_g and λ_g representing the curvature and volume in the bulk, respectively, and $(K_g, \lambda_{\partial g})$ being their boundary counterparts. The terms involving μ and μ_B are the bulk and boundary Liouville potentials, respectively. Here $\mu > 0$ and μ_B is a complex valued function on $\mathbb{R}$ which is piecewise constant in between boundary insertions. We assume the s_j are chosen in counterclockwise order on $\mathbb{R}$ and set $\mu_j = \mu_B(x)$ for $x \in (s_{j-1}, s_j)$, with convention $s_0 = -\infty$, $s_{M+1} = \infty$. We always assume $\mathrm{Re}(\mu_j) \geq 0$. To state the results below we will also need the $\sigma_j \in \mathbb{C}$ parameters related to the μ_j by the relation:

$$\mu_j = g(\sigma_j) := \sqrt{\frac{1}{\sin(\pi\frac{\gamma^2}{4})}} \cos\left(\pi\gamma(\sigma_j - \frac{Q}{2})\right). \tag{1.4}$$

We will now give a rigorous probabilistic meaning to (1.2) in the case $N = 0$ and $M = 3$. Let $P_{\mathbb{H}}$ be the probability measure corresponding to the free-boundary Gaussian free field on $\mathbb{H}$ normalized to have average zero on the semi-circle $\partial\mathbb{D} \cap \mathbb{H}$. Let the infinite measure $\mathrm{LF}_{\mathbb{H}}(d\phi)$ be the law of $\phi(z) = h(z) - 2Q\log|z|_+ + \mathbf{c}$, where $|z|_+ := \max(|z|, 1)$ and $(h, \mathbf{c})$ is sampled according to $P_{\mathbb{H}} \times [e^{-Qc}\, dc]$. We call the field ϕ sampled from $\mathrm{LF}_{\mathbb{H}}$ a Liouville field on $\mathbb{H}$. This definition of $\mathrm{LF}_{\mathbb{H}}$ corresponds to choosing the background metric in (1.3) to be $g(x) = |z|_+^{-4}$. We define the bulk and boundary Gaussian multiplicative chaos (GMC) measures of ϕ as the limits (see e.g. [Ber17, RV14]):

$$A_\phi = \lim_{\varepsilon\to 0} \epsilon^{\frac{\gamma^2}{2}} \int_{\mathbb{H}} e^{\gamma\phi_\epsilon(z)} d^2z \text{ on } \mathbb{H}, \qquad \text{and} \qquad L_\phi = \lim_{\varepsilon\to 0} \epsilon^{\frac{\gamma^2}{4}} \int_{\mathbb{R}} e^{\frac{\gamma}{2}\phi_\epsilon(z)} dz \text{ on } \mathbb{R}.$$

Given three points s_1, s_2, s_3 lying counterclockwise on $\mathbb{R}$, let L_i be the ν_ϕ-length of the counter-clock arc on $\partial\mathbb{H}$ from s_i to s_i+1, where we identify s_4 as s_1. Fix $\mu > 0$ and $\mu_1, \mu_2, \mu_3 \geq 0$. For $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ satisfying the *Seiberg bound*

$$\sum_{i=1}^{3} \beta_i > 2Q \quad \text{and} \quad \beta_i < Q,$$

the *boundary three-point correlation function* of LCFT on $\mathbb{H}$ with bulk cosmoligical constant $\mu > 0$ and boundary cosmological constants $(\mu_i)_{1\leq i\leq 3}$ is defined by:

$$\left\langle \prod_{j=1}^{3} B_{\beta_j}^{\sigma_j,\sigma_{j+1}}(s_j) \right\rangle = \int \prod_{j=1}^{3} e^{\frac{\beta_j}{2}\phi(s_j)} \cdot e^{-\mu\mu_\phi(\mathbb{H}) - \sum_{i=1}^{3}\mu_i L_i} \mathrm{LF}_{\mathbb{H}}(d\phi). \tag{1.5}$$

Here although ϕ is only a generalized function, the factor $\prod_{j=1}^{3} e^{\frac{\beta_j}{2}\phi(s_i)}$ in the integrand can be made sense of by regularization and Girsanov's theorem. Moreover, the Seiberg bound ensures that the integration in (1.5) is finite. We will review this construction in full detail in Section 2.1.

Due to conformal symmetry, the 3-point boundary correlation function has the following form:
(1.6)

$$\left\langle B_{\beta_1}^{\sigma_1,\sigma_2}(s_1) B_{\beta_2}^{\sigma_2,\sigma_3}(s_2) B_{\beta_3}^{\sigma_3,\sigma_1}(s_3) \right\rangle = \frac{H \mu^{\frac{2Q-\beta_1-\beta_2-\beta_3}{2\gamma}}}{|s_1-s_2|^{\Delta_1+\Delta_2-\Delta_3} |s_1-s_3|^{\Delta_1+\Delta_3-\Delta_2} |s_2-s_3|^{\Delta_2+\Delta_3-\Delta_1}}.$$

Here $\Delta_i = \frac{\beta_i}{2}(Q - \frac{\beta_i}{2})$, and H only depends on β_i and $\mu_i/\sqrt{\mu}$ for $1 \leq i \leq 3$. We call H the *boundary three-point structure constant* for LCFT. For simplicity, in the rest of the paper we will set the bulk cosmological constant to be $\mu = 1$ and write H as $H_{(\mu_1,\mu_2,\mu_3)}^{(\beta_1,\beta_2,\beta_3)}$.

Ponsot-Teschner [PT02] proposed a remarkable formula for H under the reparametrization

$$H \begin{pmatrix} \beta_1, \beta_2, \beta_3 \\ \sigma_1, \sigma_2, \sigma_3 \end{pmatrix} := H_{(g(\sigma_1), g(\sigma_2), g(\sigma_3))}^{(\beta_1,\beta_2,\beta_3)} \quad \text{where } g(\sigma) = \sqrt{\frac{1}{\sin(\pi\frac{\gamma^2}{4})}} \cos\left(\pi\gamma(\sigma - \frac{Q}{2})\right). \tag{1.7}$$

The Ponsot-Teschner formula is expressed in terms of the Barnes' double Gamma function $\Gamma_{\frac{\gamma}{2}}(x)$ and the double sine function $S_{\frac{\gamma}{2}}(x) = \frac{\Gamma_{\frac{\gamma}{2}}(x)}{\Gamma_{\frac{\gamma}{2}}(Q-x)}$, which are prevalent in LCFT. Both functions admit a meromorphic extension on $\mathbb{C}$ with an explicit pole structure; see Section A.2 for more details. Using these two functions and setting $\overline{\beta} = \beta_1 + \beta_2 + \beta_3$, the Ponsot-Teschner formula is given by:

(1.8)

$$H_{\mathrm{PT}}\begin{pmatrix}\beta_1,\beta_2,\beta_3\\ \sigma_1,\sigma_2,\sigma_3\end{pmatrix} = 2\pi\left(\frac{\pi(\frac{\gamma}{2})^{2-\frac{\gamma^2}{2}}\Gamma(\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\right)^{\frac{Q-\bar{\beta}/2}{\gamma}}$$
$$\times\frac{\Gamma_{\frac{\gamma}{2}}(2Q-\frac{\overline{\beta}}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\beta_1+\beta_3-\beta_2}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta_1+\beta_2-\beta_3}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta_2+\beta_3-\beta_1}{2})}{\Gamma_{\frac{\gamma}{2}}(Q)\Gamma_{\frac{\gamma}{2}}(Q-\beta_1)\Gamma_{\frac{\gamma}{2}}(Q-\beta_2)\Gamma_{\frac{\gamma}{2}}(Q-\beta_3)}$$
$$\times\frac{1}{S_{\frac{\gamma}{2}}(\frac{\beta_1}{2}+\sigma_1-\sigma_2)S_{\frac{\gamma}{2}}(\frac{\beta_3}{2}+\sigma_3-\sigma_1)S_{\frac{\gamma}{2}}(\frac{\beta_1}{2}+\sigma_1+\sigma_2-Q)S_{\frac{\gamma}{2}}(\frac{\beta_3}{2}-\sigma_1-\sigma_3+Q)}$$
$$\times\int_{\mathcal{C}}\frac{S_{\frac{\gamma}{2}}(-\frac{\beta_2}{2}+\sigma_2+\sigma_3+r)S_{\frac{\gamma}{2}}(Q-\frac{\beta_2}{2}+\sigma_3-\sigma_2+r)S_{\frac{\gamma}{2}}(\frac{\beta_3}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(Q-\frac{\beta_3}{2}+\sigma_3-\sigma_1+r)}{S_{\frac{\gamma}{2}}(Q+\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(2Q-\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(2\sigma_3+r)S_{\frac{\gamma}{2}}(Q+r)}\frac{dr}{i}.$$

Here $\mathcal{C}$ is a properly chosen contour from $-i\infty$ to $i\infty$ such that the integral is meromorphic in all of the six variables in the entire complex plane. We provide more details on H_{PT} in Appendix A.3. The first main result of our paper is the confirmation of Ponsot-Teschner formula.

Theorem 1.1. *Let $\gamma\in(0,\sqrt{2})$, $\sum_{i=1}^3\beta_i>2Q$, $\beta_i<Q$, and $\mu_i=g(\sigma_i)>0$ for $i=1,2,3$. Then*

$$2\pi H\begin{pmatrix}\beta_1,\beta_2,\beta_3\\ \sigma_1,\sigma_2,\sigma_3\end{pmatrix} = H_{\mathrm{PT}}\begin{pmatrix}\beta_1,\beta_2,\beta_3\\ \sigma_1,\sigma_2,\sigma_3\end{pmatrix}. \tag{1.9}$$

Our second main result concerns the reflection coefficient for boundary LCFT. Formally speaking, this quantity is defined as $R_{\mu_1,\mu_2}(\beta):=|s_1-s_2|^{2\Delta_\beta}\left\langle B_\beta^{\sigma_1,\sigma_2}(s_1)B_\beta^{\sigma_2,\sigma_1}(s_2)\right\rangle$ for $\beta\in\mathbb{R}$, which by conformal symmetry does not depend on $s_1,s_2\in\mathbb{R}$. Although this is in the same spirit as (1.6), there are multiple subtleties in the rigorous definition of $R_{\mu_1,\mu_2}(\beta)$. First, instead of integrating $e^{\frac{\beta}{2}\phi(s_1)}e^{\frac{\beta}{2}\phi(s_2)}\mathrm{LF}_{\mathbb{H}}(d\phi)$ following (1.5), we need to integrate against $\mathcal{M}_2^{\mathrm{disk}}(\beta)$, which is the law of the two-pointed quantum disk with β-insertion. This is because $\mathcal{M}_2^{\mathrm{disk}}(\beta)$ can be viewed as $e^{\frac{\beta}{2}\phi(s_1)}e^{\frac{\beta}{2}\phi(s_2)}\mathrm{LF}_{\mathbb{H}}(d\phi)$ modulo the redudant conformal symmetries of $\mathbb{H}$ fixing s_1,s_2. The measure $\mathcal{M}_2^{\mathrm{disk}}(\beta)$ is first introduced in [DMS14] and implicitly used in [RZ21]. It describes the law of a quantum surface with two boundary marked points. We will give its precise definition in Section 2.2.

Another subtlety in defining $R_{\mu_1,\mu_2}(\beta)$ is that we cannot directly integrate $e^{-A-\mu_1L_1-\mu L_2}$ over $\mathcal{M}_2^{\mathrm{disk}}(\beta)$ as suggested by (1.5), where A and L_1,L_2 are the area and the two boundary lengths of a sample from $\mathcal{M}_2^{\mathrm{disk}}(\beta)$, respectively. In fact, there is no β such that this integration is finite. The same issue arises in [ARS21] where the LCFT correlation function on the disk with one bulk insertion is considered. In both cases, one needs to truncate the function e^{-x} near $x=0$, after which the integral is finite for some range of β. Concretely, for $\mu_1\geq0,\mu_2\geq0$, we define

$$R_{\mu_1,\mu_2}(\beta):=\frac{2(Q-\beta)}{\gamma}\int(e^{-A-\mu_1L_1-\mu L_2}-1)\,\mathrm{d}\mathcal{M}_2^{\mathrm{disk}}(\beta)\quad\text{for }\beta\in(\frac{2}{\gamma},Q). \tag{1.10}$$

Then $R_{\mu_1,\mu_2}(\beta)$ is indeed finite. We will give the full detail of the integral in (1.10) in Section 2.2, including its finiteness. Here we still use the convention that bulk cosmological constant $\mu=1$ as in the definition of $H_{(\mu_1,\mu_2,\mu_3)}^{(\beta_1,\beta_2,\beta_3)}$. The prefactor $\frac{2(Q-\beta)}{\gamma}$ is to match with [RZ20a].

In the seminal work [FZZ00], Fateev, Zamolodchikov and Zamolodchikov proposed a formula for the boundary reflection coefficient under the same reparametrization as in (1.7):

$$R(\beta,\sigma_1,\sigma_2) := R_{g(\sigma_1),g(\sigma_2)}(\beta) \quad \text{where } g(\sigma) = \sqrt{\frac{1}{\sin(\pi\frac{\gamma^2}{4})}}\cos\left(\pi\gamma(\sigma-\frac{Q}{2})\right). \tag{1.11}$$

Their formula is

(1.12)

$$R_{\mathrm{FZZ}}(\beta,\sigma_1,\sigma_2) = \left(\frac{\pi\mu(\frac{\gamma}{2})^{2-\frac{\gamma^2}{2}}\Gamma(\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\right)^{\frac{Q-\beta}{\gamma}} \frac{\Gamma_{\frac{\gamma}{2}}(\beta-Q)S_{\frac{\gamma}{2}}(2Q-\frac{\beta}{2}-\sigma_1-\sigma_2)S_{\frac{\gamma}{2}}(\sigma_1+\sigma_2-\frac{\beta}{2})}{\Gamma_{\frac{\gamma}{2}}(Q-\beta)S_{\frac{\gamma}{2}}(\frac{\beta}{2}+\sigma_2-\sigma_1)S_{\frac{\gamma}{2}}(\frac{\beta}{2}+\sigma_1-\sigma_2)}.$$

Our second main result is the confirmation of their proposal.

Theorem 1.2. *Let $\gamma \in (0,\sqrt{2})$, $\beta \in (\frac{2}{\gamma}, Q)$, $\mu_1 \geq 0$, and $\mu_2 \geq 0$. Then*

$$R(\beta,\sigma_1,\sigma_2) = R_{\mathrm{FZZ}}(\beta,\sigma_1,\sigma_2). \tag{1.13}$$

A crucial step to our proofs of Theorem 1.1 and 1.2 is the following BPZ equations, which is of independent interest. The key novel feature of the equation is an intriguing constraint on the cosmological constants around the degenerate insertion; see Figure 1. It was conjectured in the seminal paper of Fateev, Zamolodchikov and Zamolodchikov [FZZ00, Section 4] after the numerical verification of some special cases.

Theorem 1.3. *Consider $(z_i,\alpha_i)_{1\leq i\leq N} \in \mathbb{H}\times\mathbb{R}$, $(s_j,\beta_j)_{1\leq j\leq M} \in \mathbb{R}\times\mathbb{R}$, and $(s,-\chi) \in \mathbb{R}\times\{-\frac{\gamma}{2},-\frac{2}{\gamma}\}$ satisfying the Seiberg bounds $\alpha_i < Q$, $\beta_j < Q$, and $2\sum_i \alpha_i + \sum_j \beta_j > 2Q+\chi$. When $\chi = \frac{2}{\gamma}$ let $\gamma \in (0,2)$ and when $\chi = \frac{\gamma}{2}$ assume $\gamma \in (0,\sqrt{2})$. Assume further that there exists $j_0 \in \{1,2,\dots,M+1\}$ such that $s_{j_0-1} < s < s_{j_0}$, using here the convention $s_0 = -\infty$, $s_{M+1} = +\infty$. Then we have:*

$$\left(\frac{1}{\chi^2}\partial_{ss} + \sum_{i=1}^N\left(\frac{1}{s-z_i}\partial_{z_i} + \frac{1}{s-\bar{z}_i}\partial_{\bar{z}_i}\right) + \sum_{j=1}^M \frac{1}{s-s_j}\partial_{s_j} + \sum_{i=1}^N\left(\frac{\Delta_{\alpha_i}}{(s-z_i)^2} + \frac{\Delta_{\alpha_i}}{(s-\bar{z}_i)^2}\right) + \sum_{j=1}^M \frac{\Delta_{\beta_j}}{(s-s_j)^2}\right)$$

$$\left\langle B_{-\chi}^{\sigma_{j_0}\pm\frac{\chi}{2},\sigma_{j_0}}(s)\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j\neq j_0-1} B_{\beta_j}^{\sigma_j,\sigma_{j+1}}(s_j)B_{\beta_{j_0-1}}^{\sigma_{j_0-1},\sigma_{j_0}\pm\frac{\chi}{2}}(s_{j_0})\right\rangle = 0. \tag{1.14}$$

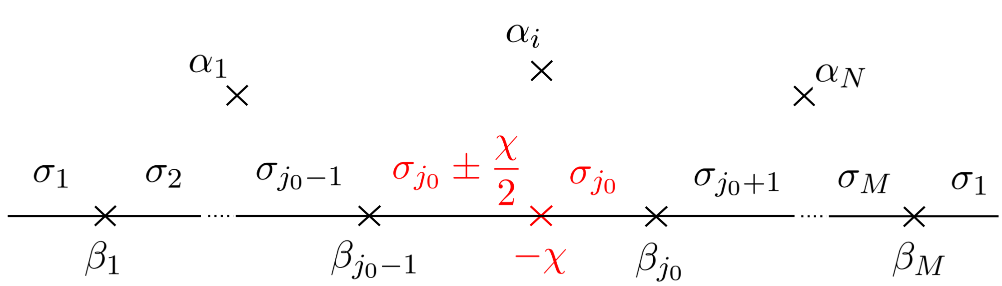

FIGURE 1. Parameters in the BPZ equation in Theorem 1.3

Remark 1.4. *In the above theorem due to a technical obstruction we have to restrict the parameter γ to $(0,\sqrt{2})$ in the case where $\chi = \frac{\gamma}{2}$. This implies that we have so far only shown Theorems 1.1 and 1.2 in the range $\gamma \in (0,\sqrt{2})$. In upcoming work in preparation M. Ang will derive Theorem*

1.3 in the full range of parameters by an alternative method. This will then allow to extend the range of validity of Theorems 1.1 and 1.2 to $\gamma \in (0,2)$.

1.2. **Novelties in the proofs.** Our proofs of Theorem 1.1 and 1.2 at the high level uses the same strategy as in the proof of DOZZ formula [KRV15, KRV20], which consists of three steps.

Step 1 Derive the BPZ equation in Theorem 1.3.

Step 2 Specializing the BPZ equation to $M = 3$ gives a hypergeometric equation. Using the solution theory for such equations and the asymptotic analysis of Gaussian multiplicative chaos, one obtains a set of functional equations for the probabilistically defined H and R called *shift equations*.

Step 3 Prove that H_{PT} and R_{FZZ} in Theorems 1.1 and 1.2 are the unique solutions to the shift equations under certain conditions that are satisfied by H and R.

Although the global strategy is clear and has been successfully used in several works [Rem20, RZ20b, RZ21] for boundary LCFT, the presence of the bulk potential poses several major difficulties that we overcome in this paper.

(1). In previous works a handy fact is that the Liouville correlation functions can be written as a moment of Gaussian multiplicative chaos times an explicit prefactor. Under this form the probabilistic definition can be extended meromorphically to a range where the shift equations make sense. This is no longer the case in our paper. We have to introduce proper truncations in the spirit of (1.10) to extend the probabilistic definition of H and R, and handle analytic issues coming with this complication.

(2). The most challenging step is the proof of the BPZ equation (1.14) in Theorem 1.3. As far as we know, there is no conceptual calculation at the physics level of rigor which explains the $(\sigma_{j_0} \pm \frac{\chi}{2}, \sigma_{j_0})$ constraint around the degenerate weight. To prove Theorem 1.3, we made a careful choice of regularization of the left hand side of (1.14) and showed that exactly under the $(\sigma_{j_0} \pm \frac{\chi}{2}, \sigma_{j_0})$ constraint, subtle cancellations occur as the regularization parameter vanishes.

(3). In the derivation of shift equations for R, we need the exact value of R for certain special cases as input. In [RZ21] where the bulk potential is absent, a desirable input was supplied by the main result of [RZ20b]. In our setting the counterpart of [RZ20b] seems as difficult as the general case. What we use instead is the value of $R(\gamma, \sigma_1, \sigma_2)$ coming from the mating of trees framework [DMS14, AG21]. Similar to [ARS21], this is an instance where integrable results from mating of tree is used in the derivation of Liouville correlation functions.

Our Sections 2, 3, and 4 are devoted to treating the above three difficulties, respectively.

2. Definitions and meromorphicity of H and R

In this section we give the probabilistic definition of H and R in a range that is large enough for our argument based on shift equations.

2.1. **Definition of H under the Seiberg bound.** Let h be the free boundary Gaussian free field on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ with covariance kernel

$$\mathbb{E}[h(x)h(y)] = G_{\mathbb{H}}(x,y) := \log \frac{1}{|x-y||x-\bar{y}|} + 2\log|x|_+ + 2\log|y|_+, \tag{2.1}$$

where $|x|_+ := \max(|x|, 1)$ and in the sense that $\mathbb{E}[(h,f)(h,g)] = \iint f(x)\mathbb{E}[h(x)h(y)]g(y)dxdy$, for smooth test functions f and g. Let $P_{\mathbb{H}}$ be the law of h, so that $P_{\mathbb{H}}$ is a probability measure on the negatively indexed Sobolev space $H^{-1}(\mathbb{H})$ ([She07, Dub09]). The particular covariance kernel corresponds to requiring the field to have average 0 on the unit circle.

We now introduce the Liouville field on $\mathbb{H}$, possibly with boundary insertions.

Definition 2.1 (Liouville field)**.** *Let* $(h, \mathbf{c})$ *be sampled from* $P_{\mathbb{H}} \times [e^{-Qc}dc]$ *and set* $\phi = h(z) - 2Q\log|z|_+ + \mathbf{c}$*. We write* $\mathrm{LF}_{\mathbb{H}}$ *as the law of* ϕ*, and call a sample from* $\mathrm{LF}_{\mathbb{H}}$ *a* Liouville field on $\mathbb{H}$.

Definition 2.2 (Liouville field with insertions)**.** *Let* $(\beta_j, s_j) \in \mathbb{R} \times \partial\mathbb{H}$ *for* $j = 1, \dots, M$*, where* $M \geq 1$ *and the* s_j *are pairwise distinct. Sample* $(h, \mathbf{c})$ *from* $C_{\mathbb{H}}^{(\beta_j,s_j)_j} P_{\mathbb{H}} \times [e^{(\frac{1}{2}\sum_j \beta_j - Q)c}dc]$ *where*

$$C_{\mathbb{H}}^{(\beta_j,s_j)_j} = \prod_{j=1}^{M} |s_j|_+^{-\beta_j(Q-\frac{\beta_j}{2})} \prod_{1\leq j<k\leq M} e^{\frac{\beta_j\beta_k}{4}G_{\mathbb{H}}(s_j,s_k)}.$$

Let $\phi(z) = h(z) - 2Q\log|z|_+ + \sum_{j=1}^{M} \frac{\beta_j}{2} G_{\mathbb{H}}(z, s_j) + \mathbf{c}$*. We write* $\mathrm{LF}_{\mathbb{H}}^{(\beta_j,s_j)_j}$ *for the law of* ϕ *and call a sample from* $\mathrm{LF}_{\mathbb{H}}^{(\beta_j,s_j)_j}$ *the* Liouville field on $\mathbb{H}$ with insertions $(\beta_j, s_j)_{1\leq j\leq M}$.

We can also define Liouville fields with an insertion at ∞. We will need the case $\mathrm{LF}_{\mathbb{H}}^{(\beta_1,0),(\beta_2,1),(\beta_3,\infty)}$, which can be defined by $\lim_{s\to\infty} |s|^{2\Delta_3}\mathrm{LF}_{\mathbb{H}}^{(\beta_1,0),(\beta_2,1),(\beta_3,s)}$ with $\Delta_3 = \frac{\beta_3}{2}(Q - \frac{\beta_3}{2})$. Here we give a more explicit definition without limiting procedure.

Definition 2.3. *Fix* $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$*. Set* $s_1 = 0, s_2 = 1, s_3 = \infty$ *and* $G_{\mathbb{H}}(z, \infty) = 2\log|z|_+$*. Sample* $(h, \mathbf{c})$ *from* $P_{\mathbb{H}} \times [e^{(\frac{1}{2}\sum_j \beta_j - Q)c}dc]$*. We write* $\mathrm{LF}_{\mathbb{H}}^{(\beta_1,0),(\beta_2,1),(\beta_3,\infty)}$ *for the law of* ϕ *where* $\phi(z) = h(z) - 2Q\log|z|_+ + \sum_{j=1}^{3} \frac{\beta_j}{2} G_{\mathbb{H}}(z, s_j) + \mathbf{c}$.

Given a sample h from $P_{\mathbb{H}}$, the associated quantum area and length measure are defined by

$$\mathcal{A}_h = \lim_{\varepsilon\to 0} \epsilon^{\frac{\gamma^2}{2}} e^{\gamma h_\epsilon(z)} d^2z, \quad \text{and} \quad \mathcal{L}_h = \lim_{\varepsilon\to 0} \epsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2} h_\epsilon(z)} dz. \tag{2.2}$$

where the limits hold in probability in the weak topology of measures. The existence of limits is well-known from Gaussian multiplicative chaos; see e.g. [Ber17, RV14].

Given a sample ϕ from $\mathrm{LF}_{\mathbb{H}}$ or as in $\mathrm{LF}_{\mathbb{H}}^{(\beta_j,s_j)_j}$, we similarly define $\mathcal{A}_\phi$ and $\mathcal{L}_\phi$ as in (2.2) with h replaced by ϕ. This allows us to rigorously define the function H in (1.6) for a certain range of parameters called the Seiberg bound.

Definition 2.4. *Let* $\mu_i \geq 0$ *for* $i = 1, 2, 3$*. Suppose* $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ *satisfy the Seiberg bound*

$$\sum_{i=1}^{3} \beta_i > 2Q \quad \textit{and} \quad \beta_i < Q, \tag{2.3}$$

Then the boundary three-point structure constant *is defined by*

$$H_{(\mu_1,\mu_2,\mu_3)}^{(\beta_1,\beta_2,\beta_3)} := \int e^{-\mathcal{A}_\phi(\mathbb{H}) - \mu_1\mathcal{L}_\phi(-\infty,0) - \mu_2\mathcal{L}_\phi(0,1) - \mu_3\mathcal{L}_\phi(1,+\infty)} \, \mathrm{LF}_{\mathbb{H}}^{(\beta_1,0),(\beta_2,1),(\beta_3,\infty)}(\mathrm{d}\phi). \tag{2.4}$$

In general, correlation functions of LCFT can be defined using Liouville field with insertions similarly as in (2.4). For example, the formal integral in (1.5) defining $\left\langle \prod_{j=1}^{3} B_{\beta_j}^{\sigma_j,\sigma_{j+1}}(s_j) \right\rangle$ should be understood as $\int e^{-\mu\mathcal{A}_\phi(\mathbb{H}) - \sum_{i=1}^{3}\mu_i L_i} \, \mathrm{LF}_{\mathbb{H}}^{(\beta_i,z_i)_3}(\mathrm{d}\phi)$. The inequality (2.3) is called the Seiberg bound, which is the condition for which the integral in (2.4) is finite [HRV18]. Using the coordinate change rule for Liouville fields (see e.g. [AHS21, Section 2.2]), Equation (1.6) holds as follows:

$$\left\langle e^{\frac{\beta_1}{2}\phi(s_1)} e^{\frac{\beta_2}{2}\phi(s_2)} e^{\frac{\beta_3}{2}\phi(s_3)} \right\rangle = \frac{H_{(\frac{\mu_1}{\sqrt{\mu}}, \frac{\mu_2}{\sqrt{\mu}}, \frac{\mu_3}{\sqrt{\mu}})}^{(\beta_1,\ \beta_2,\ \beta_3)} \mu^{\frac{2Q-\beta_1-\beta_2-\beta_3}{2\gamma}}}{|s_1 - s_2|^{\Delta_1+\Delta_2-\Delta_3}|s_1 - s_3|^{\Delta_1+\Delta_3-\Delta_2}|s_2 - s_3|^{\Delta_2+\Delta_3-\Delta_1}}.$$

Therefore throughout the paper we set $\mu = 1$ without loss of generality.

2.2. **Meromorphic extension of H.** To make sense of shift equations for H we need to meromorphically extend the range of its definition. To achieve this, we first give an alternative definition of H from Definition 2.4.

Lemma 2.5. *In the setting of Defintion 2.4, writing $s = \frac{1}{2}\sum \beta_i - Q$, we have*

$$s(s+\frac{\gamma}{2})H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} = \int (\gamma(s+\frac{\gamma}{2})A+\frac{\gamma^2}{2}A(\sum_i \mu_i L_i)+\frac{\gamma^2}{4}(\sum_i \mu_i L_i)^2)e^{-A-\sum_i \mu_i L_i}\,\mathrm{LF}_{\mathbb{H}}^{(\beta_1,0),(\beta_2,1),(\beta_3,\infty)}(\mathrm{d}\phi)$$

where $A = \mathcal{A}_\phi(\mathbb{H}), L_1 = \mathcal{L}_\phi(-\infty,0), L_2 = \mathcal{L}_\phi(0,1)$ and $L_3 = \mathcal{L}_\phi(1,\infty)$.

Proof. Let h be a Gaussian free field and $\widetilde{h} := h - 2Q\log|\cdot|_+ + \sum \frac{\beta_i}{2}G_{\mathbb{H}}(\cdot,s_i)$, where $(s_1,s_2,s_3) = (0,1,\infty)$. Write $\widetilde{A} = \mathcal{A}_{\widetilde{h}}(\mathbb{H})$ and $\widetilde{L}_1 = \mathcal{L}_{\widetilde{h}}(-\infty,0), \widetilde{L}_2 = \mathcal{L}_{\widetilde{h}}(0,1), \widetilde{L}_3 = \mathcal{L}_{\widetilde{h}}(1,+\infty)$. Since $A = e^{\gamma c}\widetilde{A}$ and $L_i = e^{\gamma c/2}\widetilde{L}_i$, by Definition 2.4, we have

$$H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2\mu_3)} = \mathbb{E}\left[\int e^{sc}\cdot e^{-e^{\gamma c}\widetilde{A}-e^{\gamma c/2}\sum_{i=1}^3 \mu_i \widetilde{L}_i}\,dc\right].$$

For $s>0$, $a>0$ and $\ell \in \mathbb{C}$ with $\Re \ell > 0$, using integration by parts twice we have

$$\begin{aligned}
&\int e^{sc}\cdot e^{-e^{\gamma c}a - e^{\gamma c/2}\ell}\,dc = -\int (\frac{1}{s}e^{sc})((-\gamma e^{\gamma c}a - \frac{\gamma}{2}e^{\gamma c/2}\ell)e^{-e^{\gamma c}a-e^{\gamma c/2}\ell})\,dc\\
=&\frac{\gamma}{s}\int e^{sc}(e^{\gamma c}a)e^{-e^{\gamma c}a-e^{\gamma c/2}\ell}\,dc + \frac{\gamma}{2s}\int e^{sc}(e^{\gamma c/2}\ell)(e^{-e^{\gamma c}a-e^{\gamma c/2}\ell}-1)\,dc\\
=&\frac{\gamma}{s}\int e^{sc}(e^{\gamma c}a)e^{-e^{\gamma c}a-e^{\gamma c/2}\ell}\,dc + \frac{\gamma}{2s}\frac{\gamma}{s+\frac{\gamma}{2}}\int e^{sc}(e^{\gamma c}a)(e^{\gamma c/2}\ell)e^{-e^{\gamma c}a-e^{\gamma c/2}\ell}\,dc \qquad (\star)\\
&+\frac{\gamma}{2s}\frac{\frac{\gamma}{2}}{s+\frac{\gamma}{2}}\int e^{sc}(e^{\gamma c/2}\ell)^2 e^{-e^{\gamma c}a-e^{\gamma c/2}\ell}\,dc.
\end{aligned}$$

Now setting $s = \frac{1}{2}\sum \beta_i - Q$, $a = \widetilde{A}$ and $\ell = \sum_i \mu_i \widetilde{L}_i$ we get

(2.5)
$$H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2\mu_3)} = \int (\frac{\gamma}{s}A+\frac{\gamma}{2s}\frac{\gamma}{s+\frac{\gamma}{2}}A(\sum_i \mu_i L_i)+\frac{\gamma}{2s}\frac{\frac{\gamma}{2}}{s+\frac{\gamma}{2}}(\sum_i \mu_i L_i)^2)e^{-A-\sum_i \mu_i L_i}\,\mathrm{LF}_{\mathbb{H}}^{(\beta_1,0),(\beta_2,1),(\beta_3,\infty)}(\mathrm{d}\phi).$$

□

This alternative description of H allows us to extend the Definition 2.4 of H as follows.

Definition 2.6. *Set $V = \{(\beta_1,\beta_2,\beta_3) : Q - \frac{1}{2}\sum \beta_i < \gamma \wedge \frac{2}{\gamma} \wedge \min_i(Q-\beta_i)$ and $\beta_i < Q$ for each $i\}$. Given $(\beta_1,\beta_2,\beta_3) \in V$ and $\Re \mu_i \geq 0$ for $i=1,2,3$, define*

(2.6)
$$\hat{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} := \int (\gamma(s+\frac{\gamma}{2})A + \frac{\gamma^2}{2}A(\sum_i \mu_i L_i) + \frac{\gamma^2}{4}(\sum_i \mu_i L_i)^2)e^{-A-\sum_i \mu_i L_i}\,\mathrm{LF}_{\mathbb{H}}^{(\beta_1,0),(\beta_2,1),(\beta_3,\infty)}(\mathrm{d}\phi),$$

$$H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} := \frac{1}{s(s+\frac{\gamma}{2})}\hat{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}, \tag{2.7}$$

where $A = \mathcal{A}_\phi(\mathbb{H}), L_1 = \mathcal{L}_\phi(-\infty,0), L_2 = \mathcal{L}_\phi(0,1)$ and $L_3 = \mathcal{L}_\phi(1,\infty)$.

Proposition 2.7. *In Definition 2.6, the integral in (2.6) defining $\hat{H}$ is absolutely convergent in the given range of parameters. For each $(\beta_1,\beta_2,\beta_3) \in V$, the function H is holomorphic on $\{(\mu_1,\mu_2,\mu_3) : \Re \mu_i > 0\}$ and continuous on $\{(\mu_1,\mu_2,\mu_3) : \Re \mu_i \geq 0\}$. For each (μ_1,μ_2,μ_3) satisfying $\Re \mu_i > 0$, the function $\hat{H}$ can be analytically extended on a complex neighborhood in $\mathbb{C}^3$ of V.*

We provide the proof Proposition 2.7 in Appendix B since our proof follows the same strategy in [KRV20] for similar questions. Under the analytic extension of $\hat{H}$ given in Proposition 2.7, we define

$$H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} = s^{-1}(s+\frac{\gamma}{2})^{-1}\hat{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} \quad \text{and} \quad H\begin{pmatrix}\beta_1,\beta_2,\beta_3\\ \sigma_1,\sigma_2,\sigma_3\end{pmatrix} := H^{(\beta_1,\beta_2,\beta_3)}_{(g(\sigma_1),g(\sigma_2),g(\sigma_3))} \tag{2.8}$$

where

$$g(\sigma) = \sqrt{\frac{1}{\sin(\pi\frac{\gamma^2}{4})}}\cos\left(\pi\gamma(\sigma-\frac{Q}{2})\right). \tag{2.9}$$

Let $\mathcal{B} = (-\frac{1}{2\gamma}+\frac{Q}{2}, \frac{1}{2\gamma}+\frac{Q}{2})\times\mathbb{R}$ and $\overline{\mathcal{B}} = [-\frac{1}{2\gamma}+\frac{Q}{2}, \frac{1}{2\gamma}+\frac{Q}{2}]\times\mathbb{R}$. Then the function $\mu = g(\sigma)$ is a holomorhpic bijection between $\{\mu\in\mathbb{C} : \Re u > 0\}$ and $\mathcal{B}^3$. Moreover, g is continuous on $\overline{\mathcal{B}}^3$. Under this change of variable Proposition 2.7 yields the following.

Proposition 2.8. *Suppose H is as in (2.8), and V and s are as in Definition 2.6. For each $(\beta_1,\beta_2,\beta_3)\in V$, the function $(\sigma_1,\sigma_2,\sigma_3)\mapsto H$ is holomorphic on $\mathcal{B}^3$ and continuous on $\overline{\mathcal{B}}^3$. For each $(\sigma_1,\sigma_2,\sigma_3)\in\mathcal{B}^3$, the function $(\beta_1,\beta_2,\beta_3)\mapsto s(s+\frac{\gamma}{2})H$ is analytic on a complex neighborhood in $\mathbb{C}^3$of V.*

Proposition 2.8 is an intermediate step to prove Theorem 1.1, namely $2\pi H\begin{pmatrix}\beta_1,\beta_2,\beta_3\\ \sigma_1,\sigma_2,\sigma_3\end{pmatrix} = H_{\mathrm{PT}}\begin{pmatrix}\beta_1,\beta_2,\beta_3\\ \sigma_1,\sigma_2,\sigma_3\end{pmatrix}$. Once establishes, Theorem 1.1 implies that H is in fact meromorphic on $\mathbb{C}^6$ in the variables $\beta_1,\beta_2,\beta_3,\sigma_1,\sigma_2,\sigma_3$.

Remark 2.9. *The meromorphic extension of H can also be done by truncations in the same spirit of (1.10). Indeed, suppose $\mu_1,\mu_2,\mu_3\in\mathbb{C}$ satisfy $\Re\mu_i\geq 0$ for $i=1,2,3$, and $(\beta_1,\beta_2,\beta_3)$ satisfy $0 < Q-\frac{1}{2}\sum\beta_i < \frac{\gamma}{2}\wedge\min_i(Q-\beta_i)$, then*

$$H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} = \int\left(e^{-\mathcal{A}_\phi(\mathbb{H})-\sum_{i=1}^3\mu_iL_i}-1\right)\,\mathrm{LF}_{\mathbb{H}}^{(\beta_1,0),(\beta_2,1),(\beta_3,\infty)}(\mathrm{d}\phi) \tag{2.10}$$

If instead $(\beta_1,\beta_2,\beta_3)$ satisfy $\frac{\gamma}{2} < Q-\frac{1}{2}\sum\beta_i < \gamma\wedge\frac{2}{\gamma}\wedge\min_i(Q-\beta_i)$, then

$$H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} = \int\left(e^{-\mathcal{A}_\phi(\mathbb{H})-\sum_{i=1}^3\mu_iL_i}-1+\sum_{i=1}^3\mu_iL_i\right)\,\mathrm{LF}_{\mathbb{H}}^{(\beta_1,0),(\beta_2,1),(\beta_3,\infty)}(\mathrm{d}\phi). \tag{2.11}$$

Both (2.10) and (2.11) follow from integration by parts. One can further extend the range of H via further truncations of e^{-x}. We left these details to interested readers.

2.3. **Probabilitic definition of R via truncations.** We first recall the definiton of two pointed quantum disk used in the definition of R. It is most convenient to describe it using the horizontal strip $\mathcal{S}=\mathbb{R}\times(0,\pi)$. Let $h_{\mathcal{S}}(z) = h_{\mathbb{H}}(e^z)\circ\exp$ where $h_{\mathbb{H}}$ is sampled from $P_{\mathbb{H}}$. We call $h_{\mathcal{S}}$ a free-boundary GFF on $\mathcal{S}$. The field $h_{\mathcal{S}}$ can be written as $h_{\mathcal{S}} = h^{\mathrm{c}}+h^{\ell}$, where h^{c} is constant on each vertical line, and h^{ℓ} has mean zero on all such lines [DMS14, Section 4.1.6]. We call h^{ℓ} the *lateral component* of the free-boundary GFF on $\mathcal{S}$.

Definition 2.10. *Fix $\beta < Q$. Let*

$$Y_t = \begin{cases} B_{2t}-(Q-\beta)t & \text{if } t\geq 0\\ \widetilde{B}_{-2t}+(Q-\beta)t & \text{if } t<0\end{cases},$$

where $(B_s)_{s\geq 0}$ is a standard Brownian motion conditioned on $B_{2s}-(Q-\beta)s<0$ for all $s>0$,[1] *and $(\widetilde{B}_s)_{s\geq 0}$ is an independent copy of $(B_s)_{s\geq 0}$. Let $h^1(z)=Y_{\Re z}$ for each $z\in\mathcal{S}$. Let $h^2_{\mathcal{S}}$ be independent of h^1 and have the law the lateral component of the free-boundary GFF on $\mathcal{S}$. Let $\psi=h^1+h^2_{\mathcal{S}}$. Let $\mathbf{c}$ be a real number sampled from $\frac{\gamma}{2}e^{(\beta-Q)c}dc$ independent of ψ and $\phi=\psi+\mathbf{c}$. Let $\mathcal{M}_2^{disk}(\beta)$ be the infinite measure describing the law of ϕ.*

For ϕ in Definition 2.10, define $\mathcal{A}_\phi=\lim_{\varepsilon\to 0}\epsilon^{\frac{\gamma^2}{2}}e^{\gamma\phi_\epsilon(z)}d^2z$ on $\mathcal{S}$ and $\mathcal{L}_\phi=\lim_{\varepsilon\to 0}\epsilon^{\frac{\gamma^2}{4}}e^{\frac{\gamma}{2}\phi_\epsilon(z)}dz$ on $\partial\mathcal{S}$. Write $A=\mathcal{A}_\phi(\mathcal{S})$ as the total $\mathcal{A}_\phi$-area, and write L_1, L_2 as the $\mathcal{L}_\phi$-length of top and bottom boundary arc of $\mathcal{S}$, respectively. It appears that $\int e^{-A-\mu_1L_1-\mu L_2}\,\mathrm{d}\mathcal{M}_2^{\mathrm{disk}}(\beta)=\infty$. For $\beta\in(\frac{2}{\gamma},Q)$, the correct definition is via truncation as in (1.10). We now extend the definition of R in the same way as for H in Definition 2.6 and Proposition 2.7.

Definition 2.11. *For $\beta\in((\frac{2}{\gamma}-\frac{\gamma}{2})\vee\frac{\gamma}{2},Q)$ and $\Re\mu_1,\Re\mu_2\geq 0$, writing $s=\beta-Q$, we define*

$$\hat{R}_{\mu_1,\mu_2}(\beta):=\frac{2(Q-\beta)}{\gamma}\int(\gamma(s+\frac{\gamma}{2})A+\frac{\gamma^2}{2}A(\sum_i\mu_iL_i)+\frac{\gamma^2}{4}(\sum_i\mu_iL_i)^2)e^{-A-\sum_i\mu_iL_i}\,d\mathcal{M}_2^{disk}(\beta),\tag{2.12}$$

$$R_{\mu_1,\mu_2}(\beta):=\frac{1}{s(s+\frac{\gamma}{2})}\hat{R}_{\mu_1,\mu_2}(\beta).\tag{2.13}$$

Here we write A,L_1,L_2 for the quantum area and boundary arc lengths of the quantum disk.

The integral in Definition 2.11 is absolutely convergent by Proposition B.4.

Proposition 2.12. *In Definition 2.11, the integral in (2.12) defining $\hat{R}$ is absolutely convergent in the given range of parameters. For $\beta\in(\frac{2}{\gamma},Q)$, it agrees with the definition of R in (1.10). For each $\beta\in((\frac{2}{\gamma}-\frac{\gamma}{2})\vee\frac{\gamma}{2},Q)$, the function $(\mu_1,\mu_2)\mapsto R$ is holomorphic on $\{\Re\mu_1>0,\Re\mu_2>0\}$ and continuous on $\{\Re\mu_1\geq 0,\Re\mu_2\geq 0\}$. For $\Re\mu_1>0,\Re\mu_2>0$, the function $\beta\mapsto R$ can be analytically extended on a complex neighborhood of $((\frac{2}{\gamma}-\frac{\gamma}{2})\vee\frac{\gamma}{2},Q)$.*

We include the proof of Proposition 2.12 in Appendix B. Proposition 2.12 yields the following counterpart of Proposition 2.8 on the change of variable.

Proposition 2.13. *Recall g in (2.9) and $\mathcal{B}=(-\frac{1}{2\gamma}+\frac{Q}{2},\frac{1}{2\gamma}+\frac{Q}{2})\times\mathbb{R}$. Let $R(\beta,\sigma_1,\sigma_2):=R_{g(\sigma_1),g(\sigma_2)}(\beta)$. For each $\beta\in((\frac{2}{\gamma}-\frac{\gamma}{2})\vee\frac{\gamma}{2},Q)$, the function $(\sigma_1,\sigma_2)\mapsto R$ is holomorphic on $\mathcal{B}^2$ and continuous on $\overline{\mathcal{B}}^2$. For each $(\sigma_1,\sigma_2)\in\mathcal{B}^2$, the function $\beta\mapsto s(s+\frac{\gamma}{2})R$ is analytic on a complex neighborhood of $((\frac{2}{\gamma}-\frac{\gamma}{2})\vee\frac{\gamma}{2},Q)$.*

Remark 2.14. *Similarly as in Remark 2.9, when $\beta\in((Q-\gamma)\vee\frac{\gamma}{2},\frac{2}{\gamma})$, we have*

$$R_{\mu_1,\mu_2}(\beta)=\frac{2(Q-\beta)}{\gamma}\int\left(e^{-A-\sum_{i=1}^2\mu_iL_i}-1+\sum_{i=1}^2\mu_iL_i\right)\,d\mathcal{M}_2^{\mathrm{disk}}(\beta).\tag{2.14}$$

3. The Belavin-Polyakov-Zamolodchikov equations

We will prove the BPZ differential equations hold in a more general setting than the one required for our purposes of deriving the formulas for H and R. Namely we will allow an arbitrary number of spectator points along with the degenerate insertion on the boundary. Therefore we will start by introducing the definition of the correlation function with arbitrary insertions, presented after

[1] Here we condition on a zero probability event. This can be made sense of via a limiting procedure.

having applied the Girsanov theorem to all the insertions. This definition was first proposed in [HRV18], modulo the fact that [HRV18] does not consider the varying boundary cosmological constant.

Definition 3.1. *(Correlation functions) Let $z_1, \dots, z_N \in \mathbb{H}$ and $s_1 < \cdots < s_M$ be the marked points on the boundary $\partial\mathbb{H}$. For $1 \le j \le M$, $\mu_j = \sqrt{\frac{1}{\sin(\pi\frac{\gamma^2}{4})}}\cos(\pi\gamma(\sigma_j - \frac{Q}{2}))$ is the boundary cosmological constant between s_j and s_{j+1} (with $s_0 = -\infty$ and $s_{M+1} = \infty$ by convention). Assume these parameters obey the constraint of the previous definition. Assume also the Seiberg bounds (3.3) hold. Define*

$$\langle \prod_{i=1}^N V_{\alpha_i}(z_i) \prod_{j=1}^M B_{\beta_j}^{\sigma_j,\sigma_{j+1}}(s_j) \rangle$$

$$= \prod_{i=1}^N |z_i - \bar{z}_i|^{-\frac{\alpha_i^2}{2}} \int_{\mathbb{R}} dc\, e^{(\sum_{i=1}^N \alpha_i + \sum_{j=1}^M \frac{\beta_j}{2} - Q)c}\, \mathbb{E}\Bigg[\prod_{i=1}^N g(z_i)^{\Delta_{\alpha_i}} e^{\alpha_i X(z_i) - \frac{\alpha_i^2}{2}\mathbb{E}[X(z_i)^2]} \prod_{j=1}^M g(s_j)^{\Delta_{\beta_j/2}} e^{\frac{\beta_j}{2} X(s_j) - \frac{\beta_j^2}{8}\mathbb{E}[X(s_j)^2]}$$

$$\exp\left(-e^{\gamma c} \int_{\mathbb{H}} e^{\gamma X(x)} \frac{g(x)}{|x-\bar{x}|^{\frac{\gamma^2}{2}}} d^2x - e^{\frac{\gamma}{2}c} \int_{\mathbb{R}} e^{\frac{\gamma}{2}X(r)} g(r)^{1/2} d\mu_B(r) \right) \Bigg],$$

where the boundary measure is defined by:

$$d\mu_B(r)/dr = \sum_{j=1}^{M-1} \mu_{j+1} \mathbf{1}_{s_j < r < s_{j+1}} + \mu_1 \mathbf{1}_{r \notin (s_1, s_M)}. \tag{3.1}$$

In the following, we will denote $e^{\gamma X(x)} d^2x$ for the GMC measure on $\mathbb{H}$ and similarly $e^{\frac{\gamma}{2}X(r)}dr$ for the GMC measure on $\mathbb{R}$. By applying the Girsanov's theorem, we obtain an equivalent expression:

$$\langle \prod_{i=1}^N V_{\alpha_i}(z_i) \prod_{j=1}^M B_{\beta_j}^{\sigma_j,\sigma_{j+1}}(s_j) \rangle$$

$$= Z_{\boldsymbol{\alpha};\boldsymbol{\beta}}(\mathbf{z};\mathbf{s}) \int_{\mathbb{R}} dc\, e^{\frac{\gamma p}{2}c} \mathbb{E}\Bigg[\exp\Bigg(- e^{\gamma c} \int_{\mathbb{H}} \frac{g(x)^{\frac{\gamma^2}{4}(-p-1)}/|x-\bar{x}|^{\frac{\gamma^2}{2}}}{\prod_{i=1}^N |x-z_i|^{\gamma\alpha_i}|x-\bar{z}_i|^{\gamma\alpha_i} \prod_{j=1}^M |x-s_j|^{\gamma\beta_j}} e^{\gamma X(x)} d^2x$$

$$- e^{\frac{\gamma c}{2}} \int_{\mathbb{R}} \frac{g(r)^{\frac{\gamma^2}{8}(-p-1)}}{\prod_{i=1}^N |r-z_i|^{\gamma\alpha_i} \prod_{j=1}^M |r-s_j|^{\frac{\gamma\beta_j}{2}}} e^{\frac{\gamma}{2}X(r)} d\mu_B(r) \Bigg) \Bigg],$$

where

$$Z_{\boldsymbol{\alpha};\boldsymbol{\beta}}(\mathbf{z};\mathbf{s}) = \prod_i |z_i - \bar{z}_i|^{-\frac{\alpha_i^2}{2}} \prod_{i<i'} |z_i - z_{i'}|^{-\alpha_i\alpha_{i'}} |z_i - \bar{z}_{i'}|^{-\alpha_i\alpha_{i'}} \prod_{i,j} |z_i - s_j|^{-\alpha_i\beta_j} \prod_{j<j'} |s_j - s_{j'}|^{-\frac{\beta_j\beta_{j'}}{2}},$$

and with:

$$p = \frac{2\sum_{i=1}^N \alpha_i + \sum_{j=1}^M \beta_j - 2Q}{\gamma}. \tag{3.2}$$

Definition 3.2. *(Probabilistic range of μ_i and α_i, β_j) We assume $\mathrm{Re}(\mu_i) \ge 0$. Given parameters $\alpha_i \in \mathbb{R}$, $\beta_j \in \mathbb{R}$, we assume the following Seiberg bounds from [HRV18]:*

$$\sum_{i=1}^N \alpha_i + \sum_{j=1}^M \frac{\beta_j}{2} > Q, \quad \alpha_i < Q, \quad \beta_j < Q. \tag{3.3}$$

We will now prove the BPZ equations of Theorem 1.3 with boundary degenerate insertion that will be used to prove our main results.

Remark 3.3. *Although we don't need it, the BPZ equation with a bulk degenerate insertion is the following. Define* $\langle z, z_i, s_j\rangle = \langle e^{\chi\phi(z)}\prod_{i=1}^N e^{\alpha_i\phi(z_i)}\prod_{j=1}^M e^{\beta_j\phi(s_j)}\rangle_{\mathbb{H}}$ *with* $z\in\mathbb{H}$, $z_i\in\mathbb{H}$, *and* $s_j\in\mathbb{R}$. *Then this quantity obeys the following PDE*

$$\Big(\frac{1}{\chi^2}\partial_{zz} + \frac{\Delta_\chi}{(z-\overline{z})^2} + \sum_i \frac{\Delta_{\alpha_i}}{(z-z_i)^2} + \sum_i \frac{\Delta_{\alpha_i}}{(z-\overline{z_i})^2} + \sum_j \frac{\Delta_{\beta_j}}{(z-s_j)^2} + \frac{1}{z-\overline{z}}\partial_{\overline{z}} + \sum_i \frac{1}{z-z_i}\partial_{z_i}$$

$$+ \sum_i \frac{1}{z-\overline{z_i}}\partial_{\overline{z_i}} + \sum_j \frac{1}{z-s_j}\partial_{s_j}\Big)\left\langle V_{-\chi}(z)\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M B_{\beta_j}^{\sigma_j,\sigma_{j+1}}(s_j)\right\rangle = 0,$$

where χ *equals* $\frac{\gamma}{2}$ *or* $\frac{2}{\gamma}$.

3.1. **Regularization.** Before applying the differential operators to the correlation function we will introduce a regularization procedure that will depend on two small parameters $\epsilon,\delta>0$. Let $\delta>0$ and consider the function $\eta_\delta = \frac{1}{\delta^2}\eta(\frac{|x|^2}{\delta^2})$ where η is a non-negative smooth function with support in $[\frac12,1]$ and satisfies $\pi\int_0^\infty \eta(r)dr = 1$. For $x\in\mathbb{C}$, we introduce the notations

$$\frac{1}{(x)_\delta} := \int_{\mathbb{C}^2} \frac{1}{x-y_1+y_2}\eta_\delta(y_1)\eta_\delta(y_2)d^2y_1d^2y_2,$$
$$\frac{1}{|x|_\delta} := \exp\left(\int_{\mathbb{C}^2}\log\frac{1}{|x-y_1+y_2|}\eta_\delta(y_1)\eta_\delta(y_2)d^2y_1d^2y_2\right).$$

Without loss of generality we can prove the BPZ equation holds in the case where $s<s_1$. Let $\mathbb{H}_\delta := \{z\in\mathbb{C}|\mathrm{Im}(z)>\delta\}$ and $-\mathbb{H}_\delta := \{z\in\mathbb{C}|\mathrm{Im}(z)<-\delta\}$. For the purposes of regularization, we will actually assume that s is not on the real line but belongs to $-\mathbb{H}_\delta$. In the case a real derivative ∂_s should now be viewed as the sum of the holomorphic and antiholomorphic derivatives $\partial_s+\partial_{\overline{s}}$. As a further smoothing we will make some assumptions about the GMC integrals. We will restrict the GMC integration over $\mathbb{H}$ to $\mathbb{H}_\delta$.

$$\begin{aligned}
&\langle B^f_{-\chi}(s)\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M B_{\beta_j}^{\sigma_{j-1},\sigma_j}(s_j)\rangle_{\epsilon,\delta}\\
=&Z_{-\chi|\boldsymbol{\alpha};\boldsymbol{\beta}}(s|\mathbf{z};\mathbf{s})\int_{\mathbb{R}} dc\, e^{(\frac{\gamma p}{2}-\frac{\chi}{2})c}\mathbb{E}\Bigg[\exp\Bigg(-e^{\gamma c}\int_{\mathbb{H}_\delta}\frac{|x-s|^{\gamma\chi}g(x)^{\frac{\gamma^2}{4}(-p+\frac{\chi}{\gamma}-1)}/|x-\bar{x}|^{\frac{\gamma^2}{2}}}{\prod_{i=1}^N|x-z_i|_\delta^{\gamma\alpha_i}|x-\bar{z}_i|_\delta^{\gamma\alpha_i}\prod_{j=1}^M|x-s_j|_\delta^{\gamma\beta_j}}e^{\gamma X(x)}d^2x\\
&-e^{\frac{\gamma c}{2}}\int_{\mathbb{R}}\frac{|r-s|^{\frac{\gamma\chi}{2}}g(r)^{\frac{\gamma^2}{8}(-p+\frac{\chi}{\gamma}-1)}}{\prod_{i=1}^N|r-z_i|_\delta^{\gamma\alpha_i}\prod_{j=1}^M|r-s_j|_\delta^{\frac{\gamma\beta_j}{2}}}e^{\frac{\gamma}{2}X(r)}d\mu^s_{B,\epsilon}(r)\Bigg)\Bigg],
\end{aligned}$$

where

$$\begin{aligned}
Z_{-\chi|\boldsymbol{\alpha};\boldsymbol{\beta}}(s|\mathbf{z};\mathbf{s}) = &\prod_i |z_i-s|^{\chi\alpha_i}\prod_j|s_j-s|^{\frac{\chi\beta_j}{2}}\prod_i|z_i-\bar{z}_i|^{-\frac{\alpha_i^2}{2}}\prod_{i<i'}|z_i-z_{i'}|^{-\alpha_i\alpha_{i'}}|z_i-\bar{z}_{i'}|^{-\alpha_i\alpha_{i'}}\\
&\times\prod_{i,j}|z_i-s_j|^{-\alpha_i\beta_j}\prod_{j<j'}|s_j-s_{j'}|^{-\frac{\beta_j\beta_{j'}}{2}},
\end{aligned}$$

and the boundary measure $d\mu^s_{B,\epsilon}(r)$ is defined in terms of f by:

$$d\mu^s_{B,\epsilon}(r)/dr = f(\arg(r-s))\mathbf{1}_{r<s_1-\epsilon} + \sum_{j=1}^M \mu_j \mathbf{1}_{s_j+\epsilon<r<s_{j+1}-\epsilon}.$$

Let us explain the reasons behind this choice. The ϵ regularization is used to smooth the boundary in the neighborhood of the spectator insertions. This is standard from the earlier works [RZ20b]. On the piece $r \in (-\infty, s_1-\epsilon)$, we originally want the σ_i parameter to go from $\sigma_0 \pm \frac{\chi}{2}$ to σ_0 when r crosses s. But in the regularization procedure we actually choose $s \in -\mathbb{H}_\delta$.

We use a differentiable function $f(\arg(r-s))$ for the cosmological constant on $(-\infty, s_1-\epsilon)$ with $f(0)=\mu_0$. Set f to be:

$$f(\theta) = \sqrt{\frac{1}{\sin(\pi\frac{\gamma^2}{4})}}\cos(\pi\gamma(\sigma_0 - \frac{Q}{2}) \pm \frac{\gamma\chi}{2}\theta). \tag{3.4}$$

The point of using this f is the following. In the case where $s \in \mathbb{R}$, when r is in between s and s_1, one has $\arg(r-s)=0$ and therefore $f(\arg(r-s)) = \mu_0$. When $r<s$, then $\arg(r-s) = \pm\pi$ and therefore $f(\arg(r-s))$ gives a value corresponding to shifting σ_0 by $\pm\frac{\chi}{2}$. Notice the two possible choices of phase correspond to the two possible BPZ equations. One also has to choose the sigma parameter to the right of the right most insertion to match with this value. The advantage of using this f is when we choose s to be complex in the lower half plane, it provides a smoothing of the shift on σ_0. Therefore when we let δ tend to 0 and have s approach the real line, we will have:

$$\langle B^f_{-\chi}(s)\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M B^{\sigma_{j-1},\sigma_j}_{\beta_j}(s_j)\rangle = \langle B^{\sigma_0\pm\frac{\chi}{2},\sigma_0}_{-\chi}(s)\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M B^{\sigma_{j-1},\sigma_j}_{\beta_j}(s_j)\rangle.$$

In the following it will be convenient to have a notation for the subset of $\mathbb{R}$ where we are using this function f:

$$I_\epsilon := (-\infty, s_1-\epsilon). \tag{3.5}$$

3.2. **Algebraic cancellation.** In this subsection we will perform what we refer to as the algebraic cancellation, which is in a sense the cancellation of the terms if one ignores the singular terms coming from the regularization procedure. The singular terms coming from the regularization will be cancelled in the next step.

When applying the differential operators of the BPZ equation to the correlation function we will encounter terms like:

$$\langle B^f_{-\chi}(s)B_\gamma(v)\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M B^{\sigma_{j-1},\sigma_j}_{\beta_j}(s_j)\rangle_{\epsilon,\delta}.$$

The notation $B_\gamma(v)$ is a boundary insertion with parameter γ, and inserting v between any s_j and s_{j+1} will keep the same boundary constant μ_j on both sides of y, therefore we take out the superscript in the notation. For simplicity, the above term will be noted as $\langle s, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$. We can extend this notation to $\langle s, y_1,\dots,y_m, v_1,\dots v_n|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$, where $y_i \in \mathbb{H}$ represents an insertion $V_\gamma(y_i)$ and $v_j\in\mathbb{R}$ stands for an insertion $B_\gamma(v_j)$.

In the following, we will work with $s \in -\mathbb{H}_\delta$. In Theorem 1.3, the ∂_{ss} is a second order real derivate with respect to a real s on $\mathbb{R}$. But since for the regularization purpose we are choosing $s\in-\mathbb{H}_\delta$, we will actually represent below the real derivative with respect to s by $\partial_s + \partial_{\bar{s}}$ where these two derivatives are respectively holomorphic and anti holomorphic with respect to s.

We now apply BPZ operator to the regularized term $\langle s, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$. We break the calculation into two lemmas.

Lemma 3.4. *The following holds*

$$\frac{1}{\chi^2}(\partial_s + \partial_{\bar{s}})^2 \langle s|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} = -C_{\epsilon,\delta} + S^{(1)}_{I,\epsilon,\delta} + o_{\delta\to 0}(1), \tag{3.6}$$

where $C_{\epsilon,\delta}$ is given by:

$$C_{\epsilon,\delta} = \left(\sum_i \frac{\alpha_i}{2\chi}(\frac{1}{(z_i-s)^2} + \frac{1}{(\bar{z}_i-\bar{s})^2}) + \sum_j \frac{\beta_j}{4\chi}(\frac{1}{(s_j-s)^2} + \frac{1}{(s_j-\bar{s})^2})\right) \langle s|\mathbf{z};\mathbf{s}\rangle_\epsilon \tag{3.7}$$

$$- \left(\sum_i \frac{\alpha_i}{2}(\frac{1}{z_i-s} + \frac{1}{\bar{z}_i-\bar{s}}) + \sum_j \frac{\beta_j}{4}(\frac{1}{s_j-s} + \frac{1}{s_j-\bar{s}})\right)^2 \langle s|\mathbf{z};\mathbf{s}\rangle_\epsilon \tag{3.8}$$

$$+ 2\left(\sum_i \frac{\alpha_i}{2}(\frac{1}{z_i-s} + \frac{1}{\bar{z}_i-\bar{s}}) + \sum_j \frac{\beta_j}{4}(\frac{1}{s_j-s} + \frac{1}{s_j-\bar{s}})\right) \tag{3.9}$$

$$\times \left(\mu\frac{\gamma}{2}\int_{\mathbb{H}_\delta}(\frac{1}{y-s} + \frac{1}{\bar{y}-\bar{s}})\langle s,y|\mathbf{z};\mathbf{s}\rangle_\epsilon d^2y + \frac{\gamma}{4}\int_{\mathbb{R}}(\frac{1}{v-s} + \frac{1}{v-\bar{s}})\langle s,v|\mathbf{z};\mathbf{s}\rangle_\epsilon d\mu^s_{B,\epsilon}(v)\right)$$

$$- \mu\frac{\gamma}{2\chi}\int_{\mathbb{H}_\delta}(\frac{1}{(y-s)^2} + \frac{1}{(\bar{y}-\bar{s})^2})\langle s,y|\mathbf{z};\mathbf{s}\rangle_\epsilon d^2y \tag{3.10}$$

$$+ \mu\frac{\gamma^2}{4}\int_{\mathbb{H}_\delta}(\frac{1}{y-s} + \frac{1}{\bar{y}-\bar{s}})^2\langle s,y|\mathbf{z};\mathbf{s}\rangle_\epsilon \tag{3.11}$$

$$- \mu^2\frac{\gamma^2}{4}\int_{\mathbb{H}_\delta^2}(\frac{1}{x-s} + \frac{1}{\bar{x}-\bar{s}})(\frac{1}{y-s} + \frac{1}{\bar{y}-\bar{s}})\langle s,x,y|\mathbf{z};\mathbf{s}\rangle_\epsilon d^2x d^2y \tag{3.12}$$

$$+ \mu\frac{\gamma^2}{4}\int_{\mathbb{H}_\delta}\int_{\mathbb{R}}(\frac{1}{y-s} + \frac{1}{\bar{y}-\bar{s}})(\frac{1}{v-s} + \frac{1}{v-\bar{s}})\langle s,y,v|\mathbf{z};\mathbf{s}\rangle_\epsilon d\mu^s_{B,\epsilon}(v)d^2y \tag{3.13}$$

$$- \frac{\gamma}{4\chi}\int_{\mathbb{R}}(\frac{1}{(v-s)^2} + \frac{1}{(v-\bar{s})^2})\langle s,v|\mathbf{z};\mathbf{s}\rangle_\epsilon d\mu^s_{B,\epsilon}(v) \tag{3.14}$$

$$+ \frac{\gamma^2}{16}\int_{\mathbb{R}}(\frac{1}{v-s} + \frac{1}{v-\bar{s}})^2\langle s,v|\mathbf{z};\mathbf{s}\rangle_\epsilon d\mu^s_{B,\epsilon}(v) \tag{3.15}$$

$$- \frac{\gamma^2}{16}\int_{\mathbb{R}^2}(\frac{1}{u-s} + \frac{1}{u-\bar{s}})(\frac{1}{v-s} + \frac{1}{v-\bar{s}})\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon d\mu^s_{B,\epsilon}(u)d\mu^s_{B,\epsilon}(v), \tag{3.16}$$

and

$$S^{(1)}_{I,\epsilon,\delta} = -\frac{\gamma}{8i\chi}\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{v-s}+\frac{1}{v-\bar{s}})(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\langle s,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dv \tag{3.17}$$

$$+\frac{\gamma}{4i\chi}\int_{I_\epsilon^2} f'(\arg(u-s))f(\arg(v-s))(\frac{1}{u-s}-\frac{1}{u-\bar{s}})(\frac{1}{v-s}+\frac{1}{v-\bar{s}})\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv \tag{3.18}$$

$$-\frac{1}{4\chi^2}\int_{I_\epsilon^2} f'(\arg(u-s))f'(\arg(v-s))(\frac{1}{u-s}-\frac{1}{u-\bar{s}})(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv \tag{3.19}$$

$$+\frac{1}{4\chi^2}\int_{I_\epsilon} f''(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})^2\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} dv \tag{3.20}$$

$$+\frac{1}{2i\chi^2}(1-\frac{\gamma\chi}{4})\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{v-s}+\frac{1}{v-\bar{s}})(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} dv. \tag{3.21}$$

Proof. One starts the computation with the following derivative, whose expression is obtained by Gaussian integration by parts.

Lemma 3.5. *The following identity holds:*

$$(\partial_s+\partial_{\bar{s}})\langle s|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} = -\left(\sum_i \frac{\chi\alpha_i}{2}(\frac{1}{z_i-s}+\frac{1}{\bar{z}_i-\bar{s}})+\sum_j\frac{\chi\beta_j}{4}(\frac{1}{s_j-s}+\frac{1}{s_j-\bar{s}})\right)\langle s|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} \tag{3.22}$$

$$+\mu\frac{\gamma\chi}{2}\int_{\mathbb{H}_\delta}(\frac{1}{y-s}+\frac{1}{\bar{y}-\bar{s}})\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y+\frac{\gamma\chi}{4}\int_{\mathbb{R}}(\frac{1}{v-s}+\frac{1}{v-\bar{s}})\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(v) \tag{3.23}$$

$$+\frac{1}{2i}\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dv. \tag{3.24}$$

with

Proof. The three lines above respectively correspond to applying the derivatives to the s dependent part of the prefactor $\prod_i |z_i-s|^{\chi\alpha_i}\prod_j|s_j-s|^{\frac{\chi\beta_j}{2}}$, applying the derivatives to the s insertion present in the GMC integrals over bulk and boundary, and lastly to applying the derivatives to the s dependence contained in the function f in the boundary GMC integral. To compute this last case one needs to use the identity:

$$(\partial_s+\partial_{\bar{s}})\arg(v-s) = \frac{1}{2i}\frac{\bar{s}-s}{(v-s)(v-\bar{s})} = \frac{1}{2i}\left(\frac{1}{v-\bar{s}}-\frac{1}{v-s}\right). \tag{3.25}$$

We now want to apply another times $(\partial_s + \partial_{\bar{s}})$. For this we need to compute the action of $(\partial_s + \partial_{\bar{s}})$ on the terms $\langle s, y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$ and $\langle s, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$, the results are given below:

$$(\partial_s + \partial_{\bar{s}})\langle s, y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$$

(3.26)
$$= -\left(\frac{\chi\gamma}{2}\left(\frac{1}{y-s} + \frac{1}{\bar{y}-\bar{s}}\right) + \sum_i \frac{\chi\alpha_i}{2}\left(\frac{1}{z_i - s} + \frac{1}{\bar{z}_i - \bar{s}}\right) + \sum_j \frac{\chi\beta_j}{4}\left(\frac{1}{s_j - s} + \frac{1}{s_j - \bar{s}}\right)\right)\langle s, y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$$

(3.27)
$$+ \mu\frac{\gamma\chi}{2}\int_{\mathbb{H}_\delta}\left(\frac{1}{x-s} + \frac{1}{\bar{x}-\bar{s}}\right)\langle s, x, y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} d^2x + \frac{\gamma\chi}{4}\int_{\mathbb{R}}\left(\frac{1}{v-s} + \frac{1}{v-\bar{s}}\right)\langle s, v, y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} d\mu^s_{B,\epsilon}(v)$$

(3.28)
$$+ \frac{1}{2i}\int_{I_\epsilon} f'(\arg(v-s))\left(\frac{1}{v-s} - \frac{1}{v-\bar{s}}\right)\langle s, v, y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} dv,$$

and:

$$(\partial_s + \partial_{\bar{s}})\langle s, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$$

(3.29)
$$= -\left(\frac{\chi\gamma}{4}\left(\frac{1}{v-s} + \frac{1}{v-\bar{s}}\right) + \sum_i \frac{\chi\alpha_i}{2}\left(\frac{1}{z_i - s} + \frac{1}{\bar{z}_i - \bar{s}}\right) + \sum_j \frac{\chi\beta_j}{4}\left(\frac{1}{s_j - s} + \frac{1}{s_j - \bar{s}}\right)\right)\langle s, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$$

(3.30)
$$+ \mu\frac{\gamma\chi}{2}\int_{\mathbb{H}_\delta}\left(\frac{1}{y-s} + \frac{1}{\bar{y}-\bar{s}}\right)\langle s, y, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} d^2y + \frac{\gamma\chi}{4}\int_{\mathbb{R}}\left(\frac{1}{u-s} + \frac{1}{u-\bar{s}}\right)\langle s, u, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} d\mu^s_{B,\epsilon}(u)$$

(3.31)
$$+ \frac{1}{2i}\int_{I_\epsilon} f'(\arg(u-s))\left(\frac{1}{u-s} - \frac{1}{u-\bar{s}}\right)\langle s, u, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} du.$$

□

Using these results we get the list of terms given by $-C_\epsilon$. We briefly state how each term arises. The term (3.7) comes from differentiating the prefactor in front of $\langle s|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$ in (3.22). The term (3.8) and half of (3.9) come from differentiating $\langle s|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$ in (3.22) which gives either (3.22) with the prefactor squared or a cross term of (3.22) and (3.23). The term (3.10) comes from differentiating the integrand of the $\int_{\mathbb{H}_\delta}$ term of (3.23). The next three terms (3.11), (3.12), (3.13) come from applying the formula (3.26) to differentiate the $\langle s, y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$ under the integral $\int_{\mathbb{H}_\delta}$ term of (3.23). More precisely the right hand side of (3.26) gives both the term (3.11) and half of the $\int_{\mathbb{H}_\delta}$ term of (3.9), while (3.27) gives both (3.12) and half of (3.13). The term (3.14) comes from differentiating the integrand of the $\int_{\mathbb{R}}$ term of (3.23). Lastly the last two terms (3.15) and (3.16) as well as half of (3.13) come from applying the formula (3.29) to compute the derivative of $\langle s, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$ under the integral $\int_{\mathbb{R}}$ term of (3.23).

We now collect all the terms we get that involve f when applying $(\partial_s+\partial_{\bar{s}})^2$. Some of the terms obtained vanish in the limit as δ goes to 0.

$$
\begin{aligned}
&-\frac{1}{2i}\left(\sum_i \frac{\chi\alpha_i}{2}(\frac{1}{z_i-s}+\frac{1}{\bar{z}_i-\bar{s}})+\sum_j\frac{\chi\beta_j}{4}(\frac{1}{s_j-s}+\frac{1}{s_j-\bar{s}})\right)\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dv\\
&+\frac{1}{2i}\mu\frac{\gamma\chi}{2}\int_{\mathbb{H}_\delta}(\frac{1}{y-s}+\frac{1}{\bar{y}-\bar{s}})\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\langle s,v,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dvd^2y\\
&+\frac{1}{2i}\frac{\gamma\chi}{4}\int_{\mathbb{R}}\left(\frac{1}{v-\bar{s}}-\frac{1}{v-s}\right)(\frac{1}{v-s}+\frac{1}{v-\bar{s}})f'(\arg(v-s))\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(v)\\
&+\frac{1}{2i}\frac{\gamma\chi}{4}\int_{\mathbb{R}}(\frac{1}{v-s}+\frac{1}{v-\bar{s}})\int_{I_\epsilon} f'(\arg(u-s))(\frac{1}{u-s}-\frac{1}{u-\bar{s}})\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dud\mu^s_{B,\epsilon}(v)\\
&+\frac{1}{4}\int_{I_\epsilon} f''(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})^2\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dv\\
&+\frac{1}{2i}\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{(v-s)^2}-\frac{1}{(v-\bar{s})^2})\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dv\\
&-\frac{1}{2i}\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\\
&\qquad\times\left(\frac{\chi\gamma}{4}(\frac{1}{v-s}+\frac{1}{v-\bar{s}})+\sum_i\frac{\chi\alpha_i}{2}(\frac{1}{z_i-s}+\frac{1}{\bar{z}_i-\bar{s}})+\sum_j\frac{\chi\beta_j}{4}(\frac{1}{s_j-s}+\frac{1}{s_j-\bar{s}})\right)\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dv\\
&+\frac{1}{2i}\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\mu\frac{\gamma\chi}{2}\int_{\mathbb{H}_\delta}(\frac{1}{y-s}+\frac{1}{\bar{y}-\bar{s}})\langle s,y,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2ydv\\
&+\frac{1}{2i}\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\frac{\gamma\chi}{4}\int_{\mathbb{R}}(\frac{1}{u-s}+\frac{1}{u-\bar{s}})\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(u)dv\\
&-\frac{1}{4}\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\int_{I_\epsilon} f'(\arg(u-s))(\frac{1}{u-s}-\frac{1}{u-\bar{s}})\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dudv
\end{aligned}
$$

The expression for $(\partial_s+\partial_{\bar{s}})\langle s|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$ has four terms. In the above list, when applying an additional $(\partial_s+\partial_{\bar{s}})$, the first term comes the first term of $(\partial_s+\partial_{\bar{s}})\langle s|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$, the second from the second, the third and fourth from the third, and the last six from the last term of $(\partial_s+\partial_{\bar{s}})\langle s|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$. It turns out that in the limit of $\delta\to 0$, the first two terms and the second to last term in the above list automatically converge to 0 and therefore can simply be ignored. The same holds for the α_i,β_j part of the seventh term. Collecting all of these terms we get the desired expression of $S^{(1)}_{I,\epsilon,\delta}$. □

Now we will prove the following lemma giving the rest of the operator.

Lemma 3.6. *One has*

$$
\begin{aligned}
&\Bigg(\sum_{i=1}^N\left(\frac{1}{s-z_i}\partial_{z_i}+\frac{1}{s-\bar{z}_i}\partial_{\bar{z}_i}+\frac{\Delta_{\alpha_i}}{(s-z_i)^2}+\frac{\Delta_{\alpha_i}}{(s-\bar{z}_i)^2}\right)\\
&+\sum_{j=1}^M\left(\frac{1}{2}(\frac{1}{s-s_j}+\frac{1}{\bar{s}-s_j})\partial_{s_j}+\frac{\Delta_{\beta_j}}{(s-s_j)^2}\right)\Bigg)\langle s|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}=C_{\epsilon,\delta}+S^{(2)}_{I,\epsilon,\delta}+S^{(2)}_{II,\epsilon}+o_{\delta\to 0}(1),
\end{aligned}
$$

where $C_{\epsilon,\delta}$ is as in the previous lemma,

$$S^{(2)}_{I,\epsilon,\delta} = \delta^{-\frac{\gamma^2}{2}} \frac{\mu}{2i} \int_{\mathbb{R}+i\delta} (\frac{1}{y-s} - \frac{1}{\bar{y}-\bar{s}}) \langle s, y | \mathbf{z}; \mathbf{s} \rangle_{\epsilon,\delta} dy \tag{3.32}$$

$$+ \mu \frac{\gamma^2}{2} \int_{\mathbb{H}_\delta} \frac{s-\bar{s}}{|y-s|^2} \frac{1}{(\bar{y}-y)} \langle s, y | \mathbf{z}; \mathbf{s} \rangle_{\epsilon,\delta} d^2 y \tag{3.33}$$

$$+ \frac{1}{4i} \int_{I_\epsilon} f'(\arg(v-s)) (\frac{1}{v-s} + \frac{1}{v-\bar{s}}) (\frac{1}{v-s} - \frac{1}{v-\bar{s}}) \langle s, v | \mathbf{z}; \mathbf{s} \rangle_\epsilon dv \tag{3.34}$$

$$- (\frac{1}{4} - \frac{\gamma}{8\chi}) \int_{\mathbb{R}} (\frac{1}{v-s} - \frac{1}{v-\bar{s}})^2 \langle s, v | \mathbf{z}; \mathbf{s} \rangle_\epsilon d\mu^s_{B,\epsilon}(v) \tag{3.35}$$

$$- \frac{\gamma^2}{16} \int_{\mathbb{R}^2} (\frac{1}{u-s} - \frac{1}{u-\bar{s}}) (\frac{1}{v-s} - \frac{1}{v-\bar{s}}) \langle s, u, v | \mathbf{z}; \mathbf{s} \rangle_\epsilon d\mu^s_{B,\epsilon}(u) d\mu^s_{B,\epsilon}(v) \tag{3.36}$$

$$+ \frac{\gamma^2}{8} \int_{\mathbb{R}^2} (\frac{1}{(u-s)(v-s)} + \frac{1}{(u-\bar{s})(v-\bar{s})}) \mathbf{1}_{|u-v|<\delta} \langle s, u, v | \mathbf{z}; \mathbf{s} \rangle_{\epsilon,\delta} d\mu^s_{B,\epsilon}(u) d\mu^s_{B,\epsilon}(v) \tag{3.37}$$

and

$$S^{(2)}_{II,\epsilon} = -\sum_{j=1}^{M} \left(\frac{\mu_{j-1}}{(s_j-s)(s_j-s-\epsilon)} \epsilon \langle s, s_j - \epsilon | \mathbf{z}; \mathbf{s} \rangle_\epsilon + \frac{\mu_j}{(s_j-s)(s_j-s+\epsilon)} \epsilon \langle s, s_j + \epsilon | \mathbf{z}; \mathbf{s} \rangle_\epsilon \right).$$

Proof. This computation is more involved than the previous as it involves performing integration by parts on certain terms. We first compute the derivatives with respect to z_k, $\bar{z}_k$ and s_j:

$$\partial_{z_k} \langle s | \mathbf{z}; \mathbf{s} \rangle_{\epsilon,\delta} = \left(\sum_{i: i \neq k} \frac{\alpha_i \alpha_k}{2(z_i - z_k)} + \sum_i \frac{\alpha_i \alpha_k}{2(\bar{z}_i - z_k)} + \sum_j \frac{\beta_j \alpha_k}{2(s_j - z_k)} - \frac{\chi \alpha_k}{2(s - z_k)} \right) \langle s | \mathbf{z}; \mathbf{s} \rangle_{\epsilon,\delta}$$
$$- \mu \frac{\gamma \alpha_k}{2} \int_{\mathbb{H}_\delta} (\frac{1}{(y-z_k)_\delta} + \frac{1}{(\bar{y}-z_k)_\delta}) \langle s, y | \mathbf{z}; \mathbf{s} \rangle_{\epsilon,\delta} d^2 y - \frac{\gamma \alpha_k}{2} \int_{\mathbb{R}} \frac{1}{(v-z_k)_\delta} \langle s, v | \mathbf{z}; \mathbf{s} \rangle_{\epsilon,\delta} d\mu^s_{B,\epsilon}(v),$$

$$\partial_{\bar{z}_k} \langle s | \mathbf{z}; \mathbf{s} \rangle_{\epsilon,\delta} = \left(\sum_i \frac{\alpha_i \alpha_k}{2(z_i - \bar{z}_k)} + \sum_{i: i \neq k} \frac{\alpha_i \alpha_k}{2(\bar{z}_i - \bar{z}_k)} + \sum_j \frac{\beta_j \alpha_k}{2(s_j - \bar{z}_k)} - \frac{\chi \alpha_k}{2(\bar{s} - \bar{z}_k)} \right) \langle s | \mathbf{z}; \mathbf{s} \rangle_{\epsilon,\delta}$$
$$- \mu \frac{\gamma \alpha_k}{2} \int_{\mathbb{H}_\delta} (\frac{1}{(y-\bar{z}_k)_\delta} + \frac{1}{(\bar{y}-\bar{z}_k)_\delta}) \langle s, y | \mathbf{z}; \mathbf{s} \rangle_{\epsilon,\delta} d^2 y - \frac{\gamma \alpha_k}{2} \int_{\mathbb{R}} \frac{1}{(v-\bar{z}_k)_\delta} \langle s, v | \mathbf{z}; \mathbf{s} \rangle_{\epsilon,\delta} d\mu^s_{B,\epsilon}(v),$$

and

$$\partial_{s_k} \langle s | \mathbf{z}; \mathbf{s} \rangle_{\epsilon,\delta} = \left(\sum_i \frac{\alpha_i \beta_k}{2} (\frac{1}{z_i - s_k} + \frac{1}{\bar{z}_i - s_k}) + \sum_{j: j \neq k} \frac{\beta_j \beta_k}{2(s_j - s_k)} - \frac{\chi \beta_k}{4} (\frac{1}{s - s_k} + \frac{1}{\bar{s} - s_k}) \right) \langle s | \mathbf{z}; \mathbf{s} \rangle_{\epsilon,\delta}$$
$$- \mu \frac{\gamma \beta_k}{2} \int_{\mathbb{H}_\delta} (\frac{1}{(y-s_k)_\delta} + \frac{1}{(\bar{y}-s_k)_\delta}) \langle s, y | \mathbf{z}; \mathbf{s} \rangle_{\epsilon,\delta} d^2 y - \frac{\gamma \beta_k}{2} \int_{\mathbb{R}} \frac{1}{(v-s_k)_\delta} \langle s, v | \mathbf{z}; \mathbf{s} \rangle_{\epsilon,\delta} d\mu^s_{B,\epsilon}(v)$$
$$- (\tilde{\mu}_{k-1} \langle s, s_k - \epsilon | \mathbf{z}; \mathbf{s} \rangle_\epsilon - \tilde{\mu}_k \langle s, s_k + \epsilon | \mathbf{z}; \mathbf{s} \rangle_\epsilon),$$

where $\tilde{\mu}_k = \mu_k$ for $1 \leq k \leq M$, and $\tilde{\mu}_0 = f(\arg(s_1 - s - \epsilon))$.

Lets first collect all the terms without integrals. We focus first on all the terms containing a z_i variable, the corresponding terms are:

$$\sum_{k=1}^{N}\frac{1}{s-z_k}\left(\sum_{i:i\neq k}\frac{\alpha_i\alpha_k}{2(z_i-z_k)}+\sum_i\frac{\alpha_i\alpha_k}{2(\overline{z}_i-z_k)}+\sum_j\frac{\beta_j\alpha_k}{2(s_j-z_k)}-\frac{\chi\alpha_k}{2(s-z_k)}\right)+\sum_{k=1}^{N}\frac{\Delta_{\alpha_k}}{(s-z_k)^2}$$
$$+\sum_{k=1}^{N}\frac{1}{s-\bar{z}_k}\sum_i\frac{\alpha_i\alpha_k}{2(z_i-\bar{z}_k)}+\frac{1}{2}\sum_{k=1}^{M}(\frac{1}{s-s_k}+\frac{1}{\bar{s}-s_k})\sum_i\frac{\alpha_i\beta_k}{2}\frac{1}{z_i-s_k}.$$

We recombine the terms using:

$$\frac{\alpha_i\alpha_k}{2}\left(\frac{1}{s-z_k}\frac{1}{z_i-z_k}+\frac{1}{s-z_i}\frac{1}{z_k-z_i}\right)=-\frac{\alpha_i\alpha_k}{2}\frac{1}{s-z_i}\frac{1}{s-z_k},$$
$$\frac{\alpha_i\alpha_k}{2}\left(\frac{1}{s-z_k}\frac{1}{\bar{z}_i-z_k}+\frac{1}{s-\bar{z}_i}\frac{1}{z_k-\bar{z}_i}\right)=-\frac{\alpha_i\alpha_k}{2}\frac{1}{s-z_k}\frac{1}{s-\bar{z}_i},$$

and similarly for the s_j part. Using this we correctly recover the z_i terms given in $C_{\epsilon,\delta}$:

$$\sum_i\frac{\alpha_i}{2\chi}\frac{1}{(z_i-s)^2}-\left(\sum_i\frac{\alpha_i}{2}\frac{1}{z_i-s}\right)^2-\sum_i\alpha_i\frac{1}{z_i-s}\left(\sum_i\frac{\alpha_i}{2}\frac{1}{\bar{z}_i-\bar{s}}+\sum_j\frac{\beta_j}{4}(\frac{1}{s_j-s}+\frac{1}{s_j-\bar{s}})\right).$$

Now we move to the terms containing integrals over $\mathbb{H}_\delta$ and $\mathbb{R}$. Lets first collect all the terms with a single integral over $\mathbb{H}_\delta$. We get the following terms:

$$-\mu\frac{\gamma\alpha_k}{2}\int_{\mathbb{H}_\delta}\frac{1}{s-z_k}(\frac{1}{(y-z_k)_\delta}+\frac{1}{(\bar{y}-z_k)_\delta})\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y$$
$$-\mu\frac{\gamma\alpha_k}{2}\int_{\mathbb{H}_\delta}\frac{1}{s-\bar{z}_k}(\frac{1}{(y-\bar{z}_k)_\delta}+\frac{1}{(\bar{y}-\bar{z}_k)_\delta})\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y$$
$$-\mu\frac{\gamma\beta_k}{2}\int_{\mathbb{H}_\delta}\frac{1}{2}\left(\frac{1}{s-s_j}+\frac{1}{\bar{s}-s_j}\right)(\frac{1}{(y-s_k)_\delta}+\frac{1}{(\bar{y}-s_k)_\delta})\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y.$$

Notice that one has:

$$-\sum_i\mu\frac{\gamma\alpha_i}{2}\int_{\mathbb{H}_\delta}(\frac{1}{s-z_i}-\frac{1}{\bar{s}-z_i})\frac{1}{(\bar{y}-z_i)_\delta}\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y=o_{\delta\to 0}(1).$$

Hence we can transform the terms above containing z_i to

$$-\sum_i\mu\frac{\gamma\alpha_i}{2}\int_{\mathbb{H}_\delta}(\frac{1}{(s-z_i)(y-z_i)_\delta}+\frac{1}{(\bar{s}-z_i)(\bar{y}-z_i)_\delta})\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y$$
$$=\sum_i\mu\frac{\gamma\alpha_i}{2}\int_{\mathbb{H}_\delta}(\frac{1}{(y-s)(y-z_i)_\delta}+\frac{1}{(\bar{y}-\bar{s})(\bar{y}-z_i)_\delta})\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y$$
$$+\sum_i\mu\frac{\gamma\alpha_i}{2}\int_{\mathbb{H}_\delta}(\frac{1}{(z_i-s)(y-s)}\frac{y-z_i}{(y-z_i)_\delta}+\frac{1}{(z_i-\bar{s})(\bar{y}-\bar{s})}\frac{\bar{y}-z_i}{(\bar{y}-z_i)_\delta})\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y.$$

In the above answer we are going to integrate by parts over y the first integral and keep the second as it appears in the answer $C_{\epsilon,\delta}$. Of course we can perform the same manipulation for the terms containing $\bar{z}_i$ and s_j. Combining the three cases we are now going to perform an integration by parts on the terms

$$\sum_i \mu \frac{\gamma\alpha_i}{2}\int_{\mathbb{H}_\delta}\left(\frac{1}{y-s}(\frac{1}{(y-z_i)_\delta}+\frac{1}{(y-\bar z_i)_\delta})+\frac{1}{\bar y-\bar s}(\frac{1}{(\bar y-z_i)_\delta}+\frac{1}{(\bar y-\bar z_i)_\delta})\right)\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y$$
$$+\sum_j \mu\frac{\gamma\beta_j}{2}\int_{\mathbb{H}_\delta}(\frac{1}{(y-s)(y-s_j)_\delta}+\frac{1}{(\bar y-\bar s)(\bar y-s_j)_\delta})\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y$$

To do this we must take in account the explicit prefactor infront of the GMC expression in $\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$ given by:

$$\frac{|y-s|^{\gamma\chi}}{|y-\bar y|^{\frac{\gamma^2}{2}}\prod_{i=1}^N|y-z_i|_\delta^{\gamma\alpha_i}|y-\bar z_i|_\delta^{\gamma\alpha_i}\prod_{j=1}^M|y-s_j|_\delta^{\gamma\beta_j}}Z_{-\chi|\boldsymbol{\alpha};\boldsymbol{\beta}}(s|\mathbf{z};\mathbf{s}).$$

Remark that $\partial_y(|y-z_i|_\delta^{-\gamma\alpha_i}|y-\bar z_i|_\delta^{-\gamma\alpha_i})=-\frac{\gamma\alpha_i}{2}(\frac{1}{(y-z_i)_\delta}+\frac{1}{(y-\bar z_i)_\delta})|y-z_i|_\delta^{-\gamma\alpha_i}|y-\bar z_i|_\delta^{-\gamma\alpha_i}$ and $\partial_{\bar y}(|y-z_i|_\delta^{-\gamma\alpha_i}|y-\bar z_i|_\delta^{-\gamma\alpha_i})=-\frac{\gamma\alpha_i}{2}(\frac{1}{(\bar y-z_i)_\delta}+\frac{1}{(\bar y-\bar z_i)_\delta})|y-z_i|_\delta^{-\gamma\alpha_i}|y-\bar z_i|_\delta^{-\gamma\alpha_i}$. We also have similar relations with $|y-s_j|_\delta^{-\gamma\beta_j}$. Recall Green's theorem, which states that for a domain $D\subset\mathbb{H}$ and two functions f,g defined on D one has:

$$\int_D \partial_z f(z)g(z)d^2z=-\int_D f(z)\partial_z g(z)d^2z+\frac{i}{2}\int_{\partial D}f(z)g(z)d\bar z,$$
$$\int_D \partial_{\bar z} f(z)g(z)d^2z=-\int_D f(z)\partial_{\bar z} g(z)d^2z-\frac{i}{2}\int_{\partial D}f(z)g(z)dz.$$

The integrations along ∂D are in the counterclockwise direction. Then we proceed with an integration by parts to obtain:

$$\begin{aligned}(3.38)\quad &\sum_i \mu \frac{\gamma\alpha_i}{2}\int_{\mathbb{H}_\delta}\left(\frac{1}{y-s}(\frac{1}{(y-z_i)_\delta}+\frac{1}{(y-\bar z_i)_\delta})+\frac{1}{\bar y-\bar s}(\frac{1}{(\bar y-z_i)_\delta}+\frac{1}{(\bar y-\bar z_i)_\delta})\right)\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y\\
&+\sum_j \mu\frac{\gamma\beta_j}{2}\int_{\mathbb{H}_\delta}(\frac{1}{(y-s)(y-s_j)_\delta}+\frac{1}{(\bar y-\bar s)(\bar y-s_j)_\delta})\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y\\
=&\delta^{-\frac{\gamma^2}{2}}\frac{\mu}{2i}\int_{\mathbb{R}+i\delta}(\frac{1}{y-s}-\frac{1}{\bar y-\bar s})\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dy+\mu(-1+\frac{\gamma\chi}{2})\int_{\mathbb{H}_\delta}(\frac{1}{(y-s)^2}+\frac{1}{(\bar y-\bar s)^2})\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y\\
&+\mu\frac{\gamma^2}{2}\int_{\mathbb{H}_\delta}(\frac{1}{y-s}-\frac{1}{\bar y-\bar s})\frac{1}{(\bar y-y)}\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y\\
&-\mu^2\frac{\gamma^2}{2}\int_{\mathbb{H}_\delta^2}\left(\frac{1}{y-s}(\frac{1}{x-y}+\frac{1}{\bar x-y})+\frac{1}{\bar y-\bar s}(\frac{1}{x-\bar y}+\frac{1}{\bar x-\bar y})\right)\langle s,x,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2xd^2y\\
&-\mu\frac{\gamma^2}{2}\int_{\mathbb{H}_\delta}\int_{\mathbb{R}}\left(\frac{1}{(y-s)(v-y)}+\frac{1}{(\bar y-\bar s)(v-\bar y)}\right)\langle s,y,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu_{B,\epsilon}^s(v)d^2y.\end{aligned}$$

Lets analyse the terms in the answer of (3.38) one by one. Note that $\delta^{-\frac{\gamma^2}{2}}\frac{\mu}{2i}\int_{\mathbb{R}+i\delta}(\frac{1}{y-s}-\frac{1}{\bar y-\bar s})\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dy$ is the boundary integration term coming from the boundary contribution in Green's theorem. The power of δ in front comes from the hyperbolic metric at the boundary of $\mathbb{H}_\delta$. We add this term to the list of singular terms $S_{I,\epsilon,\delta}^{(2)}$. The second term is expected in $C_{\epsilon,\delta}$. Next we analyse the term:

$$\mu\frac{\gamma^2}{2}\int_{\mathbb{H}_\delta}(\frac{1}{y-s}-\frac{1}{\bar y-\bar s})\frac{1}{\bar y-y}\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y=\mu\frac{\gamma^2}{2}\int_{\mathbb{H}_\delta}\frac{1}{y-s}\frac{1}{\bar y-\bar s}\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y$$
$$+\mu\frac{\gamma^2}{2}\int_{\mathbb{H}_\delta}\frac{s-\bar s}{(y-s)(\bar y-\bar s)}\frac{1}{\bar y-y}\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y.$$

Out of these two terms obtained, the first one will go in $C_{\epsilon,\delta}$ and the second one in the list of singular terms $S^{(2)}_{I,\epsilon,\delta}$. Next we move to the term containing the double integral over $\mathbb{H}^2_\delta$. Notice here that the singularity $\frac{1}{x-y}$ is integrable since we are integrating over $\mathbb{H}^2_\delta$. We can write:

$$\begin{aligned}
&-\mu^2\frac{\gamma^2}{2}\int_{\mathbb{H}^2_\delta}\left(\frac{1}{y-s}(\frac{1}{x-y}+\frac{1}{\bar{x}-y})+\frac{1}{\bar{y}-\bar{s}}(\frac{1}{x-\bar{y}}+\frac{1}{\bar{x}-\bar{y}})\right)\langle s,x,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2xd^2y\\
&=-\mu^2\frac{\gamma^2}{4}\int_{\mathbb{H}^2_\delta}(\frac{1}{x-s}+\frac{1}{\bar{x}-\bar{s}})(\frac{1}{y-s}+\frac{1}{\bar{y}-\bar{s}})\langle s,x,y|\mathbf{z};\mathbf{s}\rangle_\epsilon d^2xd^2y\\
&+\mu^2\frac{\gamma^2}{2}\int_{\mathbb{H}^2_\delta}\frac{s-\bar{s}}{(x-s)(\bar{y}-\bar{s})(x-\bar{y})}\langle s,x,y|\mathbf{z};\mathbf{s}\rangle_\epsilon d^2xd^2y+o_{\delta\to 0}(1).
\end{aligned}$$

Lastly the term containing an integral over $\mathbb{H}_\delta$ and one over $\mathbb{R}$ will be combined with a similar term coming from the integration by parts on the boundary term (see below) to get in the limit the following term present in $C_{\epsilon,\delta}$:

$$\mu\frac{\gamma^2}{2}\int_{\mathbb{H}_\delta}\int_{\mathbb{R}}\left(\frac{1}{(y-s)(v-s)}+\frac{1}{(\bar{y}-\bar{s})(v-\bar{s})}\right)\langle s,y,v|\mathbf{z};\mathbf{s}\rangle_\epsilon d\mu^s_{B,\epsilon}(v)d^2y.$$

Similarly we need to perform an integration by parts on one of the boundary integral terms. Starting from the terms

$$\begin{aligned}
&-\frac{\gamma\alpha_i}{2}\int_{\mathbb{R}}\frac{1}{s-z_i}\frac{1}{(v-z_i)_\delta}\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(v)-\frac{\gamma\alpha_i}{2}\int_{\mathbb{R}}\frac{1}{s-\bar{z}_i}\frac{1}{(v-\bar{z}_i)_\delta}\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(v)\\
&-\frac{\gamma\beta_j}{4}\int_{\mathbb{R}}\Big(\frac{1}{s-s_j}+\frac{1}{\bar{s}-s_j}\Big)\frac{1}{(v-s_j)_\delta}\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(v),
\end{aligned}$$

we add minus times the terms expected in our answer

$$-\left(\sum_i\frac{\gamma\alpha_i}{4}\left(\frac{1}{z_i-s}+\frac{1}{\bar{z}_i-s}\right)+\sum_j\frac{\gamma\beta_i}{4}\frac{1}{s_j-s}\right)\int_{\mathbb{R}}\left(\frac{1}{v-s}+\frac{1}{v-\bar{s}}\right)\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(v),$$

which then gives up to a $o(\delta)$ error, also adding the term in $\bar{s}$:

$$\frac{1}{2}\int_{\mathbb{R}}\left(\frac{1}{v-s}+\frac{1}{v-\bar{s}}\right)\left(\sum_i\frac{\gamma\alpha_i}{2}\left(\frac{1}{v-z_i}+\frac{1}{v-\bar{z}_i}\right)+\sum_j\frac{\gamma\beta_j}{2}\frac{1}{v-s_j}\right)\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(v).$$

One has $\partial_v|v-z_i|_\delta^{-\gamma\alpha_i}=\left(-\frac{\gamma\alpha_i}{2}\frac{1}{(v-z_i)_\delta}-\frac{\gamma\alpha_i}{2}\frac{1}{(v-\bar{z}_i)_\delta}\right)|v-z_i|_\delta^{-\gamma\alpha_i}$ and $\partial_v|v-s_j|_\delta^{-\frac{\gamma\beta_j}{2}}=-\frac{\gamma\beta_j}{2}\frac{1}{(v-s_j)_\delta}|v-s_j|_\delta^{-\frac{\gamma\beta_j}{2}}$. Here the ∂_v is a real derivative. We record the formulas:

$$\partial_v|v-s|^{\frac{\gamma\chi}{2}}=\frac{\gamma\chi}{4}\left(\frac{1}{v-s}+\frac{1}{v-\bar{s}}\right)|v-s|^{\frac{\gamma\chi}{2}},\quad \partial_v\arg(v-s)=\frac{1}{2i}\left(\frac{1}{v-s}-\frac{1}{v-\bar{s}}\right).$$

For this computation, before integrating by parts the integral over v, we are going to smooth the GMC measure $e^{\frac{\gamma}{2}X(v)}dv$. For this purpose, for $v\in\mathbb{R}$, we introduce the field:

$$X_\delta(v)=\frac{1}{\pi}\int_0^\pi X(v+\delta e^{i\theta})d\theta.$$

We can then compute the following covariance, for $u, v \in \mathbb{R}$

$$\begin{aligned}\mathbb{E}[X(u)X_\delta(v)] &= \frac{1}{\pi}\int_0^\pi \mathbb{E}[X(u)X(v+\delta e^{i\theta})]d\theta \\ &= \frac{2}{\pi}\int_0^\pi \log\frac{1}{|u-v+\delta e^{i\theta}|}d\theta + 2\log|v|_+ + \frac{2}{\pi}\int_0^\pi \log|u+\delta e^{i\theta}|_+ d\theta\end{aligned}$$

Notice then that $\frac{2}{\pi}\int_0^\pi \log\frac{1}{|u-v+\delta e^{i\theta}|}d\theta$ is equal to $-2\log|u-v|$ if $\delta < |u-v|$ and to $-2\log\delta$ if $\delta > |u-v|$. We also record the computation:

$$\begin{aligned}\partial_v\langle s, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} = &\left(\frac{\chi\gamma}{4}(\frac{1}{v-s}+\frac{1}{v-\bar{s}}) + \sum_i \frac{\gamma\alpha_i}{2}(\frac{1}{(z_i-v)_\delta}+\frac{1}{(\bar{z}_i-v)_\delta}) + \sum_j \frac{\gamma\beta_j}{2}\frac{1}{(s_j-v)_\delta}\right)\langle s, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} \\ &- \mu\frac{\gamma^2}{2}\int_{\mathbb{H}_\delta}(\frac{1}{y-v}+\frac{1}{\bar{y}-v})\langle s, y, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y - \frac{\gamma^2}{2}\int_{\mathbb{R}}\frac{1}{u-v}\langle s, u, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(u).\end{aligned}$$

Then by integration by parts we obtain:

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}}\left(\frac{1}{v-s}+\frac{1}{v-\bar{s}}\right)\left(\sum_i\frac{\gamma\alpha_i}{2}\left(\frac{1}{(v-z_i)_\delta}+\frac{1}{(v-\bar{z}_i)_\delta}\right)+\sum_j\frac{\gamma\beta_j}{2}\frac{1}{(v-s_j)_\delta}\right)\langle s, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(v) \\ &= \frac{1}{2}\int_{\mathbb{R}}\left(\frac{1}{v-s}+\frac{1}{v-\bar{s}}\right)\left(\frac{\chi\gamma}{4}(\frac{1}{v-s}+\frac{1}{v-\bar{s}})-\partial_v\right)\langle s, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(v) \\ &- \frac{\gamma^2}{4}\int_{\mathbb{R}}\left(\frac{1}{v-s}+\frac{1}{v-\bar{s}}\right)\left(\mu\int_{\mathbb{H}_\delta}(\frac{1}{y-v}+\frac{1}{\bar{y}-v})\langle s, y, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y + \int_{\mathbb{R}}\frac{1}{u-v}\mathbf{1}_{|v-u|>\delta}\langle s, u, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(u)\right)d\mu^s_B \\ &= \frac{\gamma\chi}{8}\int_{\mathbb{R}}\left(\frac{1}{v-s}+\frac{1}{v-\bar{s}}\right)\left(\frac{1}{v-s}+\frac{1}{v-\bar{s}}\right)\langle s, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(v) \\ &- \frac{\gamma^2}{4}\int_{\mathbb{R}}\left(\frac{1}{v-s}+\frac{1}{v-\bar{s}}\right)\left(\mu\int_{\mathbb{H}_\delta}(\frac{1}{y-v}+\frac{1}{\bar{y}-v})\langle s, y, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y + \int_{\mathbb{R}}\frac{1}{u-v}\mathbf{1}_{|v-u|>\delta}\langle s, u, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(u)\right)d\mu^s_B \\ &- \frac{1}{2}\int_{\mathbb{R}}\left(\frac{1}{(v-s)^2}+\frac{1}{(v-\bar{s})^2}\right)\langle s, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(v) \\ &+ \frac{1}{4i}\int_{I_\epsilon}\left(\frac{1}{v-s}+\frac{1}{v-\bar{s}}\right)\left(\frac{1}{v-s}-\frac{1}{v-\bar{s}}\right)f'(\arg(v-s))\langle s, v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(v).\end{aligned}$$

Lets look at how these terms are going to recombine to obtain the $C_{\epsilon,\delta}$ terms and the singular terms given by $S^{(2)}_{I,\epsilon,\delta}$. First lets look at the simple integrals over $\mathbb{R}$. The terms we want for $C_{\epsilon,\delta}$ are:

$$-\frac{\gamma}{4\chi}\int_{\mathbb{R}}(\frac{1}{(v-s)^2}+\frac{1}{(v-\bar{s})^2})\langle s, v|\mathbf{z};\mathbf{s}\rangle_\epsilon d\mu^s_{B,\epsilon}(v) + \frac{\gamma^2}{16}\int_{\mathbb{R}}(\frac{1}{v-s}+\frac{1}{v-\bar{s}})^2\langle s, v|\mathbf{z};\mathbf{s}\rangle_\epsilon d\mu^s_{B,\epsilon}(v)$$

Taking the difference between the terms we have and these terms we want to get:

$$-(\frac{1}{4}-\frac{\gamma}{8\chi})\int_{\mathbb{R}}(\frac{1}{v-s}-\frac{1}{v-\bar{s}})^2\langle s, v|\mathbf{z};\mathbf{s}\rangle_\epsilon d\mu^s_{B,\epsilon}(v).$$

Lets analyse the term with the double integral over $\mathbb{R}$. We compute

$$\begin{aligned}
&\frac{\gamma^2}{4}\int_{\mathbb{R}^2}\left(\frac{1}{v-s}+\frac{1}{v-\bar{s}}\right)\frac{1}{(v-u)}\mathbf{1}_{|u-v|>\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(u)d\mu^s_{B,\epsilon}(v)\\
&=-\frac{\gamma^2}{8}\int_{\mathbb{R}^2}(\frac{1}{(u-s)(v-s)}+\frac{1}{(u-\bar{s})(v-\bar{s})})\mathbf{1}_{|u-v|>\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(u)d\mu^s_{B,\epsilon}(v)\\
&=-\frac{\gamma^2}{16}\int_{\mathbb{R}^2}(\frac{1}{u-s}+\frac{1}{u-\bar{s}})(\frac{1}{v-s}+\frac{1}{v-\bar{s}})\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon}d\mu^s_{B,\epsilon}(u)d\mu^s_{B,\epsilon}(v)\\
&-\frac{\gamma^2}{16}\int_{\mathbb{R}^2}(\frac{1}{u-s}-\frac{1}{u-\bar{s}})(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon}d\mu^s_{B,\epsilon}(u)d\mu^s_{B,\epsilon}(v)\\
&+\frac{\gamma^2}{8}\int_{\mathbb{R}^2}(\frac{1}{(u-s)(v-s)}+\frac{1}{(u-\bar{s})(v-\bar{s})})\mathbf{1}_{|u-v|<\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(u)d\mu^s_{B,\epsilon}(v)
\end{aligned}$$

In this last line, the first term will go into $C_{\epsilon,\delta}$ and the last two will be singular terms. □

3.3. **Vanishing of singular terms.** To prove Theorem 1.3, it remains to be shown that $S^{(1)}_{I,\epsilon,\delta}+S^{(2)}_{I,\epsilon,\delta}+S^{(2)}_{II,\epsilon}\to 0$ as ϵ and δ go to 0. We first state the following result.

Lemma 3.7. *For all $1\leq j\leq M$, when $\epsilon\to 0$, $\epsilon\langle s,s_j-\epsilon|\mathbf{z};\mathbf{s}\rangle_\epsilon\to 0$ and $\epsilon\langle s,s_j+\epsilon|\mathbf{z};\mathbf{s}\rangle_\epsilon\to 0$.*

Proof. This is done in the paper [RZ20b]. □

This result implies that we have $S^{(2)}_{II,\epsilon,\delta}\to 0$ as $\epsilon\to 0$. Next $S^{(1)}_{I,\epsilon,\delta}+S^{(2)}_{I,\epsilon,\delta}$ is given by:

$$\begin{aligned}
&-\frac{\gamma}{8i\chi}\int_{I_\epsilon}f'(\arg(v-s))(\frac{1}{v-s}+\frac{1}{v-\bar{s}})(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\langle s,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dv\\
&+\frac{\gamma}{4i\chi}\int_{I_\epsilon^2}f'(\arg(u-s))f(\arg(v-s))(\frac{1}{u-s}-\frac{1}{u-\bar{s}})(\frac{1}{v-s}+\frac{1}{v-\bar{s}})\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv\\
&-\frac{1}{4\chi^2}\int_{I_\epsilon^2}f'(\arg(u-s))f'(\arg(v-s))(\frac{1}{u-s}-\frac{1}{u-\bar{s}})(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv\\
&+\frac{1}{4\chi^2}\int_{I_\epsilon}f''(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})^2\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dv\\
&+\frac{1}{2i\chi^2}(1-\frac{\gamma\chi}{4})\int_{I_\epsilon}f'(\arg(v-s))(\frac{1}{v-s}+\frac{1}{v-\bar{s}})(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dv\\
&+\delta^{-\frac{\gamma^2}{2}}\frac{\mu}{2i}\int_{\mathbb{R}+i\delta}(\frac{1}{y-s}-\frac{1}{\bar{y}-\bar{s}})\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dy\\
&+\mu\frac{\gamma^2}{2}\int_{\mathbb{H}_\delta}\frac{s-\bar{s}}{|y-s|^2}\frac{1}{(\bar{y}-y)}\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y\\
&-\frac{\gamma^2}{16}\int_{\mathbb{R}^2}(\frac{1}{u-s}-\frac{1}{u-\bar{s}})(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon d\mu^s_{B,\epsilon}(u)d\mu^s_{B,\epsilon}(v)\\
&+\frac{1}{4i}\int_{I_\epsilon}f'(\arg(v-s))(\frac{1}{v-s}+\frac{1}{v-\bar{s}})(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\langle s,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dv\\
&-(\frac{1}{4}-\frac{\gamma}{8\chi})\int_{\mathbb{R}}(\frac{1}{v-s}-\frac{1}{v-\bar{s}})^2\langle s,v|\mathbf{z};\mathbf{s}\rangle_\epsilon d\mu^s_{B,\epsilon}(v)\\
&+\frac{\gamma^2}{8}\int_{\mathbb{R}^2}(\frac{1}{(u-s)(v-s)}+\frac{1}{(u-\bar{s})(v-\bar{s})})\mathbf{1}_{|u-v|<\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{B,\epsilon}(u)d\mu^s_{B,\epsilon}(v).
\end{aligned}$$

We start with an integration by parts formula that will be use to combine all the terms in $S^{(1)}_{I,\epsilon,\delta} + S^{(2)}_{I,\epsilon,\delta}$ with a single integral into a single term involving a double integral.

Lemma 3.8. *For $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$, the following identity holds:*

$$\frac{1}{i\gamma\chi}(1-\frac{\gamma\chi}{4})\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{v-s}+\frac{1}{v-\bar{s}})(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\langle s,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dv$$
$$+\frac{1}{2\gamma\chi}\int_{I_\epsilon} f''(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})^2\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} dv$$
$$= -\frac{\gamma}{2i\chi}\int_{I_\epsilon^2} f(\arg(u-s))f'(\arg(v-s))\frac{1}{(u-v)}(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\mathbf{1}_{|u-v|>\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} du dv + o_{\delta\to 0}(1).$$

Proof. Record the identity $\partial_v \arg(v-s) = \frac{1}{2i}\left(\frac{1}{v-s}-\frac{1}{v-\bar{s}}\right)$. We then perform the integration by parts in the following way:

$$-\frac{1}{2\gamma\chi}\int_{I_\epsilon} f''(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})^2\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} dv$$
$$= -\frac{i}{\gamma\chi}\int_{I_\epsilon} \partial_v f'(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} dv$$
$$= -\frac{i}{\gamma\chi}\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})(\frac{1}{v-s}+\frac{1}{v-\bar{s}})\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} dv$$
$$+\frac{i}{\gamma\chi}\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\partial_v\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} dv.$$

We then need to compute the derivative $\partial_v\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}$, keeping in mind that several terms will be included in the $o_{\delta\to 0}(1)$. We obtain:

$$\frac{i}{4}\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})(\frac{1}{v-s}+\frac{1}{v-\bar{s}})\partial_v\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} dv$$
$$+\frac{\gamma}{2i\chi}\int_{I_\epsilon^2} f(\arg(u-s))f'(\arg(v-s))\frac{1}{(u-v)}(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\mathbf{1}_{|u-v|>\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} du dv + o_{\delta\to 0}(1).$$

This gives us the desired result. □

At this stage we are going to look separately at the cases $\chi = \frac{\gamma}{2}$ and $\chi = \frac{2}{\gamma}$.

3.3.1. *The case of $\chi = \frac{2}{\gamma}$.* When $\chi = \frac{2}{\gamma}$, the argument will be simpler because the relation on σ will be simpler and many of the singular terms directly vanish in the limit $\delta \to 0$. The ones that will remain are:

$$\frac{\gamma^2}{16}\int_{I_\epsilon} f''(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar{s}})^2\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta} dv$$
$$+\frac{1}{4i}\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{v-s}+\frac{1}{v-\bar{s}})(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\langle s,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dv$$
$$-(\frac{1}{4}-\frac{\gamma^2}{16})\int_{\mathbb{R}}(\frac{1}{v-s}-\frac{1}{v-\bar{s}})^2\langle s,v|\mathbf{z};\mathbf{s}\rangle_\epsilon d\mu^s_{B,\epsilon}(v). \tag{3.39}$$

Notice also the integration by parts formula written above up to an $o_{\delta\to 0}(1)$ reduces to:

$$\begin{aligned}&\frac{1}{i}\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{v-s}+\frac{1}{v-\bar s})(\frac{1}{v-s}-\frac{1}{v-\bar s})\langle s,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dv\\ &=-\int_{I_\epsilon} f''(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar s})^2\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dv=\int_{I_\epsilon}(\frac{1}{v-s}-\frac{1}{v-\bar s})^2\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dv.\end{aligned}$$

Combining this identity with the fact that for $\chi=\frac{2}{\gamma}$ one has $f''=-f$, one obtains that the list of terms in (3.39) sums to 0.

3.3.2. *The case of* $\chi=\frac{\gamma}{2}$. We will now focus on $\chi=\frac{\gamma}{2}$ where one has this time $f''=-\frac{\gamma^2\chi^2}{4}f=-\frac{\gamma^4}{16}f$. Let us state the integration by parts for this case. one can combine all the terms in $S^{(1)}_{I,\epsilon,\delta}+S^{(2)}_{I,\epsilon,\delta}$ with a single integral over $\mathbb{R}$ to obtain the single term:

$$\begin{aligned}&\frac{2}{i\gamma^2}(1-\frac{\gamma^2}{8})\int_{I_\epsilon} f'(\arg(v-s))(\frac{1}{v-s}+\frac{1}{v-\bar s})(\frac{1}{v-s}-\frac{1}{v-\bar s})\langle s,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dv\\ &+\frac{1}{\gamma^2}\int_{I_\epsilon} f''(\arg(v-s))(\frac{1}{v-s}-\frac{1}{v-\bar s})^2\langle s,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dv\\ &=-\frac{1}{i}\int_{I_\epsilon^2} f(\arg(u-s))f'(\arg(v-s))\frac{1}{(u-v)}(\frac{1}{v-s}-\frac{1}{v-\bar s})\mathbf{1}_{|u-v|>\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dudv.\end{aligned}$$

The last equality is obtained by applying the integration by parts lemma. Adding this last term to the remaining terms of $S^{(1)}_{I,\epsilon,\delta}+S^{(2)}_{I,\epsilon,\delta}$ containing derivatives of f we get the sum of three double integrals:

(3.40)

$$\begin{aligned}&\frac{\gamma^2}{4i}\int_{I_\epsilon^2}\cos(\frac{\gamma^2}{4}\arg(u-s))\sin(\frac{\gamma^2}{4}\arg(v-s))\frac{1}{(u-v)}(\frac{1}{v-s}-\frac{1}{v-\bar s})\mathbf{1}_{|u-v|>\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dudv\\ &-\frac{\gamma^2}{8i}\int_{I_\epsilon^2}\sin(\frac{\gamma^2}{4}\arg(u-s))\cos(\frac{\gamma^2}{4}\arg(v-s))(\frac{1}{u-s}-\frac{1}{u-\bar s})(\frac{1}{v-s}+\frac{1}{v-\bar s})\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon\mathbf{1}_{|u-v|>\delta}dudv\\ &-\frac{\gamma^2}{8i}\int_{I_\epsilon^2}\sin(\frac{\gamma^2}{4}\arg(u-s))\cos(\frac{\gamma^2}{4}\arg(v-s))(\frac{1}{u-s}-\frac{1}{u-\bar s})(\frac{1}{v-s}+\frac{1}{v-\bar s})\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon\mathbf{1}_{|u-v|<\delta}dudv\\ &-\frac{\gamma^2}{16}\int_{I_\epsilon^2}\sin(\frac{\gamma^2}{4}\arg(u-s))\sin(\frac{\gamma^2}{4}\arg(v-s))(\frac{1}{u-s}-\frac{1}{u-\bar s})(\frac{1}{v-s}-\frac{1}{v-\bar s})\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv.\end{aligned}$$

Now we also need to add the two missing terms which do not involve derivatives of f

$$\begin{aligned}&-\frac{\gamma^2}{16}\int_{\mathbb{R}^2}(\frac{1}{u-s}-\frac{1}{u-\bar s})(\frac{1}{v-s}-\frac{1}{v-\bar s})\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon d\mu^s_{\partial,\epsilon}(u)d\mu^s_{\partial,\epsilon}(v)\\ &+\frac{\gamma^2}{8}\int_{\mathbb{R}^2}(\frac{1}{(u-s)(v-s)}+\frac{1}{(u-\bar s)(v-\bar s)})\mathbf{1}_{|u-v|<\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu^s_{\partial,\epsilon}(u)d\mu^s_{\partial,\epsilon}(v),\end{aligned}\tag{3.41}$$

and

$$\mu\frac{\gamma^2}{2}\int_{\mathbb{H}_\delta}\frac{s-\bar s}{|y-s|^2}\frac{1}{(\bar y-y)}\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y+\delta^{-\frac{\gamma^2}{2}}\frac{\mu}{2i}\int_{\mathbb{R}+i\delta}(\frac{1}{y-s}-\frac{1}{\bar y-\bar s})\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dy.\tag{3.42}$$

Lets try to simplify all the terms above. First combing the last term of (3.40) with the first term of (3.41):

(3.43)

$$
\begin{aligned}
&-\frac{\gamma^2}{16}\int_{I_\epsilon^2}\cos(\frac{\gamma^2}{4}\arg(u-s)-\frac{\gamma^2}{4}\arg(v-s))(\frac{1}{u-s}-\frac{1}{u-\bar{s}})(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv\\
&=-\frac{\gamma^2}{16}\int_{I_\epsilon^2}e^{i\frac{\gamma^2}{4}\arg(u-s)-i\frac{\gamma^2}{4}\arg(v-s)}(\frac{1}{u-s}-\frac{1}{u-\bar{s}})(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\frac{|u-s|^{\frac{\gamma^2}{4}}|v-s|^{\frac{\gamma^2}{4}}}{|u-v|_\delta^{\frac{\gamma^2}{2}}}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv\\
&=-\frac{\gamma^2}{16}\int_{I_\epsilon^2}e^{i\frac{\gamma^2}{4}\arg(u-s)-i\frac{\gamma^2}{4}\arg(v-s)}(\frac{1}{u-s}-\frac{1}{u-\bar{s}})(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\frac{|u-s|^{\frac{\gamma^2}{4}}|v-s|^{\frac{\gamma^2}{4}}}{|u-v|_\delta^{\frac{\gamma^2}{2}}}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon\mathbf{1}_{|u-v|<\delta}dudv\\
&-\frac{\gamma^2}{16}\int_{I_\epsilon^2}e^{i\frac{\gamma^2}{4}\arg(u-s)-i\frac{\gamma^2}{4}\arg(v-s)}(\frac{1}{u-s}-\frac{1}{u-\bar{s}})(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\frac{|u-s|^{\frac{\gamma^2}{4}}|v-s|^{\frac{\gamma^2}{4}}}{|u-v|_\delta^{\frac{\gamma^2}{2}}}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon\mathbf{1}_{|u-v|>\delta}dudv.
\end{aligned}
$$

Next we recombine the first terms of (3.40) with the second term of (3.40) with the indicator $\mathbf{1}_{|u-v|>\delta}$ added:

$$
\begin{aligned}
&\frac{\gamma^2}{4i}\int_{I_\epsilon^2}\cos(\frac{\gamma^2}{4}\arg(u-s))\sin(\frac{\gamma^2}{4}\arg(v-s))\frac{1}{(u-v)}(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\mathbf{1}_{|u-v|>\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dudv\\
&-\frac{\gamma^2}{8i}\int_{I_\epsilon^2}\sin(\frac{\gamma^2}{4}\arg(u-s))\cos(\frac{\gamma^2}{4}\arg(v-s))(\frac{1}{u-s}-\frac{1}{u-\bar{s}})(\frac{1}{v-s}+\frac{1}{v-\bar{s}})\mathbf{1}_{|u-v|>\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv\\
&=-\frac{\gamma^2}{8}\int_{I_\epsilon^2}e^{\frac{\gamma^2}{4}\arg(v-s)-\frac{\gamma^2}{4}\arg(u-s)}\frac{s-\bar{s}}{(u-v)(v-s)(u-\bar{s})}\mathbf{1}_{|u-v|>\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv\\
&+\frac{\gamma^2}{16}\int_{I_\epsilon^2}e^{i\frac{\gamma^2}{4}\arg(v-s)-i\frac{\gamma^2}{4}\arg(u-s)}\frac{(s-\bar{s})^2}{|u-s|^2|v-s|^2}\mathbf{1}_{|u-v|>\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv.
\end{aligned}
$$

The second term in the last line above cancels with the similar term found in (3.43).

Next we regroup the terms all containing the indicator $\mathbf{1}_{|u-v|<\delta}$ coming from the second line of (3.40), the last line of (3.41), and the last line of (3.43):

$$
\begin{aligned}
&-\frac{\gamma^2}{8i}\int_{I_\epsilon^2}\sin(\frac{\gamma^2}{4}\arg(u-s))\cos(\frac{\gamma^2}{4}\arg(v-s))(\frac{1}{u-s}-\frac{1}{u-\bar{s}})(\frac{1}{v-s}+\frac{1}{v-\bar{s}})\mathbf{1}_{|u-v|<\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv\\
&+\frac{\gamma^2}{8}\int_{\mathbb{R}^2}\cos(\frac{\gamma^2}{4}\arg(u-s))\cos(\frac{\gamma^2}{4}\arg(v-s))(\frac{1}{(u-s)(v-s)}+\frac{1}{(u-\bar{s})(v-\bar{s})})\mathbf{1}_{|u-v|<\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d\mu_{\partial,\epsilon}^s(u)d\mu_{\partial,}^s\\
&-\frac{\gamma^2}{16}\int_{I_\epsilon^2}e^{i\frac{\gamma^2}{4}\arg(u-s)-i\frac{\gamma^2}{4}\arg(v-s)}(\frac{1}{u-s}-\frac{1}{u-\bar{s}})(\frac{1}{v-s}-\frac{1}{v-\bar{s}})\mathbf{1}_{|u-v|<\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv\\
&=\frac{\gamma^2}{8}\int_{I_\epsilon^2}e^{i\frac{\gamma^2}{4}\arg(u-s)+i\frac{\gamma^2}{4}\arg(v-\bar{s})}\frac{1}{(v-\bar{s})(u-s)}\mathbf{1}_{|u-v|<\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv\\
&+\frac{\gamma^2}{16}\int_{I_\epsilon^2}e^{i\frac{\gamma^2}{4}\arg(u-s)+i\frac{\gamma^2}{4}\arg(v-s)}\frac{1}{(u-s)(v-s)}\mathbf{1}_{|u-v|<\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv\\
&+\frac{\gamma^2}{16}\int_{I_\epsilon^2}e^{i\frac{\gamma^2}{4}\arg(u-\bar{s})+i\frac{\gamma^2}{4}\arg(v-\bar{s})}\frac{1}{(u-\bar{s})(v-\bar{s})}\mathbf{1}_{|u-v|<\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv\\
&=\frac{\gamma^2}{8}\int_{I_\epsilon^2}e^{i\frac{\gamma^2}{4}\arg(u-s)+i\frac{\gamma^2}{4}\arg(v-\bar{s})}\frac{1}{(v-\bar{s})(u-s)}\mathbf{1}_{|u-v|<\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv.
\end{aligned}
$$

To sum up, we are left with the last line of the above express, together with

$$
-\frac{\gamma^2}{8}\int_{I_\epsilon^2}e^{\frac{\gamma^2}{4}\arg(v-s)-\frac{\gamma^2}{4}\arg(u-s)}\frac{s-\bar{s}}{(u-v)(v-s)(u-\bar{s})}\mathbf{1}_{|u-v|>\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv
$$

and the two terms of (3.42). To analyse these final four terms, we have to divide into two cases based on the value of γ, namely $\gamma\in(0,\sqrt{2})$ or $\gamma\in(\sqrt{2},2)$. The case $\gamma=\sqrt{2}$ can be easily recovered at the end by continuity.

3.4. The case $\chi=\frac{\gamma}{2}$, $\gamma\in(0,\sqrt{2})$.

To summarize the list of terms we need to cancel is the following:

$$
\begin{aligned}
&-\frac{\gamma^2}{8}\int_{I_\epsilon^2}e^{\frac{\gamma^2}{4}\arg(v-s)-\frac{\gamma^2}{4}\arg(u-s)}\frac{s-\bar{s}}{(u-v)(v-s)(u-\bar{s})}\mathbf{1}_{|u-v|>\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv\\
&+\frac{\gamma^2}{8}\int_{I_\epsilon^2}e^{i\frac{\gamma^2}{4}\arg(u-s)+i\frac{\gamma^2}{4}\arg(v-\bar{s})}\frac{1}{(v-\bar{s})(u-s)}\mathbf{1}_{|u-v|<\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv\\
&+\mu\frac{\gamma^2}{2}\int_{\mathbb{H}_\delta}\frac{s-\bar{s}}{|y-s|^2}\frac{1}{(\bar{y}-y)}\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y+\delta^{-\frac{\gamma^2}{2}}\frac{\mu}{2i}\int_{\mathbb{R}+i\delta}(\frac{1}{y-s}-\frac{1}{\bar{y}-\bar{s}})\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dy.
\end{aligned}
$$

In this case we start with the following localization lemma.

Lemma 3.9. *For $\gamma<\sqrt{2}$, the following localization holds,*

$$
\lim_{\delta\to0}\langle s,s+\delta u,s+\delta v|\mathbf{z};\mathbf{s}\rangle_\epsilon=\lim_{\delta\to0}\langle s,s+\delta y,|\mathbf{z};\mathbf{s}\rangle_\epsilon=\langle s,s,s|\mathbf{z};\mathbf{s}\rangle_\epsilon.
$$

Next we give the following result.

Lemma 3.10. *The following identity holds, when $\gamma < \sqrt{2}$:*

$$
\begin{aligned}
&-\frac{\gamma^2}{8}\int_{I_\epsilon^2} e^{\frac{\gamma^2}{4}\arg(v-s)-\frac{\gamma^2}{4}\arg(u-s)}\frac{s-\bar s}{(u-v)(v-s)(u-\bar s)}\mathbf{1}_{|u-v|>\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv\\
&+\mu\frac{\gamma^2}{2}\int_{\mathbb{H}_\delta}\frac{s-\bar s}{|y-s|^2}\frac{1}{(\bar y-y)}\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}d^2y\\
&=\frac{\mu\gamma^2}{4\sin(\pi\frac{\gamma^2}{4})}\Bigg[e^{i\pi\frac{\gamma^2}{4}}\int_{\mathbb{R}}du_1\int_0^{\frac{\pi}{2}}d\theta e^{-i\frac{\gamma^2}{2}\theta}(u_1+\frac{1}{2}ie^{i\theta}+i)^{\frac{\gamma^2}{4}-1}(u_1-\frac{1}{2}ie^{i\theta}-i)^{\frac{\gamma^2}{4}-1}\\
&-e^{-i\pi\frac{\gamma^2}{4}}\int_{\mathbb{R}}du_1\int_0^{-\frac{\pi}{2}}d\theta e^{-i\frac{\gamma^2}{2}\theta}(u_1+\frac{1}{2}ie^{i\theta}+i)^{\frac{\gamma^2}{4}-1}(u_1-\frac{1}{2}ie^{i\theta}-i)^{\frac{\gamma^2}{4}-1}\Bigg]\langle s,s,s|\mathbf{z};\mathbf{s}\rangle+o_\delta(1).
\end{aligned}
$$

The other term in (3.42) can be rewritten

$$
\begin{aligned}
&\delta^{-\frac{\gamma^2}{2}}\frac{\mu}{2i}\int_{\mathbb{R}+i\frac{\delta}{2}}(\frac{1}{y-s}-\frac{1}{\bar y-\bar s})\langle s,y|\mathbf{z};\mathbf{s}\rangle_{\epsilon,\delta}dy\\
&=-\frac{3\mu}{2}\int_{\mathbb{R}+\frac{i}{2}}dy(y+i)^{\frac{\gamma^2}{4}-1}(\bar y-i)^{\frac{\gamma^2}{4}-1}\langle s,s|\mathbf{z};\mathbf{s}\rangle+o_\delta(1)\\
&=-\frac{3\mu}{2}\int_{\mathbb{R}}dy(y^2+\frac{9}{4})^{\frac{\gamma^2}{4}-1}\langle s,s|\mathbf{z};\mathbf{s}\rangle+o_\delta(1),
\end{aligned}
$$

and also we have left

$$
\frac{\mu}{\sin(\pi\frac{\gamma^2}{4})}\frac{\gamma^2}{8}\int_{I_\epsilon^2}e^{i\frac{\gamma^2}{4}\arg(u-s)+i\frac{\gamma^2}{4}\arg(v-\bar s)}\frac{1}{(v-\bar s)(u-s)}\mathbf{1}_{|u-v|<\delta}\langle s,u,v|\mathbf{z};\mathbf{s}\rangle_\epsilon dudv,
$$

which equals as δ goes to 0:

$$
\frac{\mu}{2\sin(\pi\frac{\gamma^2}{4})}\int_{\mathbb{R}}du(u+i)^{\frac{\gamma^2}{4}-1}\left((u+1-i)^{\frac{\gamma^2}{4}}-(u-1-i)^{\frac{\gamma^2}{4}}\right)\langle s,s,s|\mathbf{z};\mathbf{s}\rangle_\epsilon+o_\delta(1).
$$

The following lemma then completes the proof in the case where $\gamma<\sqrt{2}$.

Lemma 3.11. *For $\gamma<\sqrt{2}$, the following identity holds:*

$$
\begin{aligned}
&\frac{\mu}{2\sin(\pi\frac{\gamma^2}{4})}\int_{\mathbb{R}}du(u+i)^{\frac{\gamma^2}{4}-1}\left((u+1-i)^{\frac{\gamma^2}{4}}-(u-1-i)^{\frac{\gamma^2}{4}}\right)\\
&+\frac{\mu\gamma^2}{4\sin(\pi\frac{\gamma^2}{4})}\Bigg[e^{i\pi\frac{\gamma^2}{4}}\int_{\mathbb{R}}du_1\int_0^{\frac{\pi}{2}}d\theta e^{-i\frac{\gamma^2}{2}\theta}(u_1+\frac{1}{2}ie^{i\theta}+i)^{\frac{\gamma^2}{4}-1}(u_1-\frac{1}{2}ie^{i\theta}-i)^{\frac{\gamma^2}{4}-1}\\
&-e^{-i\pi\frac{\gamma^2}{4}}\int_{\mathbb{R}}du_1\int_0^{-\frac{\pi}{2}}d\theta e^{-i\frac{\gamma^2}{2}\theta}(u_1+\frac{1}{2}ie^{i\theta}+i)^{\frac{\gamma^2}{4}-1}(u_1-\frac{1}{2}ie^{i\theta}-i)^{\frac{\gamma^2}{4}-1}\Bigg]\\
&-\frac{3\mu}{2}\int_{\mathbb{R}}dy(y^2+\frac{9}{4})^{\frac{\gamma^2}{4}-1}\\
&=0.
\end{aligned}
$$

The proof of this lemma will be carried out in Appendix F.

4. Shift equations for H and R

In this section we derive a set of functional equations known as the shift equations on the functions H and R that will then completely specify their value. The main input to derive these shift equations is the BPZ equation from the previous Section 3. We introduce two global notations: $q = \frac{2Q-\beta_1-\beta_2-\beta_3+\chi}{\gamma}$ and

$$g_\chi(\sigma) = \left(\sin(\frac{\pi\gamma^2}{4})\right)^{-\chi/\gamma} \cos\left(2\pi\chi(\sigma - \frac{Q}{2})\right). \tag{4.1}$$

Theorem 4.1 (Shift equations for H)**.** *Let $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$ and fix $\sigma_3 \in [-\frac{1}{2\gamma} + \frac{Q}{2}, \frac{1}{2\gamma} + \frac{Q}{2} - \frac{\gamma}{4}] \times \mathbb{R}$. The function $H^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}$ can be jointly meromorphically extended to a complex neighborhood of $\mathbb{R}^3$ in $(\beta_1,\beta_2,\beta_3)$ and to $\mathbb{C}^2$ in (σ_1,σ_2). It obeys*

$$\begin{aligned} H^{(\beta_1,\beta_2-\chi,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)} &= \frac{\Gamma(\chi(\beta_1-\chi))\Gamma(1-\chi\beta_2+\chi^2)}{\Gamma(\chi(\beta_1-\chi+q\frac{\gamma}{2}))\Gamma(1-\chi\beta_2+\chi^2-q\frac{\gamma\chi}{2})} H^{(\beta_1-\chi,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2+\frac{\chi}{2},\sigma_3)} \\ &+ \frac{\chi^2\pi^{\frac{2\chi}{\gamma}}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}} \frac{\Gamma(1-\chi\beta_1)\Gamma(1-\chi\beta_2+\chi^2)\left(g_\chi(\sigma_1)-g_\chi(\sigma_2+\frac{\beta_1}{2})\right)}{\sin(\pi\chi(\chi-\beta_1))\Gamma(1+\frac{q\gamma\chi}{2})\Gamma(2-\chi(\beta_1+\beta_2-2\chi+q\frac{\gamma}{2}))} H^{(\beta_1+\chi,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2+\frac{\chi}{2},\sigma_3)}, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &\frac{\chi^2\pi^{\frac{2\chi}{\gamma}-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}}\Gamma(1-\chi\beta_2)\left(g_\chi(\sigma_3)-g_\chi(\sigma_2+\frac{\beta_2}{2})\right) H^{(\beta_1,\beta_2+\chi,\beta_3)}_{(\sigma_1,\sigma_2+\frac{\chi}{2},\sigma_3)} \\ &= \frac{\Gamma(\chi(\beta_1-\chi))}{\Gamma(-q\frac{\gamma\chi}{2})\Gamma(-1+\chi(\beta_1+\beta_2-2\chi+q\frac{\gamma}{2}))} H^{(\beta_1-\chi,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)} \\ &+ \frac{\chi^2\pi^{\frac{2\chi}{\gamma}}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}} \frac{\left(g_\chi(\sigma_1)-g_\chi(\sigma_2-\frac{\beta_1}{2}+\frac{\chi}{2})\right)\Gamma(1-\chi\beta_1)}{\sin(\pi\chi(\chi-\beta_1))\Gamma(1-\chi(\beta_1-\chi+q\frac{\gamma}{2}))\Gamma(\chi\beta_2-\chi^2+q\frac{\gamma\chi}{2})} H^{(\beta_1+\chi,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}. \end{aligned} \tag{4.3}$$

Theorem 4.2 (Shift equations for R)**.** *Let $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$. The function $R(\beta,\sigma_1,\sigma_2)$ can be jointly meromorphically extended to a complex neighborhood of $\mathbb{R}$ in β and to $\mathbb{C}^2$ in (σ_1,σ_2). It obeys*

$$\frac{R(\beta,\sigma_1,\sigma_2)}{R(\beta+\chi,\sigma_1-\frac{\chi}{2},\sigma_2)} = c_\chi(\gamma)\Gamma(-1+\chi\beta-\chi^2)\Gamma(1-\chi\beta)\left(g_\chi(\sigma_2)-g_\chi(\sigma_1-\frac{\beta}{2})\right), \tag{4.4}$$

$$\frac{R(\beta,\sigma_1,\sigma_2)}{R(\beta+\chi,\sigma_1+\frac{\chi}{2},\sigma_2)} = c_\chi(\gamma)\Gamma(-1+\chi\beta-\chi^2)\Gamma(1-\chi\beta)\left(g_\chi(\sigma_2)-g_\chi(\sigma_1+\frac{\beta}{2})\right), \tag{4.5}$$

where $c_{\frac{\gamma}{2}}(\gamma) = -\frac{1}{\Gamma(-\frac{\gamma^2}{4})}$ and $c_{\frac{2}{\gamma}}(\gamma)$ is an unknown function of γ.

Theorem 4.2 yields the exact formula for R claimed in Theorem 1.2.

Proof of Theorem 1.2 given Theorem 4.2. By combining both shift equations of Theorem 4.2 we obtain

$$\begin{aligned} \frac{R(\beta_1,\sigma_1,\sigma_2)}{R(\beta_1+2\chi,\sigma_1,\sigma_2)} &= \tilde{c}_\chi(\gamma)\Gamma(-1+\chi\beta_1-\chi^2)\Gamma(1-\chi\beta_1-\chi^2)\Gamma(1-\chi\beta_1)\Gamma(-1+\chi\beta_1) \\ &\times 4\sin(\pi\chi(\frac{\beta}{2}-\sigma_1-\sigma_2+Q))\sin(\pi\chi(\frac{\beta}{2}+\sigma_1+\sigma_2-Q))\sin(\pi\chi(\frac{\beta}{2}+\sigma_2-\sigma_1))\sin(\pi\chi(\frac{\beta}{2}+\sigma_1-\sigma_2)), \end{aligned} \tag{4.6}$$

where $\tilde{c}_{\frac{\gamma}{2}}(\gamma) = \frac{1}{\sin(\pi\frac{\gamma^2}{4})\Gamma(-\frac{\gamma^2}{4})^2}$ and where again $\tilde{c}_{\frac{2}{\gamma}}(\gamma)$ is an unknown function of γ.

Consider now the ratio $f(\beta) = \frac{R(\beta)}{R_{\mathrm{FZZ}}(\beta)}$. By using the shift equations (A.10), (A.11), (A.13) satisfied by $\Gamma_{\frac{\gamma}{2}}$ and $S_{\frac{\gamma}{2}}$, one can show that $R_{\mathrm{FZZ}}(\beta)$ satisfies the same shift equations satisfied by R, except that the constant $\tilde{c}_{\frac{2}{\gamma}}$ is now an explicit function of γ. From this we obtain that

$$f(\beta+\gamma) = f(\beta), \quad f(\beta+\frac{4}{\gamma}) = c(\gamma)f(\beta),$$

for an unknown function $c(\gamma)$ of γ. It is actually possible to see that $c(\gamma) = 1$. Indeed, when $\sigma_1, \sigma_2 \in \mathcal{B}$ and $\beta \in (\frac{2}{\gamma}, Q)$, by using (2.12) one can see that $\hat{R}_{\mu_1,\mu_2}(\beta)$ is a positive function. Since for $\beta \in (\frac{2}{\gamma}, Q)$, $s = \beta - Q < 0$ and $s + \frac{\gamma}{2} > 0$, equation (2.13) implies that $R(\beta, \sigma_1, \sigma_2) = R_{\mu_1,\mu_2}(\beta)$ is then negative for $\beta \in (\frac{2}{\gamma}, Q)$. This interval has length $\frac{\gamma}{2}$, and by using the reflection identity (4.29) it extends to an interval of length γ. The same claim can be checked on R_{FZZ} directly on the exact formula. Therefore the function f is positive on an interval of length γ, which by using the γ-periodicity implies that it is positive on $\mathbb{R}$. This in turn implies that the unknown constant $c(\gamma)$ is real and positive. Then for $\gamma^2 \notin \mathbb{Q}$, since any real number can be approximated by a linear combination $n\frac{\gamma}{2} - m\frac{2}{\gamma}$ for $n, m \in \mathbb{N}$ arbitrary large, this implies that $c(\gamma) = 1$ and that f is a constant function of β. One can then show $f(\beta) = 1$ by using the fact that R and R_{FZZ} are known to match at $\beta = Q$. Lastly the case $\gamma^2 \in \mathbb{Q}$ is easily recovered by a simple continuity argument. □

Proof of Theorem 1.1 given Theorem 4.1. It is less straightforward to see than in the case of R that the shift equations on H completely determine its value, since they contain three terms instead of two. We will use the following result proved in [RZ21, Section 3.3.3]: the solution space of the system comprised of the two shift equations (4.2), (4.3) combined with the reflection identity $H^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)} = R(\beta_1,\sigma_1,\sigma_2)H^{(2Q-\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}$ proved in Lemma 4.9 is at most one dimensional. Note that in our case since we are working with both bulk and boundary Liouville potentials the coefficients in the shift equations (4.2), (4.3) are different than the ones in [RZ21], but the result of [RZ21] still applies since the proof does not rely on the precise expression of these functions.

Now we already know H satisfies the system comprised of (4.2), (4.3) and (4.28). By the results of Lemmas A.4 and A.5 proved in Appendix A.3, we also know that H_{PT} satisfies the same three equations. Therefore the result of [RZ21] implies H and H_{PT} are a constant multiple of one another. To pin down this multiplicative constant, we will simply look at the pole of both functions at $\beta_1 = 2Q - \beta_2 - \beta_3$. It is easy to establish the following fact about H:

$$\lim_{\beta_1 \to 2Q-\beta_2-\beta_3} (\frac{\bar{\beta}}{2} - Q) H \begin{pmatrix} \beta_1, \beta_2, \beta_3 \\ \sigma_1, \sigma_2, \sigma_3 \end{pmatrix} = 1. \tag{4.7}$$

This can be seen directly on the expression of H given in Lemma 2.5. In the notations of Lemma 2.5, computing the residue of interest corresponds to multiplying H by s and then setting $s = 0$. By Lemma A.5, H_{PT} has the same residue at that point. This completes the proof of Theorem 1.1. □

We now turn to the proof of Theorems 4.1 and 4.2 which is based on the BPZ equations with the so-called operator product expansions (OPE). This type of argument has been carried out for $\mu = 0$ case in [RZ21]. The same shift equations as above are derived there except that the function g_χ is replaced by $e^{2\chi\pi i(\sigma - \frac{Q}{2})}$. We highlight two places where additional work is required. The first is contrarily to [RZ21] where H and R can reduce up to a prefactor to moments of GMC (denoted by $\overline{H}$ and $\overline{R}$ in [RZ21]), this is not possible in our case. We are forced to work with the truncation of Definitions 2.6 and 2.11 which introduces extra constraints on the parameters. The second is that at one step in the derivation of [RZ21] the value of $R(\beta,\sigma_1,\sigma_2)$ at $\mu_1=0$ is used, which was known from [RZ20b]. For us this is not available. Instead we use the mating of trees input to get an explicit formula for $R(\gamma,\sigma_1,\sigma)$ See Section 4.5.

The rest of this section is organized as follows. In Section 4.1 we introduce the specialization of the BPZ equation of Theorem 1.3 that we will need for our derivation. In Sections 4.2 and 4.3 we establish respectively Theorems 4.1 and 4.2 in the case $\chi=\frac{\gamma}{2}$ and in a limited range of parameters. In Section 4.4 we prove the reflection principle for H and R and show they can be meromorphically extended to the full range of parameters claimed in Theorems 4.1 and 4.2. Then finally in Sections 4.5 and 4.6 we establish Theorems 4.2 and 4.1 in the case $\chi=\frac{2}{\gamma}$.

4.1. **A deformation of H.** Our starting input is the BPZ equation given by Theorem 1.3 of Section 3. For the purpose of proving our main theorem we will need the following special case of the boundary BPZ equation, which corresponds to having three spectator boundary insertions along with the boundary degenerate insertion. By conformal invariance we can place the boundary spectator insertions at $0,1,\infty$. The degenerate insertion is then at a point t on the real line, and we will assume $t\in(-\infty,1)$. Recall $\mathcal{B}=(-\frac{1}{2\gamma}+\frac{Q}{2},\frac{1}{2\gamma}+\frac{Q}{2})\times\mathbb{R}$ and $\overline{\mathcal{B}}=[-\frac{1}{2\gamma}+\frac{Q}{2},\frac{1}{2\gamma}+\frac{Q}{2}]\times\mathbb{R}$. For $\chi=\frac{\gamma}{2}$ or $\frac{2}{\gamma}$, consider the following range of parameters:

$$\beta_i\in(-\infty,Q),\ \beta_1+\beta_2+\beta_3>2Q+\chi,\quad \sigma_1-\frac{\chi}{2}\in\overline{\mathcal{B}},\ \sigma_2-\frac{\chi}{2}\in\overline{\mathcal{B}},\ \sigma_1\in\overline{\mathcal{B}},\ \sigma_2\in\overline{\mathcal{B}},\ \sigma_3\in\overline{\mathcal{B}}. \tag{4.8}$$

Notice these constraints can all be simultaneously satisfied. Indeed, if all three β_i are close to Q then $\beta_1+\beta_2+\beta_3>2Q+\chi$ is true. For the σ_i, the band with its boundary included $\overline{\mathcal{B}}$ has width $\frac{1}{\gamma}$. Therefore it is possible to have both $\sigma_1-\frac{\chi}{2}$ and σ_1 in $\overline{\mathcal{B}}$, but in the case $\chi=\frac{2}{\gamma}$ this means that $\sigma_1-\frac{\chi}{2}$ and σ_1 are exactly on the boundary of $\overline{\mathcal{B}}$. The same is true for σ_2. Now under the constraint of (4.8) consider the functional

$$\begin{aligned}H_\chi(t)=\int_{\mathbb{R}}dc\,e^{-\frac{\gamma qc}{2}}\mathbb{E}\Bigg[\exp\Bigg(&-e^{\gamma c}\int_{\mathbb{H}}\frac{|x-t|^{\gamma\chi}g(x)^{\frac{\gamma^2}{4}(q-1)}/|x-\bar{x}|^{\frac{\gamma^2}{2}}}{|x|^{\gamma\beta_1}|x-1|^{\gamma\beta_2}}e^{\gamma X(x)}d^2x\\&-e^{\frac{\gamma c}{2}}\int_{\mathbb{R}}\frac{|r-t|^{\frac{\gamma\chi}{2}}g(r)^{\frac{\gamma^2}{8}(q-1)}}{|r|^{\frac{\gamma\beta_1}{2}}|r-1|^{\frac{\gamma\beta_2}{2}}}e^{\frac{\gamma}{2}X(r)}d\mu_B^t(r)\Bigg)\Bigg],\end{aligned}$$

where the boundary measure $d\mu_B^t(r)$ is defined by:

$$d\mu_B^t(r)/dr=\begin{cases}g_{\frac{\gamma}{2}}(\sigma_1-\frac{\chi}{2})\mathbf{1}_{r<0}+g_{\frac{\gamma}{2}}(\sigma_2-\frac{\chi}{2})\mathbf{1}_{0<r<t}+g_{\frac{\gamma}{2}}(\sigma_2)\mathbf{1}_{t<r<1}+g_{\frac{\gamma}{2}}(\sigma_3)\mathbf{1}_{r>1}, & \text{for } t\in(0,1),\\ g_{\frac{\gamma}{2}}(\sigma_1-\frac{\chi}{2})\mathbf{1}_{r<t}+g_{\frac{\gamma}{2}}(\sigma_1)\mathbf{1}_{t<r<0}+g_{\frac{\gamma}{2}}(\sigma_2)\mathbf{1}_{0<r<1}+g_{\frac{\gamma}{2}}(\sigma_3)\mathbf{1}_{r>1}, & \text{for } t<0.\end{cases}$$

Since the change of σ on the left and right or t can be either $+\frac{\chi}{2}$ or $-\frac{\chi}{2}$, we similarly consider the function $\tilde{H}_\chi(t)$ with the other choice, which is given by the same expression as $H_\chi(t)$ except one needs to replace $d\mu_B^t(r)$ by the following measure

$$d\tilde{\mu}_B^t(r)/dr=\begin{cases}g_{\frac{\gamma}{2}}(\sigma_1+\frac{\chi}{2})\mathbf{1}_{r<0}+g_{\frac{\gamma}{2}}(\sigma_2+\frac{\chi}{2})\mathbf{1}_{0<r<t}+g_{\frac{\gamma}{2}}(\sigma_2)\mathbf{1}_{t<r<1}+g_{\frac{\gamma}{2}}(\sigma_3)\mathbf{1}_{r>1}, & \text{for } t\in(0,1),\\ g_{\frac{\gamma}{2}}(\sigma_1+\frac{\chi}{2})\mathbf{1}_{r<t}+g_{\frac{\gamma}{2}}(\sigma_1)\mathbf{1}_{t<r<0}+g_{\frac{\gamma}{2}}(\sigma_2)\mathbf{1}_{0<r<1}+g_{\frac{\gamma}{2}}(\sigma_3)\mathbf{1}_{r>1}, & \text{for } t<0,\end{cases}$$

under this time the parameter contraints:

$$\beta_i\in(-\infty,Q),\ \beta_1+\beta_2+\beta_3>2Q+\chi,\quad \sigma_1+\frac{\chi}{2}\in\overline{\mathcal{B}},\ \sigma_2+\frac{\chi}{2}\in\overline{\mathcal{B}},\ \sigma_1\in\overline{\mathcal{B}},\ \sigma_2\in\overline{\mathcal{B}},\ \sigma_3\in\overline{\mathcal{B}}. \tag{4.9}$$

We could also of course define in a similar manner $H_\chi(t)$, $\tilde{H}_\chi(t)$ for $t\in(1,\infty)$, but this will not be needed. By a direct change of variable applied to the differential operator of Theorem 1.3, $H_\chi(t)$ and $\tilde{H}_\chi(t)$ obey the hypergeometric equation. We state this in the next proposition.

Proposition 4.3. *Under the parameter contraint of (4.8), the function H_χ obeys*

$$t(1-t)\partial_t^2 H_\chi + (C-(A+B+1)t)\partial_t H_\chi - ABH_\chi = 0, \tag{4.10}$$

with A, B, C given by:

$$A = -q\frac{\gamma\chi}{2}, \quad B = -1 + \chi(\beta_1+\beta_2-2\chi+q\frac{\gamma}{2}), \quad C = \chi(\beta_1-\chi). \tag{4.11}$$

The exact same result holds for $\tilde{H}_\chi$ under the parameter constraints (4.9).

We now state an analyticity result for $H_{\frac{\gamma}{2}}$ and $\tilde{H}_{\frac{\gamma}{2}}$ which will be used in the proof of Lemma 4.9 below.

Lemma 4.4. *Set $\chi = \frac{\gamma}{2}$. Fix the σ_i in the parameter range (4.8) but with the boundary of the band being excluded, namely $\sigma_1 - \frac{\gamma}{4} \in \mathcal{B}$, $\sigma_2 - \frac{\gamma}{4} \in \mathcal{B}$, $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{B}$. Then the function H_χ is meromorphic in all three $\beta_1, \beta_2, \beta_3$ in a complex neighborhood of the subdomain of $\mathbb{R}^3$ given by the constraints $\beta_i \in (-\infty, Q)$, $\beta_1+\beta_2+\beta_3 > 2Q+\chi$. The same claim holds for $\tilde{H}_\chi$, except this time the σ_i need to obey $\sigma_1 + \frac{\gamma}{4} \in \mathcal{B}$, $\sigma_2 + \frac{\gamma}{4} \in \mathcal{B}$, $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{B}$.*

Proof. This result follows from a direct adaptation of the proof of the claim of analyticity in the β_i for H of Proposition 2.7, which is proved in Appendix B. The fact that we have an extra insertion on the boundary poses no additional problem. Here we are also in the range of parameters where the Seiberg bounds are satisfied and thus no truncation procedure is required. ☐

Lastly we recall here the range of parameters where the functions H and R are well-defined as probabilistic quantities as performed in Section 2. For the H function the range on the β_i and σ_i is given by:

$$\left\{(\beta_1,\beta_2,\beta_3,\sigma_1,\sigma_2,\sigma_3) \,:\, \beta_i < Q,\ Q - \frac{1}{2}\sum\beta_i < \gamma \wedge \frac{2}{\gamma} \wedge \min_i(Q-\beta_i),\ \sigma_i \in \overline{\mathcal{B}}\right\}. \tag{4.12}$$

For the R function the range of β and σ_i is given by:

$$\left\{(\beta,\sigma_1,\sigma_2) \,:\, \frac{\gamma}{2} \vee (\frac{2}{\gamma}-\frac{\gamma}{2}) < \beta < Q,\ \sigma_i \in \overline{\mathcal{B}}\right\}. \tag{4.13}$$

4.2. **The $\frac{\gamma}{2}$-shift equations for H.** We start by proving the shift equations on H in the case of $\chi = \frac{\gamma}{2}$. We therefore use the functions $H_{\frac{\gamma}{2}}(t)$ and $\tilde{H}_{\frac{\gamma}{2}}(t)$. For this first lemma the parameter range is chosen such that the β_i and σ_i parameters of each H in the shift equations belong to the domain (4.12).

Lemma 4.5. *Set $\chi = \frac{\gamma}{2}$. The shift equation (4.2) holds in the parameter range*

$$\beta_1 < \frac{2}{\gamma}, \quad \beta_2,\beta_3 < Q, \quad Q - \frac{1}{2}(\beta_1+\beta_2+\beta_3-\frac{\gamma}{2}) < \gamma \wedge \frac{2}{\gamma} \wedge \min_i(Q-\beta_i), \quad \sigma_1,\sigma_2,\sigma_3 \in \mathcal{B}, \quad \sigma_2 + \frac{\gamma}{4} \in \mathcal{B}, \tag{4.14}$$

and the shift equation (4.3) holds in parameter range:

$$\beta_1,\beta_2 < \frac{2}{\gamma}, \quad \beta_3 < Q, \quad Q - \frac{1}{2}(\beta_1+\beta_2+\beta_3-\frac{\gamma}{2}) < \gamma \wedge \frac{2}{\gamma} \wedge \min_i(Q-\beta_i), \quad \sigma_1,\sigma_2,\sigma_3 \in \mathcal{B}, \quad \sigma_2 + \frac{\gamma}{4} \in \mathcal{B}. \tag{4.15}$$

Proof. Here we are always assuming $\chi = \frac{\gamma}{2}$. Lets first take a look at the parameter ranges given by (4.14) and (4.15). For $\chi = \frac{\gamma}{2}$, the shift equation (4.2) contains the following three H functions:

$$H_{(\sigma_1,\sigma_2,\sigma_3)}^{(\beta_1,\beta_2-\frac{\gamma}{2},\beta_3)}, \quad H_{(\sigma_1,\sigma_2+\frac{\gamma}{4},\sigma_3)}^{(\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}, \quad H_{(\sigma_1,\sigma_2+\frac{\gamma}{4},\sigma_3)}^{(\beta_1+\frac{\gamma}{2},\beta_2,\beta_3)}.$$

The β_i and σ_i parameters of each H must be in the range (4.12). This gives (4.14). Similarly, the second shift equations (4.3) contains the H functions

$$H^{(\beta_1,\beta_2+\frac{\gamma}{2},\beta_3)}_{(\sigma_1,\sigma_2+\frac{\gamma}{4},\sigma_3)},\quad H^{(\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)},\quad H^{(\beta_1+\frac{\gamma}{2},\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)},$$

and the β_i and σ_i then need to obey the constraint (4.15). It is easy to check that these parameter constraints are non-empty. The condition of (4.14) is weaker than (4.15), and (4.15) can be satisfied if all the β_i parameters are chosen in the interval $(\frac{2}{\gamma}-\epsilon,\frac{2}{\gamma})$ for a small $\epsilon>0$.

Lets now derive the first shift equation (4.2). We assume β_i,σ_i obey (4.8) in order for $H_{\frac{\gamma}{2}}$ to be well-defined, plus the following extra constraint on β_1:

$$\frac{\gamma}{2}<\beta_1<\frac{2}{\gamma}. \tag{4.16}$$

By Proposition 4.3, the function $t\mapsto H_{\frac{\gamma}{2}}(t)$ obeys the hypergeometric equation for $t\in(0,1)$. Using the basis of solutions of the hypergeometric equation recalled in Section A.1, we can write the following solutions around $t=0$ and $t=1$, under the assumption that neither C, $C-A-B$, or $A-B$ are integers:[2] For $t\in(0,1)$:

(4.17)
$$\begin{aligned}H_{\frac{\gamma}{2}}(t)&=C_1F(A,B,C,t)+C_2^+t^{1-C}F(1+A-C,1+B-C,2-C,t)\\&=B_1F(A,B,1+A+B-C,1-t)+B_2^-(1-t)^{C-A-B}F(C-A,C-B,1+C-A-B,1-t).\end{aligned}$$

The constants C_1,C_2^+,B_1,B_2^- are the real constants that parametrize the solution space around $t=0$ and $t=1$, we will identify them by Taylor expansion. First we note that by setting $t=0$:

$$C_1=H_{\frac{\gamma}{2}}(0)=H^{(\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(\sigma_1-\frac{\gamma}{4},\sigma_2,\sigma_3)}. \tag{4.18}$$

Next to find C_2^+ we go at higher order in the $t\to 0_+$ limit. For this we use the asymptotic expansion of $H_{\frac{\gamma}{2}}(t)-H_{\frac{\gamma}{2}}(0)$ given by Lemma E.1. Under the parameter constraints given by (4.8) and (4.16), the lemma directly tells us that

$$H_{\frac{\gamma}{2}}(t)-H_{\frac{\gamma}{2}}(0)=C_2^+t^{1-C}+o(t^{1-C}), \tag{4.19}$$

where:

$$C_2^+=-\frac{\Gamma(-1+\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta_1}{2})}{\Gamma(-\frac{\gamma^2}{4})}\left(g_{\frac{\gamma}{2}}(\sigma_1-\frac{\gamma}{4})-g_{\frac{\gamma}{2}}(\sigma_2+\frac{\beta_1}{2}-\frac{\gamma}{4})\right)H^{(\beta_1+\frac{\gamma}{2},\beta_2,\beta_3)}_{(\sigma_1-\frac{\gamma}{4},\sigma_2,\sigma_3)}. \tag{4.20}$$

Similarly by setting $t=1$ we get:

$$B_1=H^{(\beta_1,\beta_2-\frac{\gamma}{2},\beta_3)}_{(\sigma_1-\frac{\gamma}{4},\sigma_2-\frac{\gamma}{4},\sigma_3)}. \tag{4.21}$$

The connection formula (A.8) between C_1, C_2^+, and B_1 then implies the shift equation (4.2) for $\chi=\frac{\gamma}{2}$ in the range of parameters constraint by (4.8) and (4.16), after performing furthermore the replacement $\sigma_1\to\sigma_1+\frac{\gamma}{4}$ and $\sigma_2\to\sigma_2+\frac{\gamma}{4}$ (which also rotates the domain where σ_1,σ_2 belongs). To lift these constraint we then invoque the analyticity of H as a function of its parameters given by Proposition 2.8. We have thus shown that (4.2) holds for $\chi=\frac{\gamma}{2}$ in the parameter range given by (4.14).

[2]The values excluded here are recovered by a simple continuity argument in γ.

Now we repeat these steps with $\tilde{H}_{\frac{\gamma}{2}}$ to obtain the shift equation with the opposite phase. We expand $\tilde{H}_{\frac{\gamma}{2}}(t)$, for $t \in (0,1)$

(4.22)

$$\begin{aligned}\tilde{H}_{\frac{\gamma}{2}}(t) &= \tilde{C}_1 F(A,B,C,t) + \tilde{C}_2^{+} t^{1-C} F(1+A-C, 1+B-C, 2-C, t)\\ &= \tilde{B}_1 F(A,B,1+A+B-C,1-t) + \tilde{B}_2^{-}(1-t)^{C-A-B} F(C-A, C-B, 1+C-A-B, 1-t)\end{aligned}$$

and compute in the same way the values of $\tilde{C}_1, \tilde{C}_2^{+}, \tilde{B}_2^{-}$:

$$\tilde{C}_1 = H^{(\beta_1 - \frac{\gamma}{2}, \beta_2, \beta_3)}_{(\sigma_1 + \frac{\gamma}{4}, \sigma_2, \sigma_3)}, \tag{4.23}$$

$$\tilde{C}_2^{+} = -\frac{\Gamma(-1+\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta_1}{2})}{\Gamma(-\frac{\gamma^2}{4})}\left(g_{\frac{\gamma}{2}}(\sigma_1+\frac{\gamma}{4}) - g_{\frac{\gamma}{2}}(\sigma_2 - \frac{\beta_1}{2} + \frac{\gamma}{4})\right) H^{(\beta_1 + \frac{\gamma}{2}, \beta_2, \beta_3)}_{(\sigma_1 + \frac{\gamma}{4}, \sigma_2, \sigma_3)}, \tag{4.24}$$

$$\tilde{B}_2^{-} = -\frac{\Gamma(-1+\frac{\gamma\beta_2}{2}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta_2}{2})}{\Gamma(-\frac{\gamma^2}{4})}\left(g_{\frac{\gamma}{2}}(\sigma_3) - g_{\frac{\gamma}{2}}(\sigma_2 + \frac{\beta_2}{2})\right) H^{(\beta_1, \beta_2 + \frac{\gamma}{2}, \beta_3)}_{(\sigma_1 + \frac{\gamma}{4}, \sigma_2 + \frac{\gamma}{4}, \sigma_3)} \tag{4.25}$$

Then the connection formula (A.8) implies the shift equation (4.3) for $\chi = \frac{\gamma}{2}$. □

Remark 4.6. *Notice that the two shift equations we derive using respectively H_χ and $\tilde{H}_\chi$ do not have exactly the same shape, the reason being that in the proof above we used the connection formula between the coefficients C_1, C_2^{+}, B_1 to derive (4.2) and we the connection formula between $\tilde{C}_1, \tilde{C}_2^{+}, \tilde{B}_2^{-}$ to derive (4.3). One may wonder why we did not use the connection formula between $\tilde{C}_1, \tilde{C}_2^{+}, \tilde{B}_1$ in the case of $\tilde{H}_\chi$ as well. The main reason is that with this choice it is possible to perform a certain linear combination of the shift equations (4.2) and (4.3) to obtain a shift equation of H shifting only the parameter β_1. This will be crucial to be able to argue a uniqueness statement for the solutions to (4.2) and (4.3), see the proof of Theorem 1.1 at the beginning of this section and* [RZ21].

4.3. **The $\frac{\gamma}{2}$-shift equations for R.** The next step is to derive the $\frac{\gamma}{2}$-shift equation for the reflection coefficient R. The key idea is to take a suitable limit of the $\frac{\gamma}{2}$-shift equations for H that we have established in Lemma 4.5 to make R appear from H. For this purpose we give the following result expressing R as a limit of H.

Lemma 4.7. *Suppose the β_i are in the range given by (4.12) and the σ_i in $\mathcal{B}$. Suppose $Q > \beta_1 > \beta_2 \vee \frac{\gamma}{2}$ and $\beta_1 - \beta_2 < \beta_3 < Q$. Then the following limit holds:*

$$\lim_{\beta_3 \downarrow \beta_1 - \beta_2} (\beta_2 + \beta_3 - \beta_1) H^{(\beta_1, \beta_2, \beta_3)}_{(\sigma_1, \sigma_2, \sigma_3)} = 2R(\beta_1, \sigma_1, \sigma_2).$$

The proof of this lemma is carried out in Appendix C. We are now ready to prove the $\frac{\gamma}{2}$-shift equation for R.

Lemma 4.8. *Consider $\beta_1 \in (\frac{\gamma}{2} \vee (\frac{2}{\gamma} - \frac{\gamma}{2}), \frac{2}{\gamma})$ and $\sigma_1, \sigma_2 \in \mathbb{C}$ such that $\sigma_1, \sigma_2, \sigma_2 + \frac{\gamma}{4}$ all belong to $\mathcal{B}$. Then $R(\beta, \sigma_1, \sigma_2)$ obeys*

(4.26)

$$R(\beta_1, \sigma_1, \sigma_2) = -\frac{\Gamma(-1+\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta_1}{2})}{\Gamma(-\frac{\gamma^2}{4})}\left(g_{\frac{\gamma}{2}}(\sigma_1) - g_{\frac{\gamma}{2}}(\sigma_2 + \frac{\beta_1}{2})\right) R(\beta_1 + \frac{\gamma}{2}, \sigma_1, \sigma_2 + \frac{\gamma}{4}).$$

Similarly for $\beta_1 \in (0 \vee (\frac{2}{\gamma} - \gamma), \frac{2}{\gamma} - \frac{\gamma}{2})$ *and the same constraint on* σ_1, σ_2 *as previously,*

(4.27)
$$R(\beta_1 + \frac{\gamma}{2}, \sigma_1, \sigma_2 + \frac{\gamma}{4}) = -\frac{\Gamma(-1 + \frac{\gamma\beta_1}{2})\Gamma(1 - \frac{\gamma\beta_1}{2} - \frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})} \left(g_{\frac{\gamma}{2}}(\sigma_1) - g_{\frac{\gamma}{2}}(\sigma_2 - \frac{\beta_1}{2}) \right) R(\beta_1 + \gamma, \sigma_1, \sigma_2).$$

Proof. Let us derive the first shift equation (4.26) which will follow from taking a limit of (4.2). Fix a $\beta_1 \in (\frac{\gamma}{2} \vee (\frac{2}{\gamma} - \frac{\gamma}{2}), \frac{2}{\gamma})$. Consider two parameters $\epsilon, \eta > 0$ chosen small enough and set $\beta_2 = \beta_1 - \epsilon$, $\beta_3 = \beta_1 - \beta_2 + \frac{\gamma}{2} + \eta = \frac{\gamma}{2} + \epsilon + \eta$. Notice that for this parameter choice the condition (4.14) required in Lemma 4.5 is satisfied. By applying Lemma 4.7 we get:

$$\lim_{\beta_3 \downarrow \beta_1 - \beta_2 + \frac{\gamma}{2}} (\beta_2 + \beta_3 - \beta_1 - \frac{\gamma}{2}) H_{(\sigma_1,\sigma_2,\sigma_3)}^{(\beta_1, \beta_2 - \frac{\gamma}{2}, \beta_3)} = 2R(\beta_1, \sigma_1, \sigma_2),$$

$$\lim_{\beta_3 \downarrow \beta_1 - \beta_2 + \frac{\gamma}{2}} (\beta_2 + \beta_3 - \beta_1 - \frac{\gamma}{2}) \frac{\Gamma(\frac{\gamma\beta_1}{2} - \frac{\gamma^2}{4})\Gamma(1 - \frac{\gamma\beta_2}{2} + \frac{\gamma^2}{4})}{\Gamma(\frac{\gamma\beta_1}{2} + (q-1)\frac{\gamma^2}{4})\Gamma(1 - \frac{\gamma}{2}\beta_2 - (q-1)\frac{\gamma^2}{4}))} H_{(\sigma_1,\sigma_2+\frac{\gamma}{4},\sigma_3)}^{(\beta_1 - \frac{\gamma}{2}, \beta_2, \beta_3)} = 0,$$

$$\lim_{\beta_3 \downarrow \beta_1 - \beta_2 + \frac{\gamma}{2}} (\beta_2 + \beta_3 - \beta_1 - \frac{\gamma}{2}) \Bigg[\frac{\Gamma(2 + \frac{\gamma^2}{4} - \frac{\gamma\beta_1}{2})\Gamma(1 - \frac{\gamma\beta_2}{2} + \frac{\gamma^2}{4})\Gamma(-1 + \frac{\gamma\beta_1}{2} - \frac{\gamma^2}{4})\Gamma(1 - \frac{\gamma\beta_1}{2})}{\Gamma(1 + \frac{\gamma^2}{4})\Gamma(2 - \frac{\gamma}{2}(\beta_1 + \beta_2) - (q-2)\frac{\gamma^2}{4}))\Gamma(-\frac{\gamma^2}{4})}$$
$$\times \left(g_{\frac{\gamma}{2}}(\sigma_1) - g_{\frac{\gamma}{2}}(\sigma_2 + \frac{\beta_1}{2}) \right) H_{(\sigma_1,\sigma_2+\frac{\gamma}{4},\sigma_3)}^{(\beta_1 + \frac{\gamma}{2}, \beta_2, \beta_3)} \Bigg]$$
$$= 2\frac{\Gamma(-1 + \frac{\gamma\beta_1}{2} - \frac{\gamma^2}{4})\Gamma(1 - \frac{\gamma\beta_1}{2})}{\Gamma(-\frac{\gamma^2}{4})} \left(g_{\frac{\gamma}{2}}(\sigma_1) - g_{\frac{\gamma}{2}}(\sigma_2 + \frac{\beta_1}{2}) \right) R(\beta_1 + \frac{\gamma}{2}, \sigma_1, \sigma_2 + \frac{\gamma}{4}).$$

In the above three limits the second one is actually trivial, only the first and third involve using Lemma 4.7. Putting these limits together using (4.2) leads to (4.26). By using the alternative function $\tilde{H}_{\frac{\gamma}{2}}(t)$ along the same lines we obtain the relation (4.27) between $R(\beta_1 + \frac{\gamma}{2}, \sigma_1, \sigma_2 + \frac{\gamma}{4})$ and $R(\beta_1 + \gamma, \sigma_1, \sigma_2)$. Hence this implies the claim of the lemma. □

4.4. **Analytic continuation for H and R.** In this subsection we will analytically extend the functions H and R to a larger domain of parameters than the one we are currently working with. For this purpose, we must first derive the so-called reflection principle for H and R which will be obtain by performing the OPE with reflection in the case $\chi = \frac{\gamma}{2}$. We rely extensively on Lemma E.2 giving the Taylor expansion using the reflection coefficient. Finally we also extend the validity of Lemma 4.7 to a larger range of parameters.

Lemma 4.9 (Reflection principle for $H_{(\sigma_1,\sigma_2,\sigma_3)}^{(\beta_1,\beta_2,\beta_3)}$)**.** *Consider parameters* $\sigma_1, \sigma_2, \sigma_3$, $\beta_1, \beta_2, \beta_3$ *satisfying the parameter ranges* (4.12) *and* (4.13) *for* $H_{(\sigma_1,\sigma_2,\sigma_3)}^{(\beta_1,\beta_2,\beta_3)}$ *and* $R(\beta_1, \sigma_1, \sigma_2)$ *to be well-defined. Then one can meromorphically extend* $\beta_1 \mapsto H_{(\sigma_1,\sigma_2,\sigma_3)}^{(\beta_1,\beta_2,\beta_3)}$ *beyond the point* $\beta_1 = Q$ *by the following relation:*

(4.28)
$$H_{(\sigma_1,\sigma_2,\sigma_3)}^{(\beta_1,\beta_2,\beta_3)} = R(\beta_1, \sigma_1, \sigma_2) H_{(\sigma_1,\sigma_2,\sigma_3)}^{(2Q-\beta_1,\beta_2,\beta_3)}.$$

The quantity $H_{(\sigma_1,\sigma_2,\sigma_3)}^{(2Q-\beta_1,\beta_2,\beta_3)}$ *is thus well-defined as long as the parameters of* $H_{(\sigma_1,\sigma_2,\sigma_3)}^{(\beta_1,\beta_2,\beta_3)}$ *and* $R(\beta_1, \sigma_1, \sigma_2)$ *respectively obey the constraints of* (4.12) *and* (4.13)*. Similarly, for* $\sigma_1, \sigma_2 \in \mathcal{B}$*, we can analytically* *extend* $\beta_1 \mapsto R(\beta_1,\sigma_1,\sigma_2)$ *to the range* $((\frac{2}{\gamma}-\frac{\gamma}{2})\vee\frac{\gamma}{2}, Q+(\frac{2}{\gamma}\wedge\gamma))$ *thanks to the relation:*

$$R(\beta_1,\sigma_1,\sigma_2)R(2Q-\beta_1,\sigma_1,\sigma_2)=1. \tag{4.29}$$

Proof. Throughout the proof we keep the same notations as used in the proof of Lemma 4.5 for the solution space of the hypergeometric equation satisfied by $H_{\frac{\gamma}{2}}(t)$ for $t\in(0,1)$. We assume the parameters β_i, σ_i obey the condition (4.8) in order for $H_{\frac{\gamma}{2}}(t)$ to be well-defined. We also add the condition $\beta_1\in(Q-\beta_0,Q)$ so that we can apply the result of Lemma E.2 and identify the value of C_2^+ to be:

$$C_2^+ = R(\beta_1,\sigma_1-\frac{\gamma}{4},\sigma_2-\frac{\gamma}{4})H^{(2Q-\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(\sigma_1-\frac{\gamma}{4},\sigma_2,\sigma_3)}. \tag{4.30}$$

The key argument is to observe that since by Lemma 4.4 $\beta_1\mapsto H_{\frac{\gamma}{2}}(t)$ is complex analytic so is the coefficient C_2^+. By using this combined with the analyticity of R and H, we can extend the range of validity of equation (4.30) from $\beta_1\in(Q-\beta_0,Q)$ to $\beta_1\in(\frac{2}{\gamma},Q)$, still under the constraint of (4.8). Now equation (4.20) derived in the the proof of Lemma 4.5 gives us an alternative expression for C_2^+, which is valid for $\beta_1\in(\frac{\gamma}{2},\frac{2}{\gamma})$. The analycity of $\beta_1\mapsto C_2^+$ in a complex neighborhood of $\frac{2}{\gamma}$ then implies that one can "glue" together the two expressions for C_2^+. More precisely, after performing the parameter replacement $\sigma_i\to\sigma_i+\frac{\gamma}{4}$ for $i=1,2,3$, the equality

$$R(\beta_1,\sigma_1,\sigma_2)H^{(2Q-\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(\sigma_1,\sigma_2+\frac{\gamma}{4},\sigma_3+\frac{\gamma}{4})} = \frac{\Gamma(-1+\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta_1}{2})}{\Gamma(-\frac{\gamma^2}{4})}\left((g_{\frac{\gamma}{2}}(\sigma_1)-g_{\frac{\gamma}{2}}(\sigma_2+\frac{\beta_1}{2})\right)H^{(\beta_1+\frac{\gamma}{2},\beta_2,\beta_3)}_{(\sigma_1,\sigma_2+\frac{\gamma}{4},\sigma_3+\frac{\gamma}{4})}, \tag{4.31}$$

provides the desired analytic continuation of H. To land on the form of the reflection equation given in the lemma one needs to replace β_1 by $\beta_1-\frac{\gamma}{2}$. This transforms $R(\beta_1,\sigma_1,\sigma_2)$ into $R(\beta_1-\frac{\gamma}{2},\sigma_1,\sigma_2)$ which we can shift back to $R(\beta_1,\sigma_1,\sigma_2+\frac{\gamma}{4})$ using the shift equation (4.26). Lastly we perform the parameter replacement $\sigma_2+\frac{\gamma}{4}$ to σ_2 and $\sigma_3+\frac{\gamma}{4}$ to σ_3. Therefore this implies the claim of the reflection principle for H. The claim for R is then an immediate consequence. □

At this stage we will use the shift equations we have derived to analytically continue H and R both in the parameters β_i and σ_i. The analytic continuations will be defined in a larger range of parameters than (4.12) and (4.13) required for the GMC expressions to be well-defined. The proofs follow closely the ones of [RZ21]. We start with the case of R which is very straightforward.

Lemma 4.10. *(Analytic continuation of* $R(\beta_1,\sigma_1,\sigma_2)$*) For all* $\sigma_1,\sigma_2\in\mathcal{B}$*, the meromorphic function* $\beta_1\mapsto R(\beta_1,\sigma_1,\sigma_2)$ *originally defined on the interval* $((\frac{2}{\gamma}-\frac{\gamma}{2})\vee\frac{\gamma}{2},Q)$ *extends to a meromorphic function defined in a complex neighborhood of* $\mathbb{R}$ *and satisfying the shift equation:*

$$\frac{R(\beta_1,\sigma_1,\sigma_2)}{R(\beta_1+\gamma,\sigma_1,\sigma_2)} = (\frac{\gamma}{2})^4\frac{\Gamma(-1+\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta_1}{2})\Gamma(-1+\frac{\gamma\beta_1}{2})}{\sin(\pi\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma^2}{4})^2}$$
$$\times 4\sin(\frac{\gamma\pi}{2}(\frac{\beta}{2}-\sigma_1-\sigma_2+Q))\sin(\frac{\gamma\pi}{2}(\frac{\beta}{2}+\sigma_1+\sigma_2-Q))\sin(\frac{\gamma\pi}{2}(\frac{\beta}{2}+\sigma_2-\sigma_1))\sin(\frac{\gamma\pi}{2}(\frac{\beta}{2}+\sigma_1-\sigma_2)). \tag{4.32}$$

Furthermore, for a fixed β_1 *in the above complex neighborhood of* $\mathbb{R}$*, the function* $R(\beta_1,\sigma_1,\sigma_2)$ *extends to a meromorphic function of* (σ_1,σ_2) *on* $\mathbb{C}^2$*.*

Proof. This proof is performed in [RZ21] but we sketch it below as it is quite short. Fix first the parameters σ_1,σ_2 in $\mathcal{B}$. Thanks to Proposition 2.12, $\beta_1\mapsto R(\beta_1,\sigma_1,\sigma_2)$ is meromorphic in a complex neighborhood of the real interval $((\frac{2}{\gamma}-\frac{\gamma}{2})\vee\frac{\gamma}{2},Q)$, and thanks to the reflection principle given by Lemma 4.9, it is meromorphic in a complex neighborhood of $((\frac{2}{\gamma}-\frac{\gamma}{2})\vee\frac{\gamma}{2}, Q+(\frac{2}{\gamma}\wedge\gamma))$. Note that the length of this interval is strictly greater than γ.

Combining the two shift equations of Lemma 4.8, we obtain the shift equation (4.32) relating $R(\beta_1+\gamma,\sigma_1,\sigma_2)$ and $R(\beta_1,\sigma_1,\sigma_2)$, which does not shift the σ_1,σ_2 parameters. Since $R(\beta_1,\sigma_1,\sigma_2)$ is defined in a complex neighborhood of a real interval of length strictly bigger than γ, the shift equation (4.32) can be used to meromorphicaly extend $\beta_1\mapsto R(\beta_1,\sigma_1,\sigma_2)$ to a complex neighborhood of the whole real line.

Now for any β_1 fixed in this complex neighborhood of $\mathbb{R}$, we perform the analytic continuation to $\mathbb{C}^2$ in the variables σ_1,σ_2. For this one can simply apply either of the shift equation of Lemma 4.8. This is possible since the band $\mathcal{B}$ has width strictly greater than $\frac{\gamma}{4}$. Hence the result. □

Lemma 4.11. *(Analytic continuation of $H^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}$) Fix $\sigma_3\in[-\frac{1}{2\gamma}+\frac{Q}{2},\frac{1}{2\gamma}+\frac{Q}{2}-\frac{\gamma}{4}]\times\mathbb{R}$ and fix $\sigma_1,\sigma_2\in\mathcal{B}$. Then the function $(\beta_1,\beta_2,\beta_3)\mapsto H^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}$ originally defined in the parameter range given by (4.12) extends to a meromorphic function of the three variables in a small complex neighborhood of $\mathbb{R}^3$. Now fix β_1,β_2,β_3 in this complex neighborhood of $\mathbb{R}^3$, keeping σ_3 still fixed in $[-\frac{1}{2\gamma}+\frac{Q}{2},\frac{1}{2\gamma}+\frac{Q}{2}-\frac{\gamma}{4}]\times\mathbb{R}$. The function $(\sigma_1,\sigma_2)\mapsto H^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}$ then extends to a meromorphic function of $\mathbb{C}^2$.*

Proof. This proof follows exactly the proof performed in [RZ21] with one notable difference, which is that we are not able to use an extra property on the σ_i. Indeed, in the case $\mu=0$ when H has an expression reducing to a moment of a boundary GMC measure, if one adds a global constant to each of the σ_i the function H is simply changed by a global phase. In our case this property does not hold which is why we are not able to perform the extension in the variable σ_3. Notice though that once Theorem 1.1 is established, the exact formula H_{PT} implies that H extends meromorphically in all its parameters to $\mathbb{C}^6$. □

We finish this subsection by extending Lemma 4.7 to a larger range of parameters that will be required for the next subsection. This is a novel difficulty that was not present in [RZ21] because in our case we had to perform a truncation procedure to define H and R while in [RZ21] they simply reduce to a moment of GMC.

Lemma 4.12. *Suppose β_1,β_2,β_3 satisfy $Q-\frac{1}{2}\sum_i\beta_i<\frac{2}{\gamma}\wedge\min_i(Q-\beta_i)$, $Q>\beta_1>\beta_2\vee\frac{\gamma}{2}$ and $\beta_1-\beta_2<\beta_3<Q$. Let also $\sigma_1\in\overline{\mathcal{B}}$ and $\sigma_2,\sigma_3\in\mathcal{B}$. Then the following limit holds*

$$\lim_{\beta_3\to\beta_1-\beta_2}(\beta_2+\beta_3-\beta_1)H^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}=2R(\beta_1,\sigma_1,\sigma_2),$$

where the function R should be understood by the analytic continuation given in Lemma 4.10.

Proof. Lets check one step of the proof. We will use the shift equation

(4.33)

$$H^{(\beta_1,\beta_2-\frac{\gamma}{2},\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}=\frac{\Gamma(\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta_2}{2}+\frac{\gamma^2}{4})}{\Gamma(\frac{\gamma\beta_1}{2}+(q-1)\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma}{2}\beta_2-(q-1)\frac{\gamma^2}{4}))}H^{(\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(\sigma_1,\sigma_2+\frac{\gamma}{4},\sigma_3)}$$
$$-\frac{\Gamma(2+\frac{\gamma^2}{4}-\frac{\gamma\beta_1}{2})\Gamma(1-\frac{\gamma\beta_2}{2}+\frac{\gamma^2}{4})\Gamma(-1+\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta_1}{2})}{\Gamma(1+\frac{q\gamma^2}{4})\Gamma(2-\frac{\gamma}{2}(\beta_1+\beta_2)-(q-2)\frac{\gamma^2}{4}))\Gamma(-\frac{\gamma^2}{4})}\left((g_{\frac{\gamma}{2}}(\sigma_1)-g_{\frac{\gamma}{2}}(\sigma_2+\frac{\beta_1}{2})\right)H^{(\beta_1+\frac{\gamma}{2},\beta_2,\beta_3)}_{(\sigma_1,\sigma_2+\frac{\gamma}{4},\sigma_3)},$$

Lets assume that the parameters of $H^{(\beta_1+\frac{\gamma}{2},\beta_2,\beta_3)}_{(\sigma_1,\sigma_2+\frac{\gamma}{4},\sigma_3)}$ are such that the range of Proposition 2.7 is satisfied and such that $\beta_1\in((Q-\gamma)\vee\frac{\gamma}{2},Q)$. By taking a suitable limit, we can derive the limiting statement for $\beta_1\in((Q-\frac{3}{2}\gamma)\vee\frac{\gamma}{2},Q)$ which extends the probabilistic range.

A few things to be careful about. First, the term $H^{(\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(\sigma_1,\sigma_2+\frac{\gamma}{4},\sigma_3)}$ needs to disappear in the limit. For this to happen, first the parameters must be such that there are no poles in the gamma functions in the prefactor. This is true for $\beta_1 \in (\frac{\gamma}{2}, Q)$ and $\beta_2 < Q$. The fact we really need to double check is that $H^{(\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(\sigma_1,\sigma_2+\frac{\gamma}{4},\sigma_3)}$ has no poles at the given point. This should be true thanks to the shift equation as well, since there are no poles in the probabilistic range and $H^{(\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(\sigma_1,\sigma_2+\frac{\gamma}{4},\sigma_3)}$ has been defined by the shift equation which prescribes all the poles.

□

4.5. **The $\frac{2}{\gamma}$-shift equations for the reflection coefficient.** Finally we will derive the $\frac{2}{\gamma}$-shift equations on $R(\beta_1, \sigma_1, \sigma_2)$ that will completely specify its value.

Lemma 4.13. *($\frac{2}{\gamma}$-shift equations for $R(\beta_1, \sigma_1, \sigma_2)$). For all $\sigma_1, \sigma_2 \in \mathbb{C}$, the meromorphic function $\beta_1 \mapsto R(\beta_1, \sigma_1, \sigma_2)$ defined in a complex neighborhood of $\mathbb{R}$ satisfies the shift equations in Theorem 4.2 with $\chi = \frac{2}{\gamma}$.*

We are now working exclusively with the choice $\chi = \frac{2}{\gamma}$. There are two steps that will each require their own range of parameters. We first place ourselves in the following range that will allow us to apply the OPE with reflection of Lemma E.2 around the β_1 insertion at 0:

$$t \in (0,1), \;\; \beta_1, \beta_2, \beta_3 \in (Q-\epsilon, Q), \;\; \sigma_1, \sigma_2 \in i\mathbb{R} + \frac{1}{2\gamma} + \frac{Q}{2}, \;\; \sigma_3 \in \mathcal{B}. \tag{4.34}$$

In the above ϵ is chosen small enough, smaller than the constant β_0 required to apply Lemma E.2. We can thus expand $H_{\frac{2}{\gamma}}(t)$ on the basis, for $t \in (0,1)$

$$\begin{aligned} H_{\frac{2}{\gamma}}(t) &= C_1 F(A,B,C,t) + C_2^+ t^{1-C} F(1+A-C, 1+B-C, 2-C, t) \\ &= B_1 F(A,B,1+A+B-C,1-t) + B_2^-(1-t)^{C-A-B} F(C-A, C-B, 1+C-A-B, 1-t), \end{aligned} \tag{4.35}$$

where again C_1, C_2^+, B_1, B_2^- are parametrizing the solution space around the points 0 and 1. As before by sending t to 0 and to 1 one obtains:

$$C_1 = H_{\frac{2}{\gamma}}(0) = H^{(\beta_1-\frac{2}{\gamma},\beta_2,\beta_3)}_{(\sigma_1-\frac{1}{\gamma},\sigma_2,\sigma_3)}, \quad B_1 = H_{\frac{2}{\gamma}}(1) = H^{(\beta_1,\beta_2-\frac{2}{\gamma},\beta_3)}_{(\sigma_1-\frac{1}{\gamma},\sigma_2-\frac{1}{\gamma},\sigma_3)}. \tag{4.36}$$

Since the condition required for Lemma E.2, $\beta_1 \in (Q-\beta_0, Q)$, is satisfied one then derives:

$$C_2^+ = R(\beta_1, \sigma_1 - \frac{1}{\gamma}, \sigma_2 - \frac{1}{\gamma}) H^{(2Q-\beta_1-\frac{2}{\gamma},\beta_2,\beta_3)}_{(\sigma_1-\frac{1}{\gamma},\sigma_2,\sigma_3)}. \tag{4.37}$$

Similarly, we can apply Lemma E.2 around $t=1$ and get:

$$B_2^- = R(\beta_2, \sigma_2, \sigma_3) H^{(\beta_1, 2Q-\beta_2-\frac{2}{\gamma},\beta_3)}_{(\sigma_1-\frac{1}{\gamma},\sigma_2-\frac{1}{\gamma},\sigma_3)}. \tag{4.38}$$

The quantities C_1, B_2^-, C_2^+ identified above are then related by the connection formula (A.8):

$$B_2^- = \frac{\Gamma(C)\Gamma(A+B-C)}{\Gamma(A)\Gamma(B)} C_1 + \frac{\Gamma(2-C)\Gamma(A+B-C)}{\Gamma(A-C+1)\Gamma(B-C+1)} C_2^+. \tag{4.39}$$

We repeat the same procedure to identify the coefficients D_2^+, C_2^- using the same parameter ranges as before except with $t \in (-\infty, 0)$:

$$t \in (-\infty, 0), \ \ \beta_1, \beta_2, \beta_3 \in (Q - \epsilon, Q), \ \ \sigma_1, \sigma_2 \in i\mathbb{R} + \frac{1}{2\gamma} + \frac{Q}{2}, \ \ \sigma_3 \in \mathcal{B}. \tag{4.40}$$

For $t \in (-\infty, 0)$, $H_{\frac{2}{\gamma}}(t)$ can be expanded on the basis:

(4.41)

$$\begin{aligned} H_{\frac{2}{\gamma}}(t) &= C_1 F(A, B, C, t) + C_2^- t^{1-C} F(1 + A - C, 1 + B - C, 2 - C, t) \\ &= D_1 e^{i\pi A} t^{-A} F(A, 1 + A - C, 1 + A - B, t^{-1}) + D_2^+ e^{i\pi B} t^{-B} F(B, 1 + B - C, 1 + B - A, t^{-1}). \end{aligned}$$

The coefficients C_2^-, D_2^+ have expressions given by:

$$C_2^- = e^{-i\pi(1 - \frac{2\beta_1}{\gamma} + \frac{4}{\gamma^2})} R(\beta_1, \sigma_1, \sigma_2) H_{(\sigma_1 - \frac{1}{\gamma}, \sigma_2, \sigma_3)}^{(2Q - \beta_1 - \frac{2}{\gamma}, \beta_2, \beta_3)}, \tag{4.42}$$

$$D_2^+ = R(\beta_3, \sigma_1 - \frac{1}{\gamma}, \sigma_3) H_{(\sigma_1, \sigma_2, \sigma_3)}^{(\beta_1, \beta_2, 2Q - \beta_3 - \frac{2}{\gamma})}. \tag{4.43}$$

Using the connection formula (A.7) we can write this time:

$$D_2^+ = \frac{\Gamma(C)\Gamma(A - B)}{\Gamma(A)\Gamma(C - B)} C_1 + e^{i\pi(1-C)} \frac{\Gamma(2 - C)\Gamma(A - B)}{\Gamma(1 - B)\Gamma(A - C + 1)} C_2^-. \tag{4.44}$$

By eliminating the coefficient C_1 we obtain the relation:

(4.45)

$$\frac{\Gamma(B)}{\Gamma(A + B - C)} B_2^- - \frac{\Gamma(C - B)}{\Gamma(A - B)} D_2^+ = \frac{\Gamma(2 - C)}{\Gamma(A - C + 1)} \left(\frac{\Gamma(B)}{\Gamma(B - C + 1)} C_2^+ - \frac{e^{i\pi(1-C)}\Gamma(C - B)}{\Gamma(1 - B)} C_2^- \right).$$

Let us state this identity as a lemma where the constants B_2^-, D_2^+, C_2^+, and C_2^- have been replaced by their explicit expressions in terms of H and R.

Lemma 4.14. *The following identity holds:*

(4.46)

$$\begin{aligned} &\frac{\Gamma(B)}{\Gamma(A + B - C)} R(\beta_2, \sigma_2, \sigma_3) H_{(\sigma_1 - \frac{1}{\gamma}, \sigma_2 - \frac{1}{\gamma}, \sigma_3)}^{(\beta_1, 2Q - \beta_2 - \frac{2}{\gamma}, \beta_3)} - \frac{\Gamma(C - B)}{\Gamma(A - B)} R(\beta_3, \sigma_1 - \frac{1}{\gamma}, \sigma_3) H_{(\sigma_1, \sigma_2, \sigma_3)}^{(\beta_1, \beta_2, 2Q - \beta_3 - \frac{2}{\gamma})} \\ &= \frac{\Gamma(2 - C)}{\Gamma(A - C + 1)} H_{(\sigma_1 - \frac{1}{\gamma}, \sigma_2, \sigma_3)}^{(2Q - \beta_1 - \frac{2}{\gamma}, \beta_2, \beta_3)} \Bigg(\frac{\Gamma(B)}{\Gamma(B - C + 1)} R(\beta_1, \sigma_1 - \frac{1}{\gamma}, \sigma_2 - \frac{1}{\gamma}) \\ &\qquad - \frac{e^{i\pi(1-C)}\Gamma(C - B)}{\Gamma(1 - B)} e^{-i\pi(1 - \frac{2\beta_1}{\gamma} + \frac{4}{\gamma^2})} R(\beta_1, \sigma_1, \sigma_2) \Bigg) \end{aligned}$$

This identity is originally derived in the range of parameters

$$\beta_1, \beta_2, \beta_3 \in (Q - \epsilon, Q), \ \ \sigma_1, \sigma_2 \in i\mathbb{R} + \frac{1}{2\gamma} + \frac{Q}{2}, \ \ \sigma_3 \in \mathcal{B}, \tag{4.47}$$

but it can be viewed as an identity of the meromorphically extended functions H and R, the extension being provided by Lemmas 4.11 and 4.10.

The next step is to use again the limit H goes to R to derive from the above equation a relation on R. This requires to change the parameter range on the β_i in order to apply Lemma 4.12 giving R as a limit of H. Let us now take

$$\beta_1 = \beta \in (\frac{\gamma}{2}, \frac{2}{\gamma}), \quad \beta_2 = \frac{\gamma}{2} + \eta, \quad \beta_3 = Q - \beta, \ \sigma_1 \in i\mathbb{R} + \frac{1}{2\gamma} + \frac{Q}{2}, \ \sigma_2, \sigma_3 \in \mathcal{B}, \tag{4.48}$$

and study the asymptotic as $\eta \to 0$. The functions H and R appearing are viewed as defined by the meromorphic extension of Lemmas 4.11 and 4.10. For the above choice of parameters

$$q = \frac{4}{\gamma^2} - \frac{\eta}{\gamma}, \quad A = -\frac{4}{\gamma^2} + \frac{\eta}{\gamma}, \quad B = \frac{2\beta}{\gamma} - \frac{4}{\gamma^2} + \frac{\eta}{\gamma}, \quad C = \frac{2\beta}{\gamma} - \frac{4}{\gamma^2}, \tag{4.49}$$

and the two H functions that we are going to apply the Lemma 4.12 to are

$$H^{(2Q-\beta_1-\frac{2}{\gamma},\beta_2,\beta_3)}_{(\sigma_1-\frac{1}{\gamma},\sigma_2,\sigma_3)} = H^{(2Q-\beta-\frac{2}{\gamma},\frac{\gamma}{2}+\eta,Q-\beta)}_{(\sigma_1-\frac{1}{\gamma},\sigma_2,\sigma_3)}, \quad \text{and} \quad H^{(\beta_1,\beta_2,2Q-\beta_3-\frac{2}{\gamma})}_{(\sigma_1,\sigma_2,\sigma_3)} = H^{(\beta,\frac{\gamma}{2}+\eta,\frac{\gamma}{2}+\beta)}_{(\sigma_1,\sigma_2,\sigma_3)}.$$

We compute the following limits, the last one is trivial and does not require using Lemma 4.12.

(4.50)

$$\lim_{\eta\to 0} \eta D_2^+ = 2R(Q-\beta, \sigma_1 - \frac{1}{\gamma}, \sigma_3) R(\beta + \frac{\gamma}{2}, \sigma_1, \sigma_3) = \frac{2R(\beta+\frac{\gamma}{2}, \sigma_1, \sigma_3)}{R(\beta+Q, \sigma_1 - \frac{1}{\gamma}, \sigma_3)},$$

(4.51)

$$\lim_{\eta\to 0} \eta C_2^- = 2e^{-i\pi(1-\frac{2\beta}{\gamma}+\frac{4}{\gamma^2})} R(\beta, \sigma_1, \sigma_2) R(2Q-\beta-\frac{2}{\gamma}, \sigma_1 - \frac{1}{\gamma}, \sigma_2) = \frac{2e^{-i\pi(1-\frac{2\beta}{\gamma}+\frac{4}{\gamma^2})} R(\beta, \sigma_1, \sigma_2)}{R(\beta+\frac{2}{\gamma}, \sigma_1-\frac{1}{\gamma}, \sigma_2)},$$

(4.52)

$$\lim_{\eta\to 0} \eta^2 B_2^- = -4 \lim_{\eta\to 0} \eta R(\frac{\gamma}{2}+\eta, \sigma_2, \sigma_3).$$

Putting all these into (4.46), we get:

$$-4\frac{\Gamma(\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})}{\Gamma(-\frac{4}{\gamma^2})} \lim_{\eta\to 0} \eta R(\frac{\gamma}{2}+\eta, \sigma_2, \sigma_3) + \frac{2\gamma}{\Gamma(-\frac{2\beta}{\gamma})} \frac{(1-\frac{\gamma\beta}{2}+\frac{\gamma^2}{4})}{-\frac{\gamma\beta}{2}} \frac{R(\beta, \sigma_1, \sigma_3 - \frac{\gamma}{4})}{R(\beta+\frac{2}{\gamma}, \sigma_1 - \frac{1}{\gamma}, \sigma_3 - \frac{\gamma}{4})}$$
$$= \frac{2\gamma(1-\frac{2\beta}{\gamma}+\frac{4}{\gamma^2})}{\Gamma(1-\frac{2\beta}{\gamma})} \frac{R(\beta, \sigma_1, \sigma_2)}{R(\beta+\frac{2}{\gamma}, \sigma_1-\frac{1}{\gamma}, \sigma_2)}.$$

After simplifications one obtains:

$$\frac{1}{\Gamma(-1+\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})\Gamma(1-\frac{2\beta}{\gamma})} \left(\frac{R(\beta, \sigma_1, \sigma_2)}{R(\beta+\frac{2}{\gamma}, \sigma_1-\frac{1}{\gamma}, \sigma_2)} - \frac{R(\beta, \sigma_1, \sigma_3-\frac{\gamma}{4})}{R(\beta+\frac{2}{\gamma}, \sigma_1-\frac{1}{\gamma}, \sigma_3-\frac{\gamma}{4})} \right)$$
$$= \frac{2}{\gamma\Gamma(-\frac{4}{\gamma^2})} \lim_{\eta\to 0} \eta R(\frac{\gamma}{2}+\eta, \sigma_2, \sigma_3). \tag{4.53}$$

We want to determine the function on the right hand side of the above equation. It is natural to use the $\frac{\gamma}{2}$-shift equation (4.26) on R to write:

$$R(\frac{\gamma}{2}+\eta, \sigma_2, \sigma_3) = -\frac{\Gamma(-1+\frac{\gamma\eta}{2})\Gamma(1-\frac{\gamma^2}{4}-\frac{\gamma\eta}{2})}{\Gamma(-\frac{\gamma^2}{4})} \left(g_{\frac{\gamma}{2}}(\sigma_2) - g_{\frac{\gamma}{2}}(\sigma_3 + \frac{\gamma}{4} + \frac{\eta}{2}) \right) R(\gamma+\eta, \sigma_2, \sigma_3 + \frac{\gamma}{4}).$$

Simplifying the limit, one gets:

$$\lim_{\eta\to 0}\eta R(\frac{\gamma}{2}+\eta,\sigma_2,\sigma_3)=\frac{2}{\gamma}\frac{\Gamma(1-\frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})}\left(g_{\frac{\gamma}{2}}(\sigma_2)-g_{\frac{\gamma}{2}}(\sigma_3+\frac{\gamma}{4})\right)R(\gamma,\sigma_2,\sigma_3+\frac{\gamma}{4}).$$

We now introduce the shorthand notation $\mathcal{R}(\beta,\sigma_1,\sigma_2)=\frac{R(\beta,\sigma_1,\sigma_2)}{R(\beta+\frac{2}{\gamma},\sigma_1-\frac{1}{\gamma},\sigma_2)}$. By taking $\sigma_3=\sigma_2-\frac{\gamma}{4}$ in (4.53), since $\lim_{\eta\to 0}\eta R(\frac{\gamma}{2}+\eta,\sigma_2,\sigma_2-\frac{\gamma}{4})=0$, we obtain that $\mathcal{R}(\beta,\sigma_1,\sigma_2)$ is $\frac{\gamma}{2}$-periodic in σ_2. At this point we will use an extra input form the mating-of-trees framework which will give us the explicit value of $R(\gamma,\sigma_2,\sigma_3)$.

Lemma 4.15. *Suppose γ^2 is irrational. For some constant c_γ depending only on γ we have:*

$$R(\gamma,\sigma,\sigma')=c_\gamma\frac{\cos(\frac{4\pi}{\gamma}(\sigma-\frac{Q}{2}))-\cos(\frac{4\pi}{\gamma}(\sigma'-\frac{Q}{2}))}{\cos(\gamma\pi(\sigma-\frac{Q}{2}))-\cos(\gamma\pi(\sigma'-\frac{Q}{2}))}.$$

We defer the proof of this lemma to Appendix D.3. Using this lemma we can complete the proof of Lemma 4.13. Below the constant c_γ of γ can be different at every line. Using the previous lemma we can compute:

$$\begin{aligned}\lim_{\eta\to 0}\eta R(\frac{\gamma}{2}+\eta,\sigma_2,\sigma_3)&=c_\gamma\left(\cos(\pi\gamma(\sigma_2-\frac{Q}{2}))-\cos(\pi\gamma(\sigma_3+\frac{\gamma}{4}-\frac{Q}{2}))\right)R(\gamma,\sigma_2,\sigma_3+\frac{\gamma}{4})\\&=c_\gamma\sin(\frac{\pi\gamma}{2}(\sigma_2+\sigma_3+\frac{\gamma}{4}-Q))\sin(\frac{\pi\gamma}{2}(\sigma_2-\sigma_3-\frac{\gamma}{4}))\frac{\sin(\frac{2\pi}{\gamma}(\sigma_3+\frac{\gamma}{4}-\sigma_2))\sin(\frac{2\pi}{\gamma}(\sigma_3+\frac{\gamma}{4}+\sigma_2-\frac{2}{\gamma}))}{\sin(\frac{\gamma\pi}{2}(\sigma_3+\frac{\gamma}{4}-\sigma_2))\sin(\frac{\gamma\pi}{2}(\sigma_3+\frac{\gamma}{4}+\sigma_2-\frac{\gamma}{2}))}\\&=c_\gamma\sin(\frac{2\pi}{\gamma}(\sigma_3+\frac{\gamma}{4}-\sigma_2))\sin(\frac{2\pi}{\gamma}(\sigma_3+\frac{\gamma}{4}+\sigma_2-\frac{2}{\gamma}))\\&=c_\gamma\left(\cos(\frac{2\pi}{\gamma}(2\sigma_2-\frac{2}{\gamma}))-\cos(\frac{2\pi}{\gamma}(2\sigma_3+\frac{\gamma}{2}-\frac{2}{\gamma}))\right).\end{aligned}$$

Therefore we land on:

$$\begin{aligned}&\mathcal{R}(\beta,\sigma_1,\sigma_2)-\mathcal{R}(\beta,\sigma_1,\sigma_3-\frac{\gamma}{4})\\&\quad=c_\gamma\Gamma(-1+\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})\Gamma(1-\frac{2\beta}{\gamma})\left(\cos(\frac{2\pi}{\gamma}(2\sigma_2-\frac{2}{\gamma}))-\cos(\frac{2\pi}{\gamma}(2\sigma_3+\frac{\gamma}{2}-\frac{2}{\gamma}))\right).\end{aligned}\tag{4.54}$$

By setting σ_3 to any fixed value, the above equation implies the following claim

$$\mathcal{R}(\beta,\sigma_1,\sigma_2)=c_\gamma\Gamma(-1+\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})\Gamma(1-\frac{2\beta}{\gamma})\cos(\frac{2\pi}{\gamma}(2\sigma_2-\frac{2}{\gamma}))+u(\sigma_1,\beta,\gamma),\tag{4.55}$$

where $u(\sigma_1,\beta,\gamma)$ is an unknown function that does not depend on σ_2. It thus remains to evaluate this function u. For this we will use the fact that we know that $\mathcal{R}(\beta,\sigma_1,\sigma_1-\frac{\beta}{2})=0$ which can be easily deduced from the $\frac{\gamma}{2}$-shift equation (4.26). Indeed, this is clear since the right hand side of (4.26) is zero when $\sigma_2=\sigma_1-\frac{\beta}{2}$. Thus this implies that:

$$u(\sigma_1,\beta,\gamma)=-c_\gamma\Gamma(-1+\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})\Gamma(1-\frac{2\beta}{\gamma})\cos(\frac{2\pi}{\gamma}(2\sigma_1-\beta-\frac{2}{\gamma})).$$

Now renaming as $c_{\frac{2}{\gamma}}(\gamma)$ the unknown function of γ, we have thus shown that:

$$\frac{R(\beta,\sigma_1,\sigma_2)}{R(\beta+\frac{2}{\gamma},\sigma_1-\frac{1}{\gamma},\sigma_2)}=c_{\frac{2}{\gamma}}(\gamma)\Gamma(-1+\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})\Gamma(1-\frac{2\beta}{\gamma})\left(g_{\frac{2}{\gamma}}(\sigma_2)-g_{\frac{2}{\gamma}}(\sigma_1-\frac{\beta}{2})\right).\tag{4.56}$$

Similarly by working with auxiliary function $\tilde{H}_\chi(t)$ yields the shift equation:

$$\frac{R(\beta+\frac{2}{\gamma},\sigma_1-\frac{1}{\gamma},\sigma_2)}{R(\beta+\frac{4}{\gamma},\sigma_1,\sigma_2)} = c_{\frac{2}{\gamma}}(\gamma)\Gamma(-1+\frac{2\beta}{\gamma})\Gamma(1-\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})\left(g_{\frac{2}{\gamma}}(\sigma_2)-g_{\frac{2}{\gamma}}(\sigma_1+\frac{\beta}{2})\right).$$

This completes the proof of Lemma 4.13.

4.6. **Proof of the $\frac{2}{\gamma}$-shift equations for H.** In this last step we complete deriving the shift equation for H in the case where $\chi = \frac{2}{\gamma}$.

Proof of Proposition 4.1. First the claim on the meromorphic extension of H has been obtained in Lemma 4.11. Next the shift equations come from applying (A.8). The first comes from the relation

(4.57)
$$B_1 = \frac{\Gamma(\chi(\beta_1-\chi))\Gamma(1-\chi\beta_2+\chi^2)}{\Gamma(\chi(\beta_1-\chi+q\frac{\gamma}{2})\Gamma(1-\chi\beta_2+\chi^2-q\frac{\gamma\chi}{2})}C_1 + \frac{\Gamma(2-\chi\beta_1+\chi^2)\Gamma(1-\chi\beta_2+\chi^2)}{\Gamma(1+\frac{q\gamma\chi}{2})\Gamma(2-\chi(\beta_1+\beta_2-2\chi+q\frac{\gamma}{2}))}C_2^+,$$

and the second can be deduced by using:

(4.58)
$$\tilde{B}_2^- = \frac{\Gamma(\chi(\beta_1-\chi))\Gamma(-1+\chi\beta_2-\chi^2)}{\Gamma(-q\frac{\gamma\chi}{2})\Gamma(-1+\chi(\beta_1+\beta_2-2\chi+q\frac{\gamma}{2}))}\tilde{C}_1 + \frac{\Gamma(2-\chi\beta_1+\chi^2))\Gamma(-1+\chi\beta_2-\chi^2)}{\Gamma(1-\chi(\beta_1-\chi+q\frac{\gamma}{2})\Gamma(\chi\beta_2-\chi^2+q\frac{\gamma\chi}{2})}\tilde{C}_2^+.$$

We have already derived them in the case of $\chi = \frac{\gamma}{2}$ in Lemma 4.5. Setting now $\chi = \frac{2}{\gamma}$, we similarly identify the constants $B_1, C_1, C_2^+, \tilde{B}_2^-, \tilde{C}_1, \tilde{C}_2^+$. For $C_2^+, \tilde{B}_2^-, \tilde{C}_2^+$ we use the result of Lemma E.2 which gives us an expression with both an H and an R function. For instance:

$$C_2^+ = R(\beta_1,\sigma_1-\frac{1}{\gamma},\sigma_2-\frac{1}{\gamma})H^{(2Q-\beta_1-\frac{2}{\gamma},\beta_2,\beta_3)}_{(\sigma_1-\frac{1}{\gamma},\sigma_2,\sigma_3)}.$$

To obtain an expression for C_2^+ involving $H^{(\beta_1+\frac{2}{\gamma},\beta_2,\beta_3)}_{(\sigma_1-\frac{1}{\gamma},\sigma_2,\sigma_3)}$ and no R function, we will need to apply the shift equation on R given in Theorem 4.2 to simplify the ratio $\frac{R(\beta_1,\sigma_1-\frac{1}{\gamma},\sigma_2-\frac{1}{\gamma})}{R(\beta_1+\frac{2}{\gamma},\sigma_1-\frac{1}{\gamma},\sigma_2)}$ and then the reflection principle given by equation (4.28). The same strategy can be applied to $\tilde{C}_2^+$ and $\tilde{B}_2^-$. This allows us to write:

$$C_2^+ = \chi^2\pi^{\frac{2\chi}{\gamma}-1}\frac{\Gamma(-1+\chi\beta_1-\chi^2)\Gamma(1-\chi\beta_1)}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}}\left(g_\chi(\sigma_1-\frac{\chi}{2})-g_\chi(\sigma_2+\frac{\beta_1}{2}-\frac{\chi}{2})\right)H^{(\beta_1+\chi,\beta_2,\beta_3)}_{(\sigma_1-\frac{\chi}{2},\sigma_2,\sigma_3)},$$

(4.59)
$$\tilde{C}_2^+ = \chi^2\pi^{\frac{2\chi}{\gamma}-1}\frac{\Gamma(-1+\chi\beta_1-\chi^2)\Gamma(1-\chi\beta_1)}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}}\left(g_\chi(\sigma_1+\frac{\chi}{2})-g_\chi(\sigma_2-\frac{\beta_1}{2}+\frac{\chi}{2})\right)H^{(\beta_1+\chi,\beta_2,\beta_3)}_{(\sigma_1+\frac{\chi}{2},\sigma_2,\sigma_3)},$$

(4.60)
$$\tilde{B}_2^- = \chi^2\pi^{\frac{2\chi}{\gamma}-1}\frac{\Gamma(-1+\chi\beta_2-\chi^2)\Gamma(1-\chi\beta_2)}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}}\left(g_\chi(\sigma_3)-g_\chi(\sigma_2+\frac{\beta_2}{2})\right)H^{(\beta_1,\beta_2+\chi,\beta_3)}_{(\sigma_1+\frac{\chi}{2},\sigma_2+\frac{\chi}{2},\sigma_3)}.$$

Putting all these into (4.57) and (4.58) proves the shift equations stated in the proposition. ∎

Appendix A. Background on special functions

A.1. **The hypergeometric equation.** Here we recall some facts we have used on the hypergeometric equation and its solution space. For $A > 0$ let $\Gamma(A) = \int_0^\infty t^{A-1}e^{-t}dt$ denote the standard Gamma function which can then be analytically extended to $\mathbb{C} \setminus \{-\mathbb{N}\}$. Record the following properties:

$$\Gamma(A+1) = A\Gamma(A), \quad \Gamma(A)\Gamma(1-A) = \frac{\pi}{\sin(\pi A)}, \quad \Gamma(A)\Gamma(A+\frac{1}{2}) = \sqrt{\pi}2^{1-2A}\Gamma(2A). \tag{A.1}$$

Let $(A)_n := \frac{\Gamma(A+n)}{\Gamma(A)}$. For A, B, C, and t real numbers we define the hypergeometric function F by:

$$F(A,B,C,t) := \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{n!(C)_n} t^n. \tag{A.2}$$

This function can be used to solve the following hypergeometric equation:

$$\left(t(1-t)\frac{d^2}{dt^2} + (C - (A+B+1)t)\frac{d}{dt} - AB \right) f(t) = 0. \tag{A.3}$$

We can give the following three bases of solutions corresponding respectively to $t \in (0,1)$, $t \in (1,+\infty)$ and $t \in (-\infty, 0)$. Under the assumption that C and $C-A-B$ are not an integers, for $t \in (0,1)$ we can write:

$$\begin{aligned} f(t) &= C_1 F(A,B,C,t) + C_2^+ t^{1-C} F(1+A-C, 1+B-C, 2-C, t) \\ &= B_1 F(A,B,1+A+B-C,1-t) \\ &+ B_2^-(1-t)^{C-A-B} F(C-A, C-B, 1+C-A-B, 1-t). \end{aligned} \tag{A.4}$$

Moving next to $t \in (1,+\infty)$, under the assumption that $C-A-B$ and $A-B$ are not integers:

$$\begin{aligned} f(t) &= B_1 F(A,B,1+A+B-C,1-t) \\ &+ B_2^+(1-t)^{C-A-B} F(C-A, C-B, 1+C-A-B, 1-t) \\ &= D_1 t^{-A} F(A, 1+A-C, 1+A-B, t^{-1}) \\ &+ D_2^- t^{-B} F(B, 1+B-C, 1+B-A, t^{-1}). \end{aligned} \tag{A.5}$$

Lastly for $t \in (-\infty, 0)$, under the assumption that C and $A-B$ are not an integer:

$$\begin{aligned} f(t) &= C_1 F(A,B,C,t) + C_2^- t^{1-C} F(1+A-C, 1+B-C, 2-C, t) \\ &= D_1 t^{-A} F(A, 1+A-C, 1+A-B, t^{-1}) \\ &+ D_2^+ t^{-B} F(B, 1+B-C, 1+B-A, t^{-1}). \end{aligned} \tag{A.6}$$

For each of the three cases we have four real constants that parametrize the solution space, namely C_1, C_2^+, B_1, B_2^-, B_1, B_2^+, D_1, D_2^- and D_1, D_2^+, C_1, C_2^-. We thus expect to have an explicit change of basis formula that will give a link between C_1, C_2^+, B_1, B_2^-, and similarly for the two other cases. This is precisely what gives the so-called connection formulas,

$$\begin{pmatrix} C_1 \\ C_2^- \end{pmatrix} = \begin{pmatrix} \frac{\Gamma(1-C)\Gamma(A-B+1)}{\Gamma(A-C+1)\Gamma(1-B)} & \frac{\Gamma(1-C)\Gamma(B-A+1)}{\Gamma(B-C+1)\Gamma(1-A)} \\ \frac{\Gamma(C-1)\Gamma(A-B+1)}{\Gamma(A)\Gamma(C-B)} & \frac{\Gamma(C-1)\Gamma(B-A+1)}{\Gamma(B)\Gamma(C-A)} \end{pmatrix} \begin{pmatrix} D_1 \\ D_2^+ \end{pmatrix}, \tag{A.7}$$

$$\begin{pmatrix} B_1 \\ B_2^- \end{pmatrix} = \begin{pmatrix} \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} & \frac{\Gamma(2-C)\Gamma(C-A-B)}{\Gamma(1-A)\Gamma(1-B)} \\ \frac{\Gamma(C)\Gamma(A+B-C)}{\Gamma(A)\Gamma(B)} & \frac{\Gamma(2-C)\Gamma(A+B-C)}{\Gamma(A-C+1)\Gamma(B-C+1)} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2^+ \end{pmatrix}. \tag{A.8}$$

Note that in our present case we have $C_2^+ \neq C_2^-$, $B_2^+ \neq B_2^-$, $D_2^+ \neq D_2^-$, which is why we must be distinguish three cases based on which interval t belongs to.

A.2. **Double Gamma and Sine functions.** We will now provide some explanations on the functions $\Gamma_{\frac{\gamma}{2}}(x)$ and $S_{\frac{\gamma}{2}}(x)$ that we have introduced. For all $\gamma \in (0,2)$ and for $\mathrm{Re}(x) > 0$, $\Gamma_{\frac{\gamma}{2}}(x)$ is defined by the integral formula,

$$\ln \Gamma_{\frac{\gamma}{2}}(x) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-xt} - e^{-\frac{Qt}{2}}}{(1 - e^{-\frac{\gamma t}{2}})(1 - e^{-\frac{2t}{\gamma}})} - \frac{(\frac{Q}{2} - x)^2}{2} e^{-t} + \frac{x - \frac{Q}{2}}{t}\right], \tag{A.9}$$

where we have $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$. Since the function $\Gamma_{\frac{\gamma}{2}}(x)$ is continuous it is completely determined by the following two shift equations

$$\frac{\Gamma_{\frac{\gamma}{2}}(x)}{\Gamma_{\frac{\gamma}{2}}(x + \frac{\gamma}{2})} = \frac{1}{\sqrt{2\pi}} \Gamma(\frac{\gamma x}{2}) (\frac{\gamma}{2})^{-\frac{\gamma x}{2} + \frac{1}{2}}, \tag{A.10}$$

$$\frac{\Gamma_{\frac{\gamma}{2}}(x)}{\Gamma_{\frac{\gamma}{2}}(x + \frac{2}{\gamma})} = \frac{1}{\sqrt{2\pi}} \Gamma(\frac{2x}{\gamma}) (\frac{\gamma}{2})^{\frac{2x}{\gamma} - \frac{1}{2}}, \tag{A.11}$$

and by its value in $\frac{Q}{2}$, $\Gamma_{\frac{\gamma}{2}}(\frac{Q}{2}) = 1$. Furthermore $x \mapsto \Gamma_{\frac{\gamma}{2}}(x)$ admits a meromorphic extension to all of $\mathbb{C}$ with single poles at $x = -n\frac{\gamma}{2} - m\frac{2}{\gamma}$ for any $n, m \in \mathbb{N}$ and $\Gamma_{\frac{\gamma}{2}}(x)$ is never equal to 0. We have also used the double sine function defined by:

$$S_{\frac{\gamma}{2}}(x) = \frac{\Gamma_{\frac{\gamma}{2}}(x)}{\Gamma_{\frac{\gamma}{2}}(Q - x)}. \tag{A.12}$$

It obeys the following two shift equations:

$$\frac{S_{\frac{\gamma}{2}}(x + \frac{\gamma}{2})}{S_{\frac{\gamma}{2}}(x)} = 2\sin(\frac{\gamma\pi}{2}x), \quad \frac{S_{\frac{\gamma}{2}}(x + \frac{2}{\gamma})}{S_{\frac{\gamma}{2}}(x)} = 2\sin(\frac{2\pi}{\gamma}x). \tag{A.13}$$

The double sine function admits a meromorphic extension to $\mathbb{C}$ with poles at $x = -n\frac{\gamma}{2} - m\frac{2}{\gamma}$ and with zeros at $x = Q + n\frac{\gamma}{2} + m\frac{2}{\gamma}$ for any $n, m \in \mathbb{N}$. We also record the following asymptotic for $S_{\frac{\gamma}{2}}(x)$:

$$S_{\frac{\gamma}{2}}(x) \sim \begin{cases} e^{-i\frac{\pi}{2}x(x-Q)} & \text{as} \quad \mathrm{Im}(x) \to \infty, \\ e^{i\frac{\pi}{2}x(x-Q)} & \text{as} \quad \mathrm{Im}(x) \to -\infty. \end{cases} \tag{A.14}$$

A.3. **The Ponsot-Teschner function.** We detail here some properties of the exact formula H_{PT} given by equation (1.8). What needs to be understood properly is the contour integral in the expression of H_{PT}, for convenience we introduce the notation:

(A.15)

$$\mathcal{J}_{PT} = \int_{\mathcal{C}} \frac{S_{\frac{\gamma}{2}}(-\frac{\beta_2}{2} + \sigma_2 + \sigma_3 + r) S_{\frac{\gamma}{2}}(Q - \frac{\beta_2}{2} + \sigma_3 - \sigma_2 + r) S_{\frac{\gamma}{2}}(\frac{\beta_3}{2} + \sigma_3 - \sigma_1 + r) S_{\frac{\gamma}{2}}(Q - \frac{\beta_3}{2} + \sigma_3 - \sigma_1 + r)}{S_{\frac{\gamma}{2}}(Q + \frac{\beta_1}{2} - \frac{\beta_2}{2} + \sigma_3 - \sigma_1 + r) S_{\frac{\gamma}{2}}(2Q - \frac{\beta_1}{2} - \frac{\beta_2}{2} + \sigma_3 - \sigma_1 + r) S_{\frac{\gamma}{2}}(2\sigma_3 + r) S_{\frac{\gamma}{2}}(Q + r)} \frac{dr}{i}.$$

The functions H_{PT} and $\mathcal{J}_{PT}$ are then related by an explicit prefactor containing the $\Gamma_{\frac{\gamma}{2}}$ and $S_{\frac{\gamma}{2}}$. The contour of integration $\mathcal{C}$ in $\mathcal{J}_{PT}$ goes from $-i\infty$ to $i\infty$ passing to the right of the poles at $r = -(-\frac{\beta_2}{2} + \sigma_2 + \sigma_3) - n\frac{\gamma}{2} - m\frac{2}{\gamma}$, $r = -(Q - \frac{\beta_2}{2} + \sigma_3 - \sigma_2) - n\frac{\gamma}{2} - m\frac{2}{\gamma}$, $r = -(\frac{\beta_3}{2} + \sigma_3 - \sigma_1) - n\frac{\gamma}{2} - m\frac{2}{\gamma}$, $r = -(Q - \frac{\beta_3}{2} + \sigma_3 - \sigma_1) - n\frac{\gamma}{2} - m\frac{2}{\gamma}$ and to the left of the poles at $r = -(\frac{\beta_1}{2} - \frac{\beta_2}{2} + \sigma_3 - \sigma_1) + n\frac{\gamma}{2} + m\frac{2}{\gamma}$, $r = -(Q - \frac{\beta_1}{2} - \frac{\beta_2}{2} + \sigma_3 - \sigma_1) + n\frac{\gamma}{2} + m\frac{2}{\gamma}$, $r = -(2\sigma_3 - Q) + n\frac{\gamma}{2} + m\frac{2}{\gamma}$, $r = n\frac{\gamma}{2} + m\frac{2}{\gamma}$ with $m, n \in \mathbb{N}^2$. The following lemma verifies that the integral is converging at $\pm i\infty$.

Lemma A.1. *The integral in (A.15) is absolutely converging at $\pm i\infty$.*

Proof. We need to determine the asymptotic in r of the integrand of the contour integral as r goes to $\pm i\infty$. For this we simply need to use the asymptotic of $S_{\frac{\gamma}{2}}$ given by equation (A.14). Using this fact, one obtains that the integrand as $r \to +i\infty$ is equivalent to $c_1 e^{3i\pi Q r}$ and as $r \to -i\infty$ to $c_2 e^{-3i\pi Q r}$ for some constants $c_1, c_2 \in \mathbb{C}$ independent of r. Since $Q > 0$, the integral is absolutely convergent. □

We now state a lemma giving the poles of $\mathcal{J}_{PT}$, viewed as a meromorphic function over $\mathbb{C}^6$ of its six parameters $\beta_1, \beta_2, \beta_3, \sigma_1, \sigma_2, \sigma_3$.

Lemma A.2. *The poles of the function $\mathcal{J}_{PT}$ occur when $\zeta = n\frac{\gamma}{2} + m\frac{2}{\gamma}$ where $n, m \in \mathbb{N}$ and ζ is equal to any of the following*

$$\begin{array}{llll}
\frac{\beta_1}{2} - \sigma_1 - \sigma_2, & Q - \frac{\beta_1}{2} - \sigma_1 - \sigma_2, & \frac{\beta_2}{2} - \sigma_2 + \sigma_3 - Q, & \frac{\beta_2}{2} - \sigma_2 - \sigma_3, \\
-Q + \sigma_2 + \frac{\beta_1}{2} - \sigma_1, & \sigma_2 - \sigma_1 - \frac{\beta_1}{2}, & -2Q + \frac{\beta_2}{2} + \sigma_3 + \sigma_2, & -Q + \frac{\beta_2}{2} - \sigma_3 + \sigma_2, \\
\frac{\beta_1}{2} - \frac{\beta_2}{2} - \frac{\beta_3}{2}, & Q - \frac{\beta_1}{2} - \frac{\beta_2}{2} - \frac{\beta_3}{2}, & -Q - \frac{\beta_3}{2} + \sigma_3 + \sigma_1, & -\frac{\beta_3}{2} - \sigma_3 + \sigma_1, \\
-Q + \frac{\beta_1}{2} - \frac{\beta_2}{2} + \frac{\beta_3}{2}, & \frac{\beta_3}{2} - \frac{\beta_1}{2} - \frac{\beta_2}{2}, & -2Q + \frac{\beta_3}{2} + \sigma_3 + \sigma_1, & -Q + \frac{\beta_3}{2} - \sigma_3 + \sigma_1.
\end{array}$$

Proof. The proof exactly the same steps as the one given in [RZ21]. □

From here it is immediate that H_{PT} is a meromorphic function on $\mathbb{C}^6$.

Lemma A.3. *The function H_{PT} is meromorphic on $\mathbb{C}^6$ as a function of the six parameters $\beta_1, \beta_2, \beta_3, \sigma_1, \sigma_2, \sigma_3$.*

Proof. In Lemma A.2 we have established that $\mathcal{J}_{PT}$ is meromorphic on $\mathbb{C}^6$ with prescribed poles. Since H_{PT} is equal to $\mathcal{J}_{PT}$ times an explicit prefactor containing $\Gamma_{\frac{\gamma}{2}}$ and $S_{\frac{\gamma}{2}}$ functions which are meromorphic, this implies the claim. □

For the purpose of proving Theorem 1.1, we establish the following two facts about H_{PT}.

Lemma A.4. *The function H_{PT} satisfies the shift equations of Theorem 4.1 satisfied by H.*

Proof. Recalling (1.8), we introduce a function φ in the following way

$$H_{\mathrm{PT}}\begin{pmatrix}\beta_1, \beta_2, \beta_3 \\ \sigma_1, \sigma_2, \sigma_3\end{pmatrix} = \int_{\mathcal{C}} \varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(r)dr, \tag{A.16}$$

where here $\varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(r)$ contains the r dependent integrand of the contour integral $\int_{\mathcal{C}}$ present in (1.8) times the prefactor in front of the integral (which does not depend on r).

Checking that H_{PT} satisfies the shift equations of Theorem 4.1 is equivalent to checking the following shift equations,

(A.17)

$$\begin{aligned}
H_{\mathrm{PT}}\begin{pmatrix}\beta_1, \beta_2, \beta_3 \\ \sigma_1, \sigma_2, \sigma_3\end{pmatrix} =& \frac{\Gamma(\chi(\beta_1-\chi))\Gamma(1-\chi\beta_2)}{\Gamma(1-\frac{\chi}{2}(\beta_2+\beta_3-\beta_1))\Gamma(\frac{\chi}{2}(\beta_1+\beta_3-\beta_2-2\chi))} H_{\mathrm{PT}}\begin{pmatrix}\beta_1-\chi, \beta_2+\chi, \beta_3 \\ \sigma_1, \sigma_2+\frac{\chi}{2}, \sigma_3\end{pmatrix} \\
&+ \chi^2 \left(\frac{\pi\mu\Gamma(\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\right)^{\frac{\chi}{\gamma}} \frac{\Gamma(1-\chi\beta_1)\Gamma(1-\chi\beta_2)}{\sin(\pi\chi(\chi-\beta_1))\Gamma(1+\chi(Q-\frac{\bar{\beta}}{2}))\Gamma(1-\frac{\chi}{2}(\beta_1+\beta_2-\beta_3))} \\
&\times 2\sin(\pi\chi(\frac{\beta_1}{2}+\sigma_1+\sigma_2-Q))\sin(\pi\chi(\frac{\beta_1}{2}-\sigma_1+\sigma_2)) H_{\mathrm{PT}}\begin{pmatrix}\beta_1+\chi, \beta_2+\chi, \beta_3 \\ \sigma_1, \sigma_2+\frac{\chi}{2}, \sigma_3\end{pmatrix},
\end{aligned}$$

and:

(A.18)
$$\frac{\chi^2}{\pi}\left(\frac{\pi\mu\Gamma(\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\right)^{\frac{\chi}{\gamma}}\Gamma(1-\chi\beta_2)2\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2+\sigma_3-Q))\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2-\sigma_3))H_{\mathrm{PT}}\begin{pmatrix}\beta_1+\chi,\beta_2+\chi,\beta_3\\ \sigma_1,\sigma_2+\frac{\chi}{2},\sigma_3\end{pmatrix}$$
$$=\frac{\Gamma(\chi\beta_1)}{\Gamma(\frac{\chi}{2}(\bar\beta-2Q))\Gamma(\frac{\chi}{2}(\beta_1+\beta_2-\beta_3))}H_{\mathrm{PT}}\begin{pmatrix}\beta_1,\beta_2,\beta_3\\ \sigma_1,\sigma_2,\sigma_3\end{pmatrix}$$
$$-\chi^2\left(\frac{\pi\mu\Gamma(\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\right)^{\frac{\chi}{\gamma}}\frac{2\sin(\pi\chi(\frac{\beta_1}{2}-\sigma_1-\sigma_2+Q))\sin(\pi\chi(\frac{\beta_1}{2}+\sigma_1-\sigma_2))\Gamma(1-\chi\beta_1-\chi^2)}{\sin(\pi\chi\beta_1)\Gamma(\frac{\chi}{2}(\beta_2+\beta_3-\beta_1-2\chi))\Gamma(1-\frac{\chi}{2}(\beta_1+\beta_3-\beta_2))}H_{\mathrm{PT}}\begin{pmatrix}\beta_1+2\chi,\beta_2,\beta_3\\ \sigma_1,\sigma_2,\sigma_3\end{pmatrix}.$$

This formulation of the two shift equations has been derived from Theorem 4.1 by shifting β_2 to $\beta_2+\chi$ in (4.2) for the first and shifting β_1 to $\beta_1+\chi$ in (4.3) for the second. We have also used the explicit expression (4.1) for g_χ and written the difference of cosines as a product of sines. We now compute the following ratios of φ,

$$\frac{\varphi^{(\beta_1-\chi,\beta_2+\chi,\beta_3)}_{(\sigma_1,\sigma_2+\frac{\chi}{2},\sigma_3)}(r)}{\varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(r)}=\frac{\Gamma(\frac{\chi}{2}(\beta_1+\beta_3-\beta_2-2\chi))\Gamma(1-\frac{\chi}{2}(\beta_2+\beta_3-\beta_1))\Gamma(1-\chi\beta_1+\chi^2)}{\pi\Gamma(1-\chi\beta_2)}\times\sin(\pi\chi(\frac{\beta_1}{2}-\chi+\sigma_1-\sigma_2))\frac{\sin(\pi\chi(-\frac{\beta_1}{2}+\frac{\beta_2}{2}+\sigma_1-\sigma_3-r))}{\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2-\sigma_3-r))},\tag{A.19}$$

$$\frac{\varphi^{(\beta_1+\chi,\beta_2+\chi,\beta_3)}_{(\sigma_1,\sigma_2+\frac{\chi}{2},\sigma_3)}(r)}{\varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(r)}=\chi^{-2}\left(\frac{\pi\mu\Gamma(\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\right)^{-\frac{\chi}{\gamma}}\frac{\Gamma(1+\chi(Q-\frac{\bar\beta}{2}))\Gamma(1-\frac{\chi}{2}(\beta_1+\beta_2-\beta_3))}{\Gamma(1-\chi\beta_1)\Gamma(1-\chi\beta_2)}\times\frac{\sin(\pi\chi(\frac{\beta_1}{2}+\frac{\beta_2}{2}-\chi+\sigma_1-\sigma_3-r))}{2\sin(\pi\chi(\frac{\beta_1}{2}-\chi+\sigma_1+\sigma_2))\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2-\sigma_3-r))}.\tag{A.20}$$

If we plug these expressions into equation (A.17) and regroup the terms on one side we get:

$$\int_{\mathcal{C}}dr\,\varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(r)\Bigg[\frac{\sin(\pi\chi(\frac{\beta_1}{2}-\chi+\sigma_1-\sigma_2))\sin(\pi\chi(-\frac{\beta_1}{2}+\frac{\beta_2}{2}+\sigma_1-\sigma_3-r))}{\sin(\pi\chi(\beta_1-\chi))\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2-\sigma_3-r))}-1+\frac{\sin(\pi\chi(\frac{\beta_1}{2}-\sigma_1+\sigma_2))\sin(\pi\chi(\frac{\beta_1}{2}+\frac{\beta_2}{2}-\chi+\sigma_1-\sigma_3-r))}{\sin(\pi\chi(\beta_1-\chi))\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2-\sigma_3-r))}\Bigg].\tag{A.21}$$

We can verify with some algebra that the integrand of the above integral equals 0, hence (A.17) holds. To check the second shift equation, we will need additionally the ratio:

(A.22)
$$\frac{\varphi^{(\beta_1+2\chi,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(r)}{\varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(r)}=-\pi\chi^{-2}\left(\frac{\pi\mu\Gamma(\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\right)^{-\frac{\chi}{\gamma}}\frac{\Gamma(1+\chi(Q-\frac{\bar\beta}{2}))\Gamma(1-\frac{\chi}{2}(\beta_1+\beta_2-\beta_3))}{\Gamma(\frac{\chi}{2}(\beta_1+\beta_3-\beta_2))\Gamma(1-\frac{\chi}{2}(\beta_2+\beta_3-\beta_1-2\chi))\Gamma(1-\chi\beta_1-\chi^2))}$$

(A.23)
$$\times\frac{1}{\Gamma(1-\chi\beta_1)}\frac{\sin(\pi\chi(\frac{\beta_1}{2}+\frac{\beta_2}{2}-\chi+\sigma_1-\sigma_3-r))}{2\sin(\pi\chi(\frac{\beta_1}{2}+\sigma_1-\sigma_2))\sin(\pi\chi(\frac{\beta_1}{2}+\sigma_1+\sigma_2-\chi))\sin(\pi\chi(\frac{\beta_1}{2}-\frac{\beta_2}{2}+\chi+\sigma_3-\sigma_1+r))}.$$

Substituting this time into equation (A.18) and regrouping terms on one side we get:

$$\frac{\Gamma(\chi\beta_1)}{\Gamma(\frac{\chi}{2}(\overline{\beta}-2Q))\Gamma(\frac{\chi}{2}(\beta_1+\beta_2-\beta_3))}\int_{\mathcal{C}} dr\, \varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(r)$$

$$\Bigg[\frac{\sin(\pi\chi\beta_1)\sin(\pi\chi(\frac{\beta_2}{2}-\chi+\sigma_2+\sigma_3))\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2-\sigma_3))\sin(\pi\chi(\frac{\beta_1}{2}+\frac{\beta_2}{2}-\chi+\sigma_1-\sigma_3-r))}{\sin(\pi\chi(\frac{\bar\beta}{2}-\chi))\sin(\frac{\pi\chi}{2}(\beta_1+\beta_2-\beta_3))\sin(\pi\chi(\frac{\beta_1}{2}-\chi+\sigma_1+\sigma_2))\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2-\sigma_3-r))}-1$$

$$-\frac{\sin(\frac{\pi\chi}{2}(\beta_1+\beta_3-\beta_2))\sin(\frac{\pi\chi}{2}(\beta_2+\beta_3-\beta_1-2\chi))\sin(\pi\chi(\frac{\beta_1}{2}+\chi-\sigma_1-\sigma_2))\sin(\pi\chi(\frac{\beta_1}{2}+\frac{\beta_2}{2}-\chi+\sigma_1-\sigma_3-r))}{\sin(\chi(\frac{\bar\beta}{2}-\chi))\sin(\frac{\pi\chi}{2}(\beta_1+\beta_2-\beta_3))\sin(\pi\chi(\frac{\beta_1}{2}-\chi+\sigma_1+\sigma_2))\sin(\pi\chi(\frac{\beta_1}{2}-\frac{\beta_2}{2}+\chi+\sigma_3-\sigma_1+r))}\Bigg].$$

After some algebra we can write this quantity in the form:

$$\frac{\Gamma(\chi\beta_1)}{\Gamma(\frac{\chi}{2}(\overline{\beta}-2Q))\Gamma(\frac{\chi}{2}(\beta_1+\beta_2-\beta_3))}\frac{\sin(\pi\chi\beta_1)}{\sin(\pi\chi(\frac{\overline{\beta}}{2}-\chi))\sin(\frac{\pi\chi}{2}(\beta_1+\beta_2-\beta_3))\sin(\pi\chi(\frac{\beta_1}{2}-\chi+\sigma_1+\sigma_2))}$$

$$\times\int_{\mathcal{C}} dr\, \varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(r)\Bigg[\frac{\sin(\pi\chi(\frac{\beta_1}{2}+\frac{\beta_2}{2}-\chi+\sigma_1-\sigma_3-r))\sin(\pi\chi r)\sin(\pi\chi(2\sigma_3-\chi+r))}{\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2-\sigma_3-r))}$$

$$-\frac{\sin(\pi\chi(\frac{\beta_3}{2}+\sigma_3-\sigma_1+r))\sin(\pi\chi(\frac{\beta_3}{2}-\chi+\sigma_1-\sigma_3-r))\sin(\pi\chi(\frac{\beta_2}{2}-\sigma_2-\sigma_3-r))}{\sin(\pi\chi(\frac{\beta_1}{2}-\frac{\beta_2}{2}+\chi+\sigma_3-\sigma_1+r))}\Bigg]$$

$$=\frac{\Gamma(\chi\beta_1)}{\Gamma(\frac{\chi}{2}(\overline{\beta}-2Q))\Gamma(\frac{\chi}{2}(\beta_1+\beta_2-\beta_3))}$$

$$\times\frac{\sin(\pi\chi\beta_1)\sin(\pi\chi(\frac{\beta_3}{2}+\sigma_3-\sigma_1))}{\sin(\pi\chi(\frac{\overline{\beta}}{2}-\chi))\sin(\frac{\pi\chi}{2}(\beta_1+\beta_2-\beta_3))\sin(\pi\chi(\frac{\beta_1}{2}-\chi+\sigma_1-\sigma_2))\sin(\pi\chi(\frac{\beta_3}{2}-\sigma_1-\sigma_3))}$$

$$\times\int_{\mathcal{C}} dr\,(T_{-\chi}-1)\left(\frac{\sin(\pi\chi(\frac{\beta_1}{2}+\frac{\beta_2}{2}-\chi+\sigma_1-\sigma_3-r))\sin(\pi\chi(2\sigma_3-\chi+r))\sin(\pi\chi(2\sigma_3-2\chi+r))}{\sin(\pi\chi(\frac{\beta_2}{2}+\chi-\sigma_2-\sigma_3-r))}\varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2+\chi,\sigma_3+\chi)}(r)\right),$$

Here we have used the notation $T_{-\chi}$ for the operator that shifts the argument of the function it is applied to by $-\chi$, the variable we are shifting being r. Since we have the combination $(T_{-\chi}-1)$ in the integrand, this corresponds to integrating r over two contours that are separated by a shift horizontal shift of χ and which are in the opposite direction. Provided that there are no poles in between these two contours, the whole contour integral will be equal to 0. This is indeed the case thanks to the way the contour $\mathcal{C}$ has been chosen, which is to the right of the half-lattice of poles extending in the $-\infty$ direction, and to the left of the half-lattice of poles extending in the $+\infty$ direction. □

Next we move on to showing:

Lemma A.5. *The function H_{PT} satisfies the following property:*

$$\lim_{\beta_1\to 2Q-\beta_2-\beta_3}(\frac{\bar\beta}{2}-Q)H_{\text{PT}}\begin{pmatrix}\beta_1,\beta_2,\beta_3\\ \sigma_1,\sigma_2,\sigma_3\end{pmatrix}=1. \tag{A.24}$$

Furthermore H_{PT} satisfies the reflection principle (4.28) of Lemma 4.9 satisfied by H, namely:

$$H_{\text{PT}}{}^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}=R_{\text{FZZ}}(\beta_1,\sigma_1,\sigma_2)H_{\text{PT}}{}^{(2Q-\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}. \tag{A.25}$$

Proof. It is rather direct to observe that H_{PT} satisfies the reflection principle (4.28), since the contour integral is not changed when applying the transform $\beta_1\to 2Q-\beta_1$. The evaluation of the residue is an easy algebra using the shift equations of $\Gamma_{\frac{\gamma}{2}}$ and $S_{\frac{\gamma}{2}}$. When β_1 approaches $2Q-\beta_2-\beta_3$ from the right hand side, the two poles at $r=-(\frac{\beta_3}{2}+\sigma_3-\sigma_1)$ and $r=-(Q-\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1)$ in the contour integral will collapse. To extract the divergent term, we can slightly modify the contour to let it go from the right hand side of $r = -(Q - \frac{\beta_1}{2} - \frac{\beta_2}{2} + \sigma_3 - \sigma_1)$, this allows us to pick up the divergent term by residue theorem:

$$\int_{\mathcal{C}} \frac{S_{\frac{\gamma}{2}}(-\frac{\beta_2}{2} + \sigma_2 + \sigma_3 + r) S_{\frac{\gamma}{2}}(Q - \frac{\beta_2}{2} + \sigma_3 - \sigma_2 + r) S_{\frac{\gamma}{2}}(\frac{\beta_3}{2} + \sigma_3 - \sigma_1 + r) S_{\frac{\gamma}{2}}(Q - \frac{\beta_3}{2} + \sigma_3 - \sigma_1 + r)}{S_{\frac{\gamma}{2}}(Q + \frac{\beta_1}{2} - \frac{\beta_2}{2} + \sigma_3 - \sigma_1 + r) S_{\frac{\gamma}{2}}(2Q - \frac{\beta_1}{2} - \frac{\beta_2}{2} + \sigma_3 - \sigma_1 + r) S_{\frac{\gamma}{2}}(2\sigma_3 + r) S_{\frac{\gamma}{2}}(Q + r)} \frac{dr}{i}$$

$$\underset{\beta_1 \to 2Q - \beta_2 - \beta_3}{\sim} \frac{1}{2\pi(\frac{\overline{\beta}}{2} - Q)} \frac{S_{\frac{\gamma}{2}}(\frac{\beta_1}{2} + \sigma_1 + \sigma_2 - Q) S_{\frac{\gamma}{2}}(\frac{\beta_1}{2} + \sigma_1 - \sigma_2) S_{\frac{\gamma}{2}}(Q - \beta_3)}{S_{\frac{\gamma}{2}}(\beta_1) S_{\frac{\gamma}{2}}(-\frac{\beta_3}{2} + \sigma_1 + \sigma_3) S_{\frac{\gamma}{2}}(Q - \frac{\beta_3}{2} + \sigma_1 - \sigma_3)}.$$

We can check that when $\beta_1 \to 2Q - \beta_2 - \beta_3$, the preceding term is equivalent to

$$2\pi(\frac{\overline{\beta}}{2} - Q) \frac{S_{\frac{\gamma}{2}}(\beta_1) S_{\frac{\gamma}{2}}(\beta_3)}{S_{\frac{\gamma}{2}}(\frac{\beta_1}{2} + \sigma_1 - \sigma_2) S_{\frac{\gamma}{2}}(\frac{\beta_3}{2} + \sigma_3 - \sigma_1) S_{\frac{\gamma}{2}}(\frac{\beta_1}{2} + \sigma_1 + \sigma_2 - Q) S_{\frac{\gamma}{2}}(\frac{\beta_3}{2} - \sigma_1 - \sigma_3 + Q)} \tag{A.26}$$

This proves that $\lim_{\beta_1 \to 2Q - \beta_2 - \beta_3} (\frac{\bar{\beta}}{2} - Q) H_{\mathrm{PT}} \begin{pmatrix} \beta_1, \beta_2, \beta_3 \\ \sigma_1, \sigma_2, \sigma_3 \end{pmatrix} = 1.$

□

Appendix B. Analytic continuation in β

Proposition B.1. *Fix μ_i with $\Re \mu_i > 0$ for $i = 1, 2, 3$. For the domain*

$$V := \{(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 \, : \, Q - \frac{1}{2} \sum \beta_i < \gamma \wedge \frac{2}{\gamma} \wedge \min_i (Q - \beta_i) \text{ and } \beta_i < Q \text{ for } i = 1, 2, 3\}, \tag{B.1}$$

each of the functions

$$(\beta_1, \beta_2, \beta_3) \mapsto \int A e^{-A - \sum \mu_i L_i} \, \mathrm{LF}_{\mathbb{H}}^{(\beta_1, 0)(\beta_2, 1), (\beta_3, \infty)}(d\phi),$$

$$(\beta_1, \beta_2, \beta_3) \mapsto \int A (\sum \mu_i L_i) e^{-A - \sum \mu_i L_i} \, \mathrm{LF}_{\mathbb{H}}^{(\beta_1, 0)(\beta_2, 1), (\beta_3, \infty)}(d\phi)$$

$$(\beta_1, \beta_2, \beta_3) \mapsto \int (\sum \mu_i L_i)^2 e^{-A - \sum \mu_i L_i} \, \mathrm{LF}_{\mathbb{H}}^{(\beta_1, 0)(\beta_2, 1), (\beta_3, \infty)}(d\phi)$$

is well-defined in the sense that the integrals converge absolutely, and extends analytically to a neighborhood of V in $\mathbb{C}^3$. Here, we write $A = \mathcal{A}_\phi(\mathbb{H})$, $L_1 = \mathcal{L}_\phi(-\infty, 0)$, $L_2 = \mathcal{L}_\phi(0, 1)$ and $L_3 = \mathcal{L}_\phi(1, \infty)$.

Proof. We will prove the result for the second function; the others are proved similarly, with the same inputs. Moreover, we will assume that $\mu_1 = \mu_2 = \mu_3 = \mu$ for simplicity; the general case follows by a similar argument. By changing coordinates [AHS21, Lemma 2.20, Proposition 2.16], setting $(p_1, p_2, p_3) = (0, 2, -2)$, it suffices to prove that the function $f : V \mapsto \mathbb{C}$ given by

$$f(\beta_1, \beta_2, \beta_3) = \int \mathcal{A}_\phi(\mathbb{H}) \mathcal{L}_\phi(\mathbb{R}) e^{-\mathcal{A}_\phi(\mathbb{H}) - \mu \mathcal{L}_\phi(\mathbb{R})} \, \mathrm{LF}_{\mathbb{H}}^{(\beta_j, p_j)_j}(d\phi)$$

extends analytically to a neighborhood of V in $\mathbb{C}^3$.

To show this, we will approximate f using truncations of $\mathbb{H}$ away from the insertions. For $r \geq 1$, let $\mathbb{H}_r := \mathbb{H} \backslash \bigcup_i B_{e^{-r}}(p_i)$ and $\mathbb{R}_r := \mathbb{R} \backslash \bigcup_i B_{e^{-r}}(p_i)$. For a sample $h \sim P_{\mathbb{H}}$, write $A_r := \mathcal{A}_{h - 2Q \log |\cdot|_+}(\mathbb{H}_r)$ and $L_r := \mathcal{L}_{h - 2Q \log |\cdot|_+}(\mathbb{R}_r)$, and for $x \in \mathbb{R}$ write $h_r(x)$ for the average of h on $\partial B_{e^{-r}}(x) \cap \mathbb{H}$. For $r \geq 1$, define the function

$$f_r(\beta_1, \beta_2, \beta_3) := \int_{\mathbb{R}} dc \, e^{(\frac{1}{2} \sum \beta_i - Q)c} \mathbb{E} \left[\prod_i e^{\frac{\beta_i}{2} h_r(x_i) - \frac{\beta_i^2}{8} \mathbb{E}[h_r(x_i)^2]} (e^{\gamma c} A_r)(e^{\frac{\gamma}{2} c} L_r) \exp(-e^{\gamma c} A_r - \mu e^{\frac{\gamma}{2} c} L_r) \right]$$

on a neighborhood of V in $\mathbb{C}^3$ to be specified later. Clearly f_r is analytic.

Step 1: Showing $f_r|_V$ converges pointwise to f. By Girsanov's theorem, this is equivalent to showing that for all $(\beta_1,\beta_2,\beta_3)\in V$ we have

$$\lim_{r\to\infty}\int \mathcal{A}_\phi(\mathbb{H}_r)\mathcal{L}_\phi(\mathbb{R}_r)e^{-\mathcal{A}_\phi(\mathbb{H}_r)-\mu\mathcal{L}_\phi(\mathbb{R}_r)}\,\mathrm{LF}_{\mathbb{H}}^{(\beta_j,p_j)_j}(d\phi)=f(\beta_1,\beta_2,\beta_3).$$

We will produce a function $g(\phi)$ such that $g(\phi)\geq \mathcal{A}_\phi(\mathbb{H}_r)\mathcal{L}_\phi(\mathbb{R}_r)e^{-\mathcal{A}_\phi(\mathbb{H}_r)-\mu\mathcal{L}_\phi(\mathbb{R}_r)}$ and $\int g(\phi)\,\mathrm{LF}_{\mathbb{H}}^{(\beta_j,p_j)_j}(d\phi)<\infty$. Then, since $\lim_{r\to\infty}\mathcal{A}_\phi(\mathbb{H}_r)=\mathcal{A}_\phi(\mathbb{H})$ and $\lim_{r\to\infty}\mathcal{L}_\phi(\mathbb{R}_r)=\mathcal{L}_\phi(\mathbb{R})$, the dominated convergence theorem completes this step.

Since the function $\ell\mapsto \ell e^{-\mu\ell}$ is bounded, it suffices to dominate $\mathcal{A}_\phi(\mathbb{H}_r)e^{-\mathcal{A}_\phi(\mathbb{H}_r)}$ by $g(\phi)$. We choose $g(\phi)=\mathcal{A}_\phi(\mathbb{H})e^{-\mathcal{A}_\phi(\mathbb{H})}+\mathcal{A}_\phi(\mathbb{H}_1)e^{-\mathcal{A}_\phi(\mathbb{H}_1)}+1_{\mathcal{A}_\phi(\mathbb{H}_1)<1<\mathcal{A}_\phi(\mathbb{H})}$; this dominates because $\mathcal{A}_\phi(\mathbb{H}_r)\in[\mathcal{A}_\phi(\mathbb{H}_1),\mathcal{A}_\phi(\mathbb{H})]$, and $a\mapsto e^{-a}$ is increasing on $[0,1]$ and decreasing on $[1,\infty)$. Each of $\mathcal{A}_\phi(\mathbb{H})$ and $\mathcal{A}_\phi(\mathbb{H}_1)$ has a power law with exponent $\frac{s}{\gamma}-1$ (REF), where $s=\frac{1}{2}\sum\beta_1-Q>-\gamma$. Thus each of $\mathcal{A}_\phi(\mathbb{H})e^{-\mathcal{A}_\phi(\mathbb{H})}$ and $\mathcal{A}_\phi(\mathbb{H}_1)e^{-\mathcal{A}_\phi(\mathbb{H}_1)}$ are integrable, and when $s>0$ (resp. $s<0$), the function $1_{\mathcal{A}_\phi(\mathbb{H}_1)<1}$ (resp. $1_{1<\mathcal{A}_\phi(\mathbb{H})}$) is integrable and dominates $1_{\mathcal{A}_\phi(\mathbb{H}_1)<1<\mathcal{A}_\phi(\mathbb{H})}$. Therefore, when $s\neq 0$ the function g is integrable, and continuity settles the case when $s=0$.

Step 2: Showing that each point in V has an open neighborhood in $\mathbb{C}^3$ on which f_r converges uniformly as $r\to\infty$.

If we condition on $h|_{\mathbb{H}_r}$, then $(h_{r+1}(p_i)-h_r(p_i))_{i=1,2,3}$ is a triple of conditionally independent Gaussians. Therefore we have the alternative expression

$$f_r(\beta_1,\beta_2,\beta_3)=\int_{\mathbb{R}}dc\,e^{(\frac{1}{2}\sum\beta_i-Q)c}\mathbb{E}\left[\prod_i e^{\frac{\beta_i}{2}h_{r+1}(x_i)-\frac{\beta_i^2}{8}\mathbb{E}[h_{r+1}(x_i)^2]}(e^{\gamma c}A_r)(e^{\frac{\gamma}{2}c}L_r)\exp(-e^{\gamma c}A_r-\mu e^{\frac{\gamma}{2}c}L_r)\right].$$

Write each $\beta_j=x_j+iy_j$. Let $g(a,\ell)=ae^{-a}\ell e^{-\mu\ell}$, and write $\widetilde{A}_r:=\mathcal{A}_{\widetilde{h}}(\mathbb{H}\backslash\bigcup_i B_{e^{-r}}(x_i))$ and $\widetilde{L}_r:=\mathcal{L}_{\widetilde{h}}(\mathbb{R}\backslash\bigcup_i B_{e^{-r}}(x_i))$ where $\widetilde{h}=h+\sum\frac{x_i}{2}G_{\mathbb{H}}(\cdot,x_i)-2Q\log|\cdot|_+$. By Girsanov's theorem,

$$\begin{aligned}
&|f_{r+1}(\beta_1,\beta_2,\beta_3)-f_r(\beta_1,\beta_2,\beta_3)|\\
&=\left|\int_{\mathbb{R}}dc\,e^{(\frac{1}{2}\sum\beta_i-Q)c}\mathbb{E}\left[\prod_i e^{\frac{\beta_i}{2}h_{r+1}(x_i)-\frac{\beta_i^2}{8}\mathbb{E}[h_{r+1}(x_i)^2]}(g(e^{\gamma c}A_{r+1},e^{\frac{\gamma}{2}c}L_{r+1})-g(e^{\gamma c}A_r,e^{\frac{\gamma}{2}c}L_r))\right]\right|\\
&\leq\int_{\mathbb{R}}dc\,e^{(\frac{1}{2}\sum x_i-Q)c}\mathbb{E}\left[\left|\prod_i e^{\frac{\beta_i}{2}h_{r+1}(x_i)-\frac{\beta_i^2}{8}\mathbb{E}[h_{r+1}(x_i)^2]}(g(e^{\gamma c}A_{r+1},e^{\frac{\gamma}{2}c}L_{r+1})-g(e^{\gamma c}A_r,e^{\frac{\gamma}{2}c}L_r))\right|\right]\\
&\leq Ce^{\frac{r}{4}\sum y_i^2}\int_{\mathbb{R}}dc\,e^{(\frac{1}{2}\sum x_i-Q)c}\mathbb{E}\left[|g(e^{\gamma c}\widetilde{A}_{r+1},e^{\frac{\gamma}{2}c}\widetilde{L}_{r+1})-g(e^{\gamma c}\widetilde{A}_r,e^{\frac{\gamma}{2}c}\widetilde{L}_r)|\right].
\end{aligned}$$

Now, using the triangle inequality, and the existence of constants $C,C'>0$ for which $|e^{-\mu x}-e^{-\mu x'}|\leq C'|e^{-Cx}-e^{-Cx'}|$ for all $x,x'>0$, we have for any $a_{r+1}>a_r>0$ and $\ell_{r+1}>\ell_r>0$ that

$$|g(a_{r+1},\ell_{r+1})-g(a_r,\ell_r)|\lesssim(a_{r+1}-a_r)e^{-a_{r+1}}+(\ell_{r+1}-\ell_r)e^{-(\Re\mu)\ell_{r+1}}+a_r(e^{-a_r}-e^{-a_{r+1}})+\ell_r|e^{-C\ell_r}-e^{-C\ell_{r+1}}|.$$

Now we have four terms to bound. Writing $s = \frac{1}{2}\sum x_i - Q$,

$$\int_{\mathbb{R}} dc\, e^{(s+\gamma)c}\mathbb{E}[(\widetilde{A}_{r+1} - \widetilde{A}_r)e^{-e^{\gamma c}\widetilde{A}_{r+1}}] = \frac{1}{\gamma}\Gamma(\frac{s}{\gamma}+1)\mathbb{E}[(\widetilde{A}_{r+1} - \widetilde{A}_r)\widetilde{A}_{r+1}^{-\frac{s}{\gamma}-1}],$$

$$\int_{\mathbb{R}} dc\, e^{(s+\frac{\gamma}{2})c}\mathbb{E}[(\widetilde{L}_{r+1} - \widetilde{L}_r)e^{-(\Re\mu)e^{\frac{\gamma}{2}c}\widetilde{L}_{r+1}}] = \frac{2}{\gamma}\Gamma(\frac{2s}{\gamma}+1)(\Re\mu)^{-\frac{2s}{\gamma}-1}\mathbb{E}[(\widetilde{L}_{r+1} - \widetilde{L}_r)\widetilde{L}_{r+1}^{-\frac{2s}{\gamma}-1}],$$

$$\int_{\mathbb{R}} dc\, e^{(s+\gamma)c}\mathbb{E}[\widetilde{A}_r(e^{-e^{\gamma c}\widetilde{A}_r} - e^{-e^{\gamma c}\widetilde{A}_{r+1}})] = \frac{1}{\gamma}\Gamma(\frac{s}{\gamma}+1)\mathbb{E}[\widetilde{A}_r(\widetilde{A}_r^{-\frac{s}{\gamma}-1} - \widetilde{A}_{r+1}^{-\frac{s}{\gamma}-1})],$$

$$\int_{\mathbb{R}} dc\, e^{(s+\frac{\gamma}{2})c}\mathbb{E}[\widetilde{L}_r(e^{-Ce^{\frac{\gamma}{2}c}\widetilde{L}_r} - e^{-Ce^{\frac{\gamma}{2}c}\widetilde{L}_{r+1}})] = \frac{2}{\gamma}\Gamma(\frac{2s}{\gamma}+1)C^{-\frac{2s}{\gamma}-1}\mathbb{E}[\widetilde{L}_r(\widetilde{L}_r^{-\frac{2s}{\gamma}-1} - \widetilde{L}_{r+1}^{-\frac{2s}{\gamma}-1})].$$

For the first integral, we are using $\int e^{tc}e^{-e^{\gamma c}a}\,\mathrm{d}c = \frac{1}{\gamma}\Gamma(\frac{t}{\gamma})a^{-\frac{t}{\gamma}}$ which holds for $t, a > 0$. For the third integral we use $\int e^{tc}(e^{-e^{\gamma c}a} - e^{-e^{\gamma c}a'})\,\mathrm{d}c = \frac{1}{\gamma}\Gamma(\frac{t}{\gamma})(a^{-\frac{t}{\gamma}} - (a')^{-\frac{t}{\gamma}})$ which holds for $t > -\gamma$ and $a, a' > 0$, and follows from the former integral via integration by parts. The second and fourth integrals are proved similarly.

By Lemma B.2, for each point in V, there is a neighborhood O and a constant $C > 0$ such that these terms are uniformly bounded by $Ce^{-r/C}$ on O. Thus

$$|f_{r+1}(\beta_1,\beta_2,\beta_3) - f_r(\beta_1,\beta_2,\beta_3)| \lesssim e^{(\frac{1}{4}\sum b_i^2 - C^{-1})r} \quad \text{uniformly in } O.$$

Step 3: Showing that $\lim_{r\to\infty} f_r$ is an analytic continuation of f. By Step 2, there is a neighborhood of V on which $\lim_{r\to\infty} f_r$ exists and is holomorphic. By Step 1, this limit agrees with f on V, hence is the desired analytic continuation of f. □

Lemma B.2. *For each point in (B.1), there is a neighborhood $U \subset \mathbb{R}^3$ and a constant $C > 0$ so that the following holds. For $(\beta_1,\beta_2,\beta_3) \in U$ and $r \geq 1$, the four expectations*

$$\mathbb{E}[\widetilde{A}_{r+1}^{-\frac{s}{\gamma}-1}(\widetilde{A}_{r+1}-\widetilde{A}_r)], \quad \mathbb{E}[\widetilde{A}_r(\widetilde{A}_r^{-\frac{s}{\gamma}-1}-\widetilde{A}_{r+1}^{-\frac{s}{\gamma}-1})], \quad \mathbb{E}[\widetilde{L}_{r+1}^{-\frac{2s}{\gamma}-1}(\widetilde{L}_{r+1}-\widetilde{L}_r)], \quad \mathbb{E}[\widetilde{L}_r|\widetilde{L}_r^{-\frac{2s}{\gamma}-1}-\widetilde{L}_{r+1}^{-\frac{2s}{\gamma}-1}|]$$

are all bounded by $Ce^{-r/C}$. Here, we set $s = \frac{1}{2}\sum\beta_i - Q$, $\widetilde{h} = h + \sum\frac{\beta_i}{2}G_{\mathbb{H}}(\cdot,x_i) - 2Q\log|\cdot|_+$ and $\widetilde{A}_r := \mu_{\widetilde{h}}(\mathbb{H}\setminus\bigcup_i B_{e^{-r}}(p_i))$, $\widetilde{L}_r := \nu_{\widetilde{h}}(\mathbb{R}\setminus\bigcup_i B_{e^{-r}}(p_i))$ with $(p_1,p_2,p_3) = (0,2,-2)$.

Proof. We explain the exponential bounds for each $(\beta_1,\beta_2,\beta_3)$ (rather than uniformly in U); all inputs in this argument vary continuously in $(\beta_1,\beta_2,\beta_3)$ so we get uniform bounds in neighborhoods U.

First inequality. We have $\mathbb{E}[\widetilde{A}_{r+1}^\lambda] < \mathbb{E}[\widetilde{A}_1^\lambda] \wedge \mathbb{E}[\widetilde{A}_\infty^\lambda] < \infty$ for $\lambda < \frac{2}{\gamma^2} \wedge \frac{1}{\gamma}\min(Q-\beta_i)$, where finiteness follows from the proof of [HRV18, Corollary 6.11]. Thus, for any $\varepsilon > 0$ and $p, q > 1$ satisfying $\frac{1}{p}+\frac{1}{q} = 1$ and p sufficiently small,

$$\begin{aligned}\mathbb{E}[\widetilde{A}_{r+1}^{-\frac{s}{\gamma}-1}(\widetilde{A}_{r+1}-\widetilde{A}_r)] &\leq \varepsilon\mathbb{E}[\widetilde{A}_{r+1}^{-\frac{s}{\gamma}-1}] + \mathbb{E}[1_{\widetilde{A}_{r+1}-\widetilde{A}_r>\varepsilon}\widetilde{A}_{r+1}^{-\frac{s}{\gamma}}] \\ &\lesssim \varepsilon + \mathbb{E}[\widetilde{A}_{r+1}^{-\frac{s}{\gamma}p}]^{1/p}\mathbb{P}[\widetilde{A}_{r+1}-\widetilde{L}_r > \varepsilon]^{1/q} \lesssim \varepsilon + \mathbb{P}[\widetilde{A}_{r+1}-\widetilde{A}_r > \varepsilon]^{1/q}.\end{aligned}$$

Here, we use the fact that $-\frac{s}{\gamma}-1, -\frac{s}{\gamma}p < \frac{2}{\gamma^2}\wedge\frac{1}{\gamma}\min(Q-\beta_i)$ which holds since $(\beta_1,\beta_2,\beta_3)$ lie in B.1. Now, for sufficiently small $m > 0$, by the multifractal spectrum of LQG (see e.g. [BP, Theorem 3.23]) we have

$$\mathbb{P}[\widetilde{A}_{r+1}-\widetilde{A}_r > \varepsilon] \leq \varepsilon^{-m}\mathbb{E}[(\widetilde{A}_{r+1}-\widetilde{A}_r)^m] \lesssim \varepsilon^{-m}e^{-m\frac{\gamma}{2}(Q-\max_i\beta_i - m\gamma)r} \tag{B.2}$$

where the implicit constant depends on m and the term $e^{m\frac{\gamma}{2}r\max_i\beta_i}$ comes from the log singularities added to the field. Choosing $\varepsilon = e^{-r/C}$ for large C and taking $m > 0$ small then yields the result.

Second inequality. Since $(\beta_1, \beta_2, \beta_3)$ lies in (B.1) we have $-\frac{s}{\gamma} - 1 < 0$. For any $a, \varepsilon > 0$ we have $a(a^{-\frac{s}{\gamma}-1} - (a+\varepsilon)^{-\frac{s}{\gamma}-1}) \lesssim \varepsilon a^{-\frac{s}{\gamma}-1}$ with implicit constant depending only on s, and so

$$\mathbb{E}[\widetilde{A}_r(\widetilde{A}_r^{-\frac{s}{\gamma}-1} - \widetilde{A}_{r+1}^{-\frac{s}{\gamma}-1})] \lesssim \varepsilon \mathbb{E}[\widetilde{A}_r^{-\frac{s}{\gamma}-1}] + \mathbb{E}[1_{\widetilde{A}_{r+1}-\widetilde{A}_r > \varepsilon} \widetilde{A}_r^{-\frac{s}{\gamma}}].$$

The argument is then identical to that of the first inequality.

Third and fourth inequalities. The third inequality is proved identically to the first inequality. If $-\frac{2s}{\gamma} - 1 < 0$ then the fourth inequality is proved the same way as the second. Now we prove the fourth inequality in the regime $-\frac{2s}{\gamma} - 1 \geq 0$; since we are in (B.1) we also have $-\frac{2s}{\gamma} - 1 < 1$, so $\widetilde{L}_r(\widetilde{L}_{r+1}^{-\frac{2s}{\gamma}-1} - \widetilde{L}_r^{-\frac{2s}{\gamma}-1}) \lesssim \widetilde{L}_{r+1}^{-\frac{2s}{\gamma}-1}(\widetilde{L}_{r+1} - \widetilde{L}_r)$. Now the second inequality applies, completing the proof.

□

Proof of Proposition 2.7. Analyticity in μ is immediate from Morera's theorem. The β-merophicity immediately follows from Proposition B.1, where we proved the desired properties for each term in $\hat{H}$. □

The proof of Proposition B.4 is easier to explain when we parametrize in the strip $\mathcal{S} := \mathbb{R} \times (0, \pi)$. The Gaussian free field h on $\mathcal{S}$ is defined via $h = h_{\mathbb{H}} \circ \exp$ where $h_{\mathbb{H}} \sim P_{\mathbb{H}}$. It has covariance

$$G_{\mathcal{S}}(z, w) = \dots.$$

Definition B.3. *Suppose $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$. Sample $(h, \mathbf{c}) \sim P_{\mathcal{S}} \times [e^{(\frac{1}{2}\sum \beta_i - Q)c} dc]$, and let*

$$\widetilde{h}(z) = h(z) - (Q - \beta_1)(0 \vee \Re z) - (Q - \beta_2)(0 \vee (-\Re z)) + \frac{\beta_3}{2} G_{\mathcal{S}}(z, 0).$$

Let $\phi = \widetilde{h} + \mathbf{c}$. Let $\mathrm{LF}_{\mathcal{S}}^{(\beta_i, p_i)}$ be the law of ϕ where $(p_1, p_2, p_3) = (+\infty, -\infty, 0)$.

Now, we state the analogous holomorphicity statement result for Proposition 2.12.

Proposition B.4. *Fix μ_i with $\Re \mu_i > 0$ for $i = 1, 2, 3$, and consider the following three maps from $((Q - \gamma) \vee \frac{\gamma}{2}, Q)$ to $\mathbb{C}$:*

$$\beta \mapsto \int \frac{1}{L} A e^{-A - \mu_1 L_1 - \mu_2 L_2} \, \mathrm{LF}_{\mathbb{H}}^{(\beta,0),(\gamma,1),(\beta,\infty)}(d\phi),$$

$$\beta \mapsto \int \frac{1}{L_2} A(\mu_1 L_1 + \mu_2 L_2) e^{-A - \mu_1 L_1 - \mu_2 L_2} \, \mathrm{LF}_{\mathbb{H}}^{(\beta,0),(\gamma,1),(\beta,\infty)}(d\phi),$$

$$\beta \mapsto \int \frac{1}{L_2} (\mu_1 L_1 + \mu_2 L_2)^2 e^{-A - \mu_1 L_1 - \mu_2 L_2} \, \mathrm{LF}_{\mathbb{H}}^{(\beta,0),(\gamma,1),(\beta,\infty)}(d\phi),$$

where we write $A = \mathcal{A}_\phi(\mathbb{H})$, $L_1 = \mathcal{L}_\phi(-\infty, 0)$ and $L_2 = \mathcal{L}_\phi(0, \infty)$. Then each map is well-defined in the sense that the integrals converge absolutely, and extends analytically to a neighborhood of $((Q - \gamma) \vee \frac{\gamma}{2}, Q)$ in $\mathbb{C}$.

Proof. We only prove the result for the second function, but the others are proved identically. We work in $(\mathcal{S}, -\infty, 0, +\infty)$ rather than $(\mathbb{H}, 0, 1, \infty)$ since the symmetry between the two β-insertions is more apparent. Moreover, to simplify notation we assume $\mu_1 = \mu_2 = \mu$; the argument works the same way when $\mu_1 \neq \mu_2$. It suffices to prove that the function

$$f(\beta) = \int \frac{\mathcal{A}_\phi(\mathcal{S}) \mathcal{L}_\phi(\partial \mathcal{S})}{\mathcal{L}_\phi(\mathbb{R})} e^{-\mathcal{A}_\phi(\mathcal{S}) - \mu \mathcal{L}_\phi(\partial \mathcal{S})} \mathrm{LF}_{\mathcal{S}}^{(\beta, \pm\infty),(\gamma,0)}(d\phi)$$

extends analytically to a neighborhood of $((Q - \gamma) \vee \frac{\gamma}{2}, Q)$ in $\mathbb{C}$.

For $r \geq 1$ let $\mathcal{S}_r := (-r,r)\times(0,\pi)$, $I_r = (-r,r)\times\{\pi\}$ and $J_r = (-r,r)\times\{0\}$. For a sample $h \sim P_{\mathcal{S}}$, write $A_r := \mathcal{A}_{h-Q|\Re\cdot|}(\mathcal{S}_r)$, $K_r = \mathcal{L}_{h-Q|\Re\cdot|}(I_r\cup J_r)$ and $L_r = \mathcal{L}_{h-Q|\Re\cdot|}(J_r)$. Write X_t for the average of h on $\{x\}\times(0,\pi)$. Then define for $r\geq 1$ the function

$$f_r(\beta) = \int_{\mathbb{R}} dce^{(\beta+\frac{\gamma}{2}-Q)c}\mathbb{E}\left[\prod_{t=\pm r} e^{\frac{\beta}{2}X_t - \frac{\beta^2}{8}\mathbb{E}[X_t^2]}\frac{(e^{\gamma c}A_r)(e^{\frac{\gamma}{2}c}K_r)}{e^{\frac{\gamma}{2}c}L_r}e^{-A_r-\mu K_r}\right]$$

on a neighborhood of $((Q-\gamma)\vee\frac{\gamma}{2},Q)$ in $\mathbb{C}$ to be specified later. Clearly f_r is analytic.

Step 1: Showing $\lim_{r\to\infty} f_r(\beta) = f(\beta)$ when $\beta\in((Q-\gamma)\vee\frac{\gamma}{2},Q)$.
By Girsanov's theorem, this is equivalent to showing that for all $\beta\in((Q-\gamma)\vee\frac{\gamma}{2},Q)$ we have

$$\lim_{r\to\infty}\int \frac{\mathcal{A}_\phi(\mathcal{S}_r)\mathcal{L}_\phi((-r,r)\times\{0,\pi\})}{\mathcal{L}_\phi(-r,r)}e^{-\mathcal{A}_\phi(\mathcal{S}_r)-\mu\mathcal{L}_\phi((-r,r)\times\{0,\pi\})}\mathrm{LF}_{\mathcal{S}}^{(\beta,\pm\infty),(\gamma,0)}(d\phi) = f(\beta).$$

This follows from the dominated convergence theorem: for some deterministic constant C,

$$\left|\frac{\mathcal{A}_\phi(\mathcal{S}_r)\mathcal{L}_\phi((-r,r)\times\{0,\pi\})}{\mathcal{L}_\phi(-r,r)}e^{-\mathcal{A}_\phi(\mathcal{S}_r)-\mu\mathcal{L}_\phi((-r,r)\times\{0,\pi\})}\right| \leq C\frac{e^{-(\Re\mu/2)\mathcal{L}_\phi((-r,r))}}{\mathcal{L}_\phi(-r,r)} \leq C\frac{e^{-(\Re\mu/2)\mathcal{L}_\phi((-1,1))}}{\mathcal{L}_\phi(-1,1)},$$

and

$$\int\frac{e^{-(\Re\mu/2)\mathcal{L}_\phi((-1,1))}}{\mathcal{L}_\phi(-1,1)}\mathrm{LF}_{\mathcal{S}}^{(\beta,\pm\infty),(\gamma,0)}(d\phi) = \frac{2}{\gamma}\Gamma(\frac{2}{\gamma}(\beta-Q))(\Re\mu/2)^{\frac{2}{\gamma}(Q-\beta)}\mathbb{E}[\mathcal{L}_{\widetilde{h}}(-1,1)^{\frac{2}{\gamma}(Q-\beta)-1}]<\infty.$$

Here we write $\widetilde{h} = h-(Q-\beta)|\Re\cdot| + \frac{\gamma}{2}G_{\mathcal{S}}(\cdot,0)$, and the moment bound comes from [RZ20a, Proposition 1.10].

Step 2: Showing that each point in $((Q-\gamma)\vee\frac{\gamma}{2},Q)$ has an open neighborhood in $\mathbb{C}$ on which f_r converges uniformly as $r\to\infty$.
We have the alternative expression

$$f_r(\beta) = \int_{\mathbb{R}} e^{(\beta+\frac{\gamma}{2}-Q)c}\mathbb{E}\left[\prod_{t=\pm(r+1)} e^{\frac{\beta}{2}X_t-\frac{\beta^2}{8}\mathbb{E}[X_t^2]}\frac{(e^{\gamma c}A_r)(e^{\frac{\gamma}{2}c}K_r)}{e^{\frac{\gamma}{2}c}L_r}e^{-A_r-\mu K_r}\right]dc.$$

Let $g(a,k,\ell) = \frac{ak}{\ell}e^{-a-\mu k}$, and let $\widetilde{A}_r = \mathcal{A}_{\widetilde{h}}(S_r), \widetilde{K}_r = \mathcal{L}_{\widetilde{h}}(I_r\cup J_r), \widetilde{L}_r = \mathcal{L}_{\widetilde{h}}(J_r)$ where $\widetilde{h} = h-(Q-x)|\Re\cdot|+\frac{\gamma}{2}G_{\mathcal{S}}(\cdot,0)$. Write $\beta = x+iy$. As before, we have

$$|f_{r+1}(\beta)-f_r(\beta)| \leq Ce^{\frac{r}{2}y}\int_{\mathbb{R}} e^{(x+\frac{\gamma}{2}-Q)}\mathbb{E}\left[|g(e^{\gamma c}\widetilde{A}_{r+1},e^{\frac{\gamma}{2}c}\widetilde{K}_{r+1},e^{\frac{\gamma}{2}c}\widetilde{L}_{r+1}) - g(e^{\gamma c}\widetilde{A}_r,e^{\frac{\gamma}{2}c}\widetilde{K}_r,e^{\frac{\gamma}{2}c}\widetilde{L}_r)|\right]dc.$$

Now, we can find a constant C so that for all $a_{r+1}>a_r, k_{r+1}>k_r,\ell_{r+1}>\ell_r$ we have

$$\begin{aligned}&|g(a_{r+1},k_{r+1},\ell_{r+1})-g(a_r,k_r,\ell_r)|\\ &\leq |(a_{r+1}-a_r)\frac{k_{r+1}}{\ell_{r+1}}e^{-\mu k_{r+1}}| + |(\frac{1}{\ell_r}-\frac{1}{\ell_{r+1}})k_{r+1}e^{-\mu k_{r+1}}| + |\frac{k_{r+1}-k_r}{\ell_r}e^{-\mu k_{r+1}}| + |\frac{k_r}{\ell_r}(e^{-\mu k_r}-e^{-\mu k_{r+1}})|\\ &\lesssim |(a_{r+1}-a_r)\frac{k_{r+1}}{\ell_{r+1}}e^{-(\Re\mu)k_{r+1}}| + |(\frac{1}{\ell_r}-\frac{1}{\ell_{r+1}})k_{r+1}e^{-(\Re\mu)k_{r+1}}| + |\frac{k_{r+1}-k_r}{\ell_r}e^{-(\Re\mu)k_{r+1}}| + |\frac{k_r}{\ell_r}(e^{-Ck_r}-e^{-Ck_{r+1}})|.\end{aligned}$$

This gives us four terms to bound. Writing $s = x + \frac{\gamma}{2} - Q \in ((-\frac{\gamma}{2}) \vee (\frac{\gamma}{2} - \frac{2}{\gamma}), \frac{\gamma}{2})$,

$$\int dc\, e^{(s+\gamma)c}\mathbb{E}[(\widetilde{A}_{r+1} - \widetilde{A}_r)\frac{\widetilde{K}_{r+1}}{\widetilde{L}_{r+1}} e^{-e^{\gamma c}(\Re\mu)\widetilde{K}_r}] = \frac{2}{\gamma}\Gamma(\frac{2s}{\gamma} + 2)\mathbb{E}[(\widetilde{A}_{r+1} - \widetilde{A}_r)\frac{\widetilde{K}_{r+1}^{-\frac{2s}{\gamma}-1}}{\widetilde{L}_{r+1}}],$$

$$\int dc\, e^{sc}\mathbb{E}[(\frac{1}{\widetilde{L}_r} - \frac{1}{\widetilde{L}_{r+1}})\widetilde{K}_{r+1} e^{-e^{\gamma c/2}(\Re\mu)\widetilde{K}_{r+1}}] = \frac{2}{\gamma}\Gamma(\frac{2s}{\gamma})(\Re\mu)^{-\frac{2s}{\gamma}}\mathbb{E}[(\frac{1}{\widetilde{L}_r} - \frac{1}{\widetilde{L}_{r+1}})\widetilde{K}_{r+1}^{-\frac{2s}{\gamma}+1}],$$

$$\int dc\, e^{sc}\mathbb{E}[\frac{\widetilde{K}_{r+1} - \widetilde{K}_r}{\widetilde{L}_r} e^{-e^{\gamma c/2}(\Re\mu)\widetilde{K}_{r+1}}] = \frac{2}{\gamma}\Gamma(\frac{2s}{\gamma})(\Re\mu)^{-\frac{2s}{\gamma}}\mathbb{E}[\frac{\widetilde{K}_{r+1} - \widetilde{K}_r}{\widetilde{L}_r}\widetilde{K}_{r+1}^{-\frac{2s}{\gamma}}],$$

$$\int dc\, e^{sc}\mathbb{E}[\frac{\widetilde{K}_r}{\widetilde{L}_r}(e^{-e^{\gamma c/2}C\widetilde{K}_r} - e^{-e^{\gamma c/2}C\widetilde{K}_{r+1}})] = \frac{2}{\gamma}\Gamma(\frac{2s}{\gamma})C^{-\frac{2s}{\gamma}}\mathbb{E}[\frac{\widetilde{K}_r}{\widetilde{L}_r}(\widetilde{K}_r^{-\frac{2s}{\gamma}} - \widetilde{K}_{r+1}^{-\frac{2s}{\gamma}})].$$

For the first three integrals, we are using $\int e^{tc}e^{-e^{\gamma c/2}k}\,dc = \frac{2}{\gamma}\Gamma(\frac{2t}{\gamma})k^{-\frac{2t}{\gamma}}$ which holds for $t, \ell > 0$. For the fourth integral we use $\int e^{tc}(e^{-e^{\gamma c/2}k} - e^{-e^{\gamma c/2}k'})\,dc = \frac{2}{\gamma}\Gamma(\frac{2t}{\gamma})(k^{-\frac{2t}{\gamma}} - (k')^{-\frac{2t}{\gamma}})$ which holds for $t > -\frac{\gamma}{2}$ and $k, k' > 0$, and follows from the former integral via integration by parts.

By Lemma B.5 and the deterministic inequality $|a^p - b^p| \le p|a-b|\max(a^{p-1}, b^{p-1})$ for $a, b > 0$ and $p \in \mathbb{R}$, for each point in $((Q-\gamma) \vee \frac{\gamma}{2}, Q)$ there is a neighborhood O and a constant $C > 0$ such that these four terms are uniformly bounded by $Ce^{-r/C}$ on O, so

$$|f_{r+1}(\beta) - f_r(\beta)| \lesssim e^{(\frac{1}{2}b^2 - C^{-1})r} \qquad \text{uniformly in } O.$$

Step 3: Showing that $\lim_{r\to\infty} f_r$ is an analytic continuation of f. By Step 2, there is a neighborhood of $((Q-\gamma) \vee \frac{\gamma}{2}, Q)$ on which $\lim_{r\to\infty} f_r$ exists and is holomorphic. By Step 1, this limit agrees with f on $((Q-\gamma) \vee \frac{\gamma}{2}, Q)$, hence is the desired analytic continuation of f. □

Lemma B.5. *For each point in $((Q-\gamma) \vee \frac{\gamma}{2}, Q)$ there is a neighborhood $U \subset \mathbb{R}$ and a constant $C > 0$ such that the following holds. For $\beta \in U$ and $r \ge$, writing $s = \beta + \frac{\gamma}{2} - Q$, the expectations*

$$\mathbb{E}[(\widetilde{A}_{r+1} - \widetilde{A}_r)\frac{\widetilde{K}_{r+1}^{-\frac{2s}{\gamma}-1}}{\widetilde{L}_{r+1}}], \quad \mathbb{E}[(\frac{1}{\widetilde{L}_r} - \frac{1}{\widetilde{L}_{r+1}})\widetilde{K}_{r+1}^{-\frac{2s}{\gamma}+1}], \quad \mathbb{E}[\frac{\widetilde{K}_{r+1} - \widetilde{K}_r}{\widetilde{L}_r}\widetilde{K}_{r+1}^{-\frac{2s}{\gamma}}], \quad \mathbb{E}[\frac{\widetilde{K}_{r+1} - \widetilde{K}_r}{\widetilde{L}_r}\widetilde{K}_r^{-\frac{2s}{\gamma}}]$$

are each bounded by $Ce^{-r/C}$. Here, we set $\widetilde{h} = h - (Q-\beta)|\Re\cdot| + \frac{\gamma}{2}G_{\mathcal{S}}(\cdot, 0)$ and $\widetilde{A}_r = \mathcal{A}_{\widetilde{h}}((-r,r) \times (0,\pi))$, $\widetilde{K}_r = \mathcal{L}_{\widetilde{h}}((-r,r) \times \{0,\pi\})$ and $\widetilde{L}_r = \mathcal{L}_{\widetilde{h}}(-r,r)$.

Proof. For the first inequality: Let $\widehat{h} = h - (Q-\beta)|\Re\cdot|$, so away from the origin $|\widetilde{h} - \widehat{h}|$ is bounded. Since $\widehat{h}|_{(1,2)\times(0,\pi)} \stackrel{d}{=} \widehat{h}(\cdot - (r-1))|_{(r,r+1)\times(0,\pi)} + (Q-\beta)(r-1)$,

$$\begin{aligned}\mathbb{E}[(\widetilde{A}_{r+1} - \widetilde{A}_r)\frac{\widetilde{K}_{r+1}^{-\frac{2s}{\gamma}-1}}{\widetilde{L}_{r+1}}] &\lesssim \mathbb{E}[\frac{\mathcal{A}_{\widehat{h}}((r,r+1)\times(0,\pi))}{\mathcal{L}_{\widehat{h}}((r,r+1)\times\{0,\pi\})^{\frac{2s}{\gamma}+1}\mathcal{L}_{\widehat{h}}(r,r+1)}] \\ &= \mathbb{E}[\frac{\mathcal{A}_{\widehat{h}}((1,2)\times(0,\pi))}{\mathcal{L}_{\widehat{h}}((1,2)\times\{0,\pi\})^{\frac{2s}{\gamma}+1}\mathcal{L}_{\widehat{h}}(1,2)}]e^{-(\gamma - \frac{\gamma}{2}(\frac{2s}{\gamma}+2))(Q-\beta)(r-1)}.\end{aligned}$$

By Lemma B.7 the last expectation is bounded. Moreover $(\gamma - \frac{\gamma}{2}(\frac{2s}{\gamma}+2))(Q-\beta) > 0$, so the first inequality is shown.

For the second inequality, take $p > 1$ small and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$, then for $\varepsilon > 0$,

$$\begin{aligned}\mathbb{E}[\frac{\widetilde{L}_{r+1} - \widetilde{L}_r}{\widetilde{L}_r \widetilde{L}_{r+1}} \widetilde{K}_{r+1}^{-\frac{2s}{\gamma}+1}] &\leq \varepsilon \mathbb{E}[\frac{\widetilde{K}_{r+1}^{-\frac{2s}{\gamma}+1}}{\widetilde{L}_r^2}] + \mathbb{E}[1_{\widetilde{L}_{r+1}-\widetilde{L}_r > \varepsilon} \frac{\widetilde{L}_{r+1} - \widetilde{L}_r}{\widetilde{L}_r \widetilde{L}_{r+1}} \widetilde{K}_{r+1}^{-\frac{2s}{\gamma}+1}] \\ &\leq \varepsilon \mathbb{E}[\frac{\widetilde{K}_{r+1}^{-\frac{2s}{\gamma}+1}}{\widetilde{L}_r^2}] + \mathbb{E}\left[\left(\frac{1}{\widetilde{L}_r} \widetilde{K}_{r+1}^{-\frac{2s}{\gamma}+1}\right)^p\right]^{1/p} \mathbb{P}[\widetilde{L}_{r+1} - \widetilde{L}_r > \varepsilon]^{1/q}.\end{aligned}$$

By REF the expectations are both uniformly bounded in r. Writing $\widehat{h} = h - (Q-\beta)|\Re \cdot|$ and using $\widehat{h}|_{(0,1)\times(0,\pi)} \stackrel{d}{=} \widehat{h}(\cdot - r)|_{(r,r+1)\times(0,\pi)} + (Q-\beta)r$, we have

$$\mathbb{P}[\widetilde{L}_{r+1} - \widetilde{L}_r > \varepsilon] \leq \varepsilon^{-1}\mathbb{E}[\widetilde{L}_{r+1} - \widetilde{L}_r] \lesssim \varepsilon^{-1}\mathbb{E}[\mathcal{L}_{\widehat{h}}(r, r+1)] = \varepsilon^{-1} e^{-\frac{\gamma}{2}(Q-\beta)r}\mathbb{E}[\mathcal{L}_{\widehat{h}}(0,1)].$$

We conclude that $\mathbb{E}[\frac{\widetilde{L}_{r+1}-\widetilde{L}_r}{\widetilde{L}_r\widetilde{L}_{r+1}} \widetilde{K}_{r+1}^{-\frac{2s}{\gamma}+1}] \lesssim \varepsilon + \varepsilon^{-1/q} e^{\frac{\gamma}{2}(Q-\beta)r/q}$. Take $\varepsilon = e^{-r/C}$ with C sufficiently large to conclude.

For the third inequality, choose $p > 1$ small and q such that $\frac{1}{p} + \frac{1}{q} = 1$, then for $\varepsilon > 0$,

$$\mathbb{E}[\frac{\widetilde{K}_{r+1} - \widetilde{K}_r}{\widetilde{L}_r} \widetilde{K}_{r+1}^{-\frac{2s}{\gamma}}] \leq \varepsilon \mathbb{E}[\frac{\widetilde{K}_{r+1}^{-\frac{2s}{\gamma}+1}}{\widetilde{L}_r^2}] + \mathbb{E}\left[\left(\frac{\widetilde{K}_{r+1} - \widetilde{K}_r}{\widetilde{L}_r} \widetilde{K}_{r+1}^{-\frac{2s}{\gamma}}\right)^p\right]^{1/p} \mathbb{P}[\widetilde{K}_{r+1} - \widetilde{K}_r > \varepsilon]^{1/q}.$$

The same argument as before applies to show the desired $\mathbb{E}[\frac{\widetilde{K}_{r+1}-\widetilde{K}_r}{\widetilde{L}_r} \widetilde{K}_{r+1}^{-\frac{2s}{\gamma}}] \lesssim e^{-r/C}$. The fourth inequality has essentially the same proof. □

Lemma B.6. *Suppose $p > 0$ and $q \in (0, \frac{4}{\gamma^2})$ satisfy $q - p < \frac{2}{\gamma}(Q-\beta) - 1$. Then there is a continuous function $C(\beta, p, q)$ such that for $\widetilde{h} = h - (Q-\beta)|\Re \cdot| + \frac{\gamma}{2}G_{\mathcal{S}}(\cdot, 0)$ we have*

$$\mathbb{E}[\frac{\mathcal{L}_{\widetilde{h}}((-(r+1), r+1) \times \{0,\pi\})^q}{\mathcal{L}_{\widetilde{h}}(-r,r)^p}] < C(\beta, p, q).$$

Proof. We will just show that this expectation is finite; all the inputs in the proof vary continuously so the statement about $C(\beta,p,q)$ follows. First, note that $\mathbb{E}[\mathcal{L}_{\widetilde{h}}(-1,1)^{q-p}] < \infty$ since $q-p < \frac{2}{\gamma}(Q-\beta)-1 < \frac{4}{\gamma^2}-1$, so we are done if we show $\mathbb{E}[\frac{(\mathcal{L}_{\widetilde{h}}((-r,r)\times\{0,\pi\})-\mathcal{L}_{\widetilde{h}}(-1,1))^q}{\mathcal{L}_{\widetilde{h}}(-r,r)^p}] < \infty$. Let $\widehat{h} = h - (Q-\beta)|\Re\cdot|$, then

$$\mathbb{E}[\frac{(\mathcal{L}_{\widetilde{h}}((-(r+1), r+1) \times \{0,\pi\}) - \mathcal{L}_{\widetilde{h}}(-1,1))^q}{\mathcal{L}_{\widetilde{h}}(-r,r)^p}] \lesssim \mathbb{E}[\frac{\mathcal{L}_{\widehat{h}}((-(r+1), r+1) \times \{0,\pi\})^q}{\mathcal{L}_{\widehat{h}}(-r,r)^p}] \lesssim \mathbb{E}[\frac{\mathcal{L}_{\widehat{h}}((0, r+1) \times \{0,\pi\})^q}{\mathcal{L}_{\widehat{h}}(0,r)^p}$$

Let X_t be the average value of $\widehat{h}$ on $\{t\} \times (0,\pi)$, and let M_r be the maximum value of $(X_t)_{[0,r-1]}$. We claim that

$$\mathbb{E}[\mathcal{L}_{\widehat{h}}(0,r)^{-a} \mid M_r] \lesssim e^{-\frac{\gamma}{2}aM_r} \quad \text{for } a > 0, \tag{B.3}$$

$$\mathbb{E}[\mathcal{L}_{\widehat{h}}((0,r+1)\times\{0,\pi\})^b \mid M_r] \lesssim e^{\frac{\gamma}{2}(1+\varepsilon)bM_r}, \quad \text{for } b \in (0, \frac{4}{\gamma^2}), \varepsilon > 0 \tag{B.4}$$

where the implicit constants depend only on a, b, ε, β but not on r.

Assuming (B.3) and (B.4), we complete the proof. Let $\lambda > 1$ be small and λ' satisfy $\frac{1}{\lambda} + \frac{1}{\lambda'} = 1$, then

$$\mathbb{E}[\frac{\mathcal{L}_{\widehat{h}}((0,r+1)\times\{0,\pi\})^q}{\mathcal{L}_{\widehat{h}}(0,r)^p} \mid M_r] \leq \mathbb{E}[\mathcal{L}_{\widehat{h}}((0,r+1)\times\{0,\pi\})^{q\lambda} \mid M_r]^{1/\lambda}\mathbb{E}[\mathcal{L}_{\widehat{h}}(0,r)^{-p\lambda'} \mid M_r]^{1/\lambda'}$$
$$\lesssim e^{(\frac{\gamma}{2}(1+\varepsilon)q - \frac{\gamma}{2}p)M_r},$$

which is bounded by 1 if $\frac{\gamma}{2}(1+\varepsilon)q - \frac{\gamma}{2}pM \leq 0$, and otherwise bounded by

$$\mathbb{E}[\frac{\mathcal{L}_{\widehat{h}}((0,r+1)\times\{0,\pi\})^q}{\mathcal{L}_{\widehat{h}}(0,r)^p}] \lesssim \mathbb{E}[e^{(\frac{\gamma}{2}(1+\varepsilon)q-\frac{\gamma}{2}p)M_\infty}] = \frac{Q-\beta}{Q-\beta-\frac{\gamma}{2}((1+\varepsilon)q-p)} < \infty,$$

where the last equality holds since, by a standard Brownian motion calculation (Lemma C.2), the law of M_∞ is exponential with variance $(Q-\beta)^2$.

Finally, we prove (B.3) and (B.4). For $t \geq 0$ let X_t be the average of $\widehat{h}$ on $\{t\}\times(0,\pi)$.

For (B.3): if we further condition on the time $t_* \in (0, r-1)$ at which X_{t_*} is maximal, then the conditional law of $(X_t)_{[t_*,r-1]}$ is variance 2 Brownian motion with drift $-(Q-\beta)$ conditioned to stay below M_r, and $(X_t)_{[r-1,r]}$ then evolves as unconditioned variance 2 Brownian motion with drift $-(Q-\beta)$. Then $Y = M_r - \inf_{[t_*,t_*+1]} X_t$ is a nonnegative sub-Gaussian random variable independent of M_r. Let h_2 be the projection of h to $\mathcal{H}_2(\mathcal{S})$, then h_2 is independent of (X_t) and hence (M_r, t_*), so

$$\mathbb{E}[\mathcal{L}_{\widehat{h}}(0,r)^{-a} \mid M_r, t_*] \leq \mathbb{E}[\mathcal{L}_{\widehat{h}}(t_*, t_*+1)^{-a} \mid M_r, t_*] \leq \mathbb{E}[\mathcal{L}_{h_2+M_r-Y}(t_*,t_*+1)^{-a} \mid M_r, t_*]$$
$$\leq e^{-\frac{\gamma}{2}aM_r}\mathbb{E}[e^{\frac{\gamma}{2}aY} \mid t_*]\mathbb{E}[\mathcal{L}_{h_2}(0,1)^{-a}] \lesssim e^{-\frac{\gamma}{2}aM_r}.$$

For (B.4), we split cases based on the value of M_r. In the case where $M_r \leq \frac{2}{\gamma}\varepsilon^{-1}\log r$, an argument similar to that of [DMS14, Lemma A.5] proves the following: writing j^* for the value of t in $[j, j+1]$ maximizing X_t,

$$\mathbb{E}[(\sum_{j=0}^{r} e^{\frac{\gamma}{2}X_{j^*}})^b \mid M_r] \lesssim e^{\frac{\gamma}{2}bM_r} \leq e^{\frac{\gamma}{2}(1+\varepsilon)bM_r} \quad \text{if } M_r \leq \frac{2}{\gamma}\varepsilon^{-1}\log r.$$

If instead $M_r > \frac{2}{\gamma}\varepsilon^{-1}\log r$, then $r^b < e^{\frac{\gamma}{2}b\varepsilon M_r}$. Let Y be the maximum value of variance 2 Brownian motion with drift $-(Q-\beta)$ on $[0,1]$ independent of M_r. Then

$$\mathbb{E}[(\sum_{j=0}^{r} e^{\frac{\gamma}{2}X_{j^*}})^b \mid M_r] \leq \mathbb{E}[(re^{\frac{\gamma}{2}M_r} + e^{\frac{\gamma}{2}(M_r+Y)})^b \mid M_r] \lesssim e^{\frac{\gamma}{2}(1+\varepsilon)bM_r} \quad \text{if } M_r > \frac{2}{\gamma}\varepsilon^{-1}\log r.$$

In either case we get the same bound. Then, because h_2 and (X_t) are independent, an application of Jensen's inequality as in the proof of [DMS14, (A.11)] yields (B.4). ☐

Lemma B.7. *Let $I \subset (0 \vee (2-\frac{4}{\gamma^2}), \infty)$ be a closed interval. Then there exists $C > 0$ such that for $h \sim P_{\mathcal{S}}$,*

$$\mathbb{E}[\frac{\mathcal{A}_h((1,2)\times(0,\pi))}{\mathcal{L}_h((1,2)\times\{0,\pi\})^\lambda \mathcal{L}_h(1,2)}] < C \quad \textit{for } \lambda \in I.$$

Proof. We map to the half-plane via the map $z \mapsto e^{i\pi - z}$, so it suffices to prove a uniform bound for

$$\mathbb{E}[\frac{\mathcal{A}_h(\mathbb{H}\cap(e^{-1}\mathbb{D}\backslash e^{-2}\mathbb{D}))}{\mathcal{L}_h((-e^{-1},-e^{-2})\cup(e^{-2},e^{-1}))^\lambda \mathcal{L}_h(e^{-2},e^{-1})}], \quad \text{where } h \sim P_{\mathbb{H}}.$$

Let $S=\mathbb{H}\cap(e^{-1}\mathbb{D}\backslash e^{-2}\mathbb{D})$, $J_+=(e^{-2},e^{-1})$, and $J=(-e^{-1},-e^{-2})\cup J_+$. Fix also $\lambda_0\in(0\vee(2-\frac{4}{\gamma^2}),1)$ lying to the left of I. Then for any $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}\mathbb{E}[\frac{\mathcal{A}_h(S)}{\mathcal{L}_h(J)^\lambda\mathcal{L}_h(J_+)}] &= \int_S (2\Im z)^{-\gamma^2/2}\mathbb{E}[\frac{1}{\mathcal{L}_{h+\gamma G_{\mathbb{H}}(\cdot,z)}(J)^q\mathcal{L}_{h+\gamma G_{\mathbb{H}}(\cdot,z)}(J_+)}]\,dz\\ &\lesssim \int_S (2\Im z)^{-\gamma^2/2}\mathbb{E}_z[\mathcal{L}_{h+\gamma G_{\mathbb{H}}(\cdot,z)}(J)^{-\lambda_0}]^{1/p}\mathbb{E}[\mathcal{L}_h(J)^{-q(\lambda-\lambda_0)-\lambda_0}\mathcal{L}_h(J_+)^q]^{1/q}\,dz.\end{aligned}$$

The equality holds since weighting by the quantum area of S corresponds to adding a γ-log insertion at a random point in S, via the so-called *rooted GMC measure*; see e.g. [DS11, Section 3.3]. In the second term in the inequality, we drop the $+\gamma G_{\mathbb{H}}(\cdot,z)$ terms since they are bounded from below on $\mathbb{H}$. We will choose $p>1$ small later.

For each $z\in S$, let J_z be an interval of length $\frac{1}{100}\Im z$ such that the distance from z to J_z is at most $\Im z$. By the multifractal spectrum of GMC (see e.g. [BP, Theorem 3.23] with $d=1$; the discrepancy of a factor of 2 arises since $h|_{\mathbb{R}}$ has correlations $\mathbb{E}[h(x)h(y)]\asymp -2\log|x-y|$ when $|x-y|$ is small) we have

(B.5)
$$\mathbb{E}[\mathcal{L}_{h+\gamma G_{\mathbb{H}}(\cdot,z)}(J)^{-\lambda_0}]\le \mathbb{E}[\mathcal{L}_{h+\gamma G_{\mathbb{H}}(\cdot,z)}(J_z)^{-\lambda_0}]\lesssim (\Im z)^{\frac{\gamma}{2}\cdot 2\gamma\lambda_0}\mathbb{E}[\mathcal{L}_h(J_z)^{-\lambda_0}]\lesssim (\Im z)^{-\frac{\gamma^2}{4}\lambda_0^2+(\frac{3\gamma^2}{4}-1)\lambda_0}.$$

Plugging (B.5) into the previous bound, and using the existence of all negative GMC moments, we have

$$\mathbb{E}[\frac{\mathcal{A}_h(S)}{\mathcal{L}_h(J)^\lambda\mathcal{L}_h(J_+)}]\lesssim \int_S (\Im z)^{-\frac{\gamma^2}{2}+\frac{1}{p}(-\frac{\gamma^2}{4}\lambda_0^2+(\frac{3\gamma^2}{4}-1)\lambda_0)}\,dz<\infty,$$

where we choose $p>1$ sufficiently small that $-\frac{\gamma^2}{2}+\frac{1}{p}(-\frac{\gamma^2}{4}\lambda_0^2+(\frac{3\gamma^2}{4}-1)\lambda_0)>-1$. (This is possible since $f(t)=-\frac{\gamma^2}{2}+(-\frac{\gamma^2}{4}t^2+(\frac{3\gamma^2}{4}-1)t)+1=-\frac{\gamma^2}{4}(t-(2-\frac{4}{\gamma^2}))(t-1)$ is positive on $(2-\frac{4}{\gamma^2},1)$.) □

Proof of Proposition 2.12. [AHS21, Proposition 2.28] relates the two-pointed quantum disk weighted by the quantum length of a side to the three-point Liouville field: writing $A=\mathcal{A}_\phi(\mathbb{H}), L_1=\mathcal{L}_\phi(-\infty,0), L_2=\mathcal{L}_\phi(0,\infty)$ and $s=\beta-Q$, we have

$$R_{\mu_1,\mu_2}(\beta)=\frac{1}{Q-\beta}\int(\frac{\gamma}{s}A+\frac{\gamma}{2s}\frac{\gamma}{s+\frac{\gamma}{2}}A(\sum_i\mu_iL_i)+\frac{\gamma}{2s}\frac{\frac{\gamma}{2}}{s+\frac{\gamma}{2}}(\sum_i\mu_iL_i)^2)\frac{e^{-A-\sum_i\mu_iL_i}}{L_2}\mathrm{LF}_{\mathbb{H}}^{(\beta,0),(\gamma,1),(\beta,\infty)}(d\phi).$$

Therefore Proposition B.4 gives the analyticity of $R_{\mu_1,\mu_2}(\beta)$ in β. □

Appendix C. Reflection coefficient as the limit of H: proof of Lemma 4.7

In this section we prove Lemma 4.7. We first prove a version of Lemma 4.7 when we condition on the field average maximum, then prove the full statement.

C.1. Convergence when conditioning on field maximum.

The goal of this section is to prove the following. Recall the Liouville field on the strip $\mathcal{S}$ defined in Definition B.3. Let $(x_1,x_2,x_3)=(+\infty,-\infty,0)\subset\partial\mathcal{S}$.

Proposition C.1. *Fix $\beta_1\in(\frac{\gamma}{2}\vee(Q-\gamma),Q)$, $\beta_2\in(0,\beta_1)$ and $\beta_3\in(\beta_1-\beta_2,\beta_1-\frac{1}{2}\beta_2)$. When ϕ is sampled from $\mathrm{LF}_{\mathcal{S}}^{(\beta_i,x_i)}(d\phi)$ and conditioned on $M_\phi=m$, then $(\mathcal{A}_\phi(\mathcal{S}),\mathcal{L}_\phi((0,\infty)\times\{0\}),\mathcal{L}_\phi(\mathbb{R}\times\{\pi\}),\mathcal{L}_\phi((-\infty,0)\times\{0\}))$ converges in distribution to $(\mathcal{A}_{\psi+m}(\mathcal{S}),\mathcal{L}_{\psi+m}(\mathbb{R}\times\{0\}),\mathcal{L}_{\psi+m}(\mathbb{R}\times\{\pi\}),0)$ as $\beta_3\downarrow\beta_1-\beta_2$. Here ψ is as in Definition 2.10.*

The key to obtaining this convergence is the following elementary Brownian motion calculation; see e.g. REF.

Lemma C.2. *Let $(B_t)_{t\geq 0}$ be standard Brownian motion and let $b>0$. Then*

$$\mathbb{P}[\sup_{t\geq 0} B_{2t} - bt \geq m] = \min(e^{-bm}, 1).$$

In the rest of this section we freely use the decomposition $\phi = \widetilde{h} + \mathbf{c}$ of Definition B.3: Recall that if we sample $(h, \mathbf{c}) \sim P_{\mathcal{S}} \times [e^{(\frac{1}{2}\sum \beta_i - Q)c}\, dc]$, let

$$\widetilde{h}(z) = h(z) - (Q-\beta_1)(0 \vee \Re z) - (Q-\beta_2)(0 \vee (-\Re z)) + \frac{\beta_3}{2} G_{\mathcal{S}}(z, 0), \tag{C.1}$$

then we can set $\phi = \widetilde{h} + \mathbf{c}$.

Lemma C.3. *Fix $\beta_1 \in (\frac{\gamma}{2} \vee (Q-\gamma), Q)$ and $\beta_2 \in (0, \beta_1)$, $\beta_3 \in (\beta_1 - \beta_2, \beta_1)$. Consider the measure $(\beta_2 + \beta_3 - \beta_1)\mathrm{LF}_{\mathcal{S}}^{(\beta_i, x_i)}(d\phi)$. Let $s = \frac{1}{2}\sum \beta_i - Q$ and let M_ϕ be the supremum over $t>0$ of the average of ϕ on $\{t\} \times (0,\pi)$. Then*

$$(\beta_2 + \beta_3 - \beta_1)\mathrm{LF}_{\mathcal{S}}^{(\beta_i, x_i)}[M_\phi \in dm] = 2(Q-\beta_1)e^{sm}.$$

Moreover, the conditional law of $\mathbf{c}$ given $M_\phi = m$ is

$$\mathbb{P}[\mathbf{c} \in dc \mid M_\phi = m] = 1_{c<m} \frac{1}{2}(\beta_2 + \beta_3 - \beta_1) e^{\frac{1}{2}(\beta_2+\beta_3-\beta_1)(c-m)}.$$

Proof. Write $M_{\widetilde{h}}$ for the supremum over $t>0$ of the average of $\widetilde{h}$ on $\{t\} \times (0,\pi)$; Lemma C.2 describes the law of $M_{\widetilde{h}}$. Then

$$\mathrm{LF}_{\mathcal{S}}^{(\beta_i, x_i)}[M_\phi \geq m \text{ and } \mathbf{c} \in dc] = e^{sc}\mathbb{P}[M_{\widetilde{h}} \geq m - c] = 1_{m>c} e^{sc - (Q-\beta_1)(m-c)} + 1_{m\leq c} e^{sc},$$

$$\mathrm{LF}_{\mathcal{S}}^{(\beta_i, x_i)}[M_\phi \in dm \text{ and } \mathbf{c} \in dc] = 1_{m>c}(Q-\beta_1) e^{sc-(Q-\beta_1)(m-c)},$$

$$\mathrm{LF}_{\mathcal{S}}^{(\beta_i, x_i)}[M_\phi \in dm] = \int 1_{m>c}(Q-\beta_1)e^{sc-(Q-\beta_1)(m-c)}\, dc = \frac{2(Q-\beta_1)}{\beta_2+\beta_3-\beta_1} e^{sm},$$

$$\mathbb{P}[\mathbf{c} \in dc \mid M_\phi = m] = \frac{\mathrm{LF}_{\mathcal{S}}^{(\beta_i, x_i)}[M_\phi \in dm \text{ and } \mathbf{c} \in dc]}{\mathrm{LF}_{\mathcal{S}}^{(\beta_i, x_i)}[M_\phi \in dm]} = 1_{c<m}\frac{1}{2}(\beta_2+\beta_3-\beta_1)e^{\frac{1}{2}(\beta_2+\beta_3-\beta_1)(c-m)}.$$

□

Lemma C.4. *Fix $\beta_1 \in (\frac{\gamma}{2} \vee (Q-\gamma), Q)$ and $\beta_2 \in (0, \beta_1)$, $\beta_3 \in (\beta_1 - \beta_2, \beta_1)$. When $\widetilde{h}$ from (C.1) is conditioned on $M_{\widetilde{h}} = M$, then $(\mathcal{A}_{\widetilde{h}-M}(\mathcal{S}), \mathcal{L}_{\widetilde{h}-M}(0,\infty), \mathcal{L}_{\widetilde{h}-M}(\mathbb{R} \times \{\pi\}), \mathcal{L}_{\widetilde{h}-M}(-\infty, 0))$ converges in distribution as $M \to \infty$ to $(\mathcal{A}_\psi(\mathcal{S}), \mathcal{L}_\psi(\mathbb{R}), \mathcal{L}_\psi(\mathbb{R}\times\{\pi\}), 0)$. Moreover this convergence is uniform for $\beta_3 \in (\beta_1 - \beta_2, \beta_1 - \frac{1}{2}\beta_2)$.*

Proof. Define

$$\tau = \inf\{t \,:\, X_t^{\widetilde{h}} > M - \sqrt{M}\}, \quad \sigma = \inf\{t \,:\, X_t^{\psi} > -\sqrt{M}\}.$$

Then we can couple $(X_{t+\tau}^{\widetilde{h}} - M)_{t\geq 0}$ and $(X_{t+\sigma}^{\psi})_{t\geq 0}$ to agree almost surely; indeed both of these have the law of $(B_{2t} - (Q-\beta_1)t)_{t\geq 0}$ conditioned to have maximum value $\sqrt{M}$, where B_t is standard Brownian motion.

Now, we write fields with the subscript 2 to denote projection of the field to the space $\mathcal{H}_2(\mathcal{S})$ of distributions with mean zero on $\{t\} \times (0,\pi)$ for all t. We have $\tau \to \infty$ in probability as $M \to \infty$, so with high probability the Dirichlet energy of $\gamma G_{\mathcal{S}}(\cdot, 0)$ on $(\tau, \infty) \times (0,\pi)$ is close to zero. Thus the law of $(h_2 + \gamma G_{\mathcal{S}}(\cdot, 0))|_{(\tau,\infty)\times(0,\pi)}$ is close in total variation to that of $h_2|_{(\tau,\infty)\times(0,\pi)}$ [MS17, Proposition 2.9], so we can further couple $\widetilde{h}_2(\cdot + \tau)|_{\mathcal{S}_+}$ and $\psi_2(\cdot+\sigma)|_{\mathcal{S}_+}$ to agree with high probability.

We conclude that we can couple $\widetilde{h}$ and ψ so that with probability $1 - o_M(1)$ the distributions $(\widetilde{h}(\cdot - \tau) - M)|_{\mathcal{S}_+}$ and $\psi(\cdot - \sigma)|_{\mathcal{S}_+}$ agree.

Now, all that remains is to show that $\mathcal{A}_{\widetilde{h}-M}((-\infty, \tau) \times (0, \pi)), \mathcal{L}_{\widetilde{h}-M}((-\infty, \tau) \times \{0, \pi\}) \to 0$ in probability as $M \to \infty$. This holds because the field average of $\widetilde{h} - M$ in $(-\infty, \tau) \times (0, \pi)$ is bounded above by $-\sqrt{M}$. □

Proof of Proposition C.1. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a bounded continuous function, so by Lemma C.4, writing $T_{\widetilde{h}-M} = (\mathcal{A}_{\widetilde{h}-M}(\mathcal{S}), \mathcal{L}_{\widetilde{h}-M}(0, \infty), \mathcal{L}_{\widetilde{h}-M}(\mathbb{R} \times \{\pi\}), \mathcal{L}_{\widetilde{h}-M}(-\infty, 0))$, we have

$$\lim_{M\to\infty} \mathbb{E}[f(T_{\widetilde{h}-M})) \mid M_{\widetilde{h}} = M] = \mathbb{E}[f(\mathcal{A}_\psi(\mathcal{S}), \mathcal{L}_\psi(\mathbb{R}), \mathcal{L}_\psi(\mathbb{R} \times \{\pi\}), 0)].$$

Let $\delta = \frac{1}{2}(\beta_2 + \beta_3 - \beta_1)$. Lemma C.3 says that conditioned on $M_\phi = m$, the conditional law of $\mathbf{c}$ is $1_{c<m} \delta e^{\delta(c-m)}\, dc$. Thus, defining $T_{\phi-m}$ as above,

$$\mathbb{E}[f(T_{\phi-m}) \mid M_\phi = m] = \int_{-\infty}^{m} \delta e^{\delta(c-m)} \mathbb{E}[f(T_{\widetilde{h}-(m-c)}) \mid M_{\widetilde{h}} = m - c]\, dc.$$

As $\beta_3 \downarrow \beta_1 - \beta_2$ we have $\delta \downarrow 0$, which combined with the first limit gives the desired

$$\lim_{\beta_3 \downarrow \beta_1 - \beta_2} \mathbb{E}[f(T_{\phi-m}) \mid M_\phi = m] = \mathbb{E}[f(\mathcal{A}_\psi(\mathcal{S}), \mathcal{L}_\psi(\mathbb{R}), \mathcal{L}_\psi(\mathbb{R} \times \{\pi\}), 0)].$$

□

C.2. **Completing the proof.** We first state without proof a technical estimate, then prove Lemma 4.7, and finally prove the estimate.

Lemma C.5. *Fix $\beta_1 \in (\frac{\gamma}{2} \vee (Q - \gamma), Q)$, $\beta_2 \in (0, \beta_1)$ and $\beta_3 \in (\beta_1 - \beta_2, \beta_1 - \frac{1}{2}\beta_2)$. Let $\Re \mu_i > 0$ for $i = 1, 2, 3$. Then with $(x_1, x_2, x_3) = (+\infty, -\infty, 0)$ we have*

$$\lim_{C\to\infty} (\beta_2 + \beta_3 - \beta_1) \int 1_{|M_\phi| > C} |\mathfrak{H}_\phi| \operatorname{LF}_{\mathcal{S}}^{(\beta_i, x_i)}(\mathrm{d}\phi) = 0 \qquad \textit{uniformly in } \beta_3,$$

where we define, writing $s = \frac{1}{2}\sum \beta_i - Q, A = \mathcal{A}_\phi(\mathcal{S}), L_1 = \mathcal{L}_\phi(0, \infty), L_2 = \mathcal{L}_\phi(\mathbb{R} \times \{\pi\}), L_3 = \mathcal{L}_\phi(-\infty, 0)$,

$$\mathfrak{H}_\phi := (\gamma(s + \frac{\gamma}{2})A + \frac{\gamma^2}{2} A (\sum_{i=1}^{3} \mu_i L_i) + \frac{\gamma^2}{4} (\sum_{i=1}^{3} \mu_i L_i)^2) e^{-A - \sum_{i=1}^{3} \mu_i L_i}. \tag{C.2}$$

Proof of Lemma 4.7. For the field ψ on $\mathcal{S}$ from Definition 2.10, with $A' = \mathcal{A}_\psi(\mathcal{S}), L_1' = \mathcal{L}_\psi(\mathbb{R}), L_2' = \mathcal{L}_\psi(\mathbb{R} \times \{\pi\})$, define

$$\mathfrak{R}_\psi := (Q - \beta)(\gamma(s + \frac{\gamma}{2})A' + \frac{\gamma^2}{2} A' (\sum_{i=1}^{2} \mu_i L_i') + \frac{\gamma^2}{4} (\sum_{i=1}^{2} \mu_i L_i')^2) e^{-A' - \sum_{i=1}^{2} \mu_i L_i'}.$$

Let $C > 0$. By Proposition C.1 we have, with $\mathfrak{H}_\phi$ defined in (C.2),

$$\lim_{\beta_3 \downarrow \beta_1 - \beta_2} \mathbb{E}[\mathfrak{H}_\phi \mid M_\phi = m] = \frac{1}{Q - \beta_1} \mathbb{E}[\mathfrak{R}_{\psi + m}] \qquad \text{uniformly in } m \in (-C, C),$$

Integrating this against $1_{m \in (-C,C)} 2(Q - \beta_1) e^{sm}\, dm$ and using the law of M_ϕ given in Lemma C.3, we conclude

$$\lim_{\beta_3 \downarrow \beta_1 - \beta_2} (\beta_2 + \beta_3 - \beta_1) \int 1_{|M_\phi| \le C} \mathfrak{H}_\phi \operatorname{LF}_{\mathcal{S}}^{(\beta_i, x_i)}(d\phi) = 2 \int_{-C}^{C} e^{(\beta_1 - Q)m} \mathbb{E}[\mathfrak{R}_{g+m}]\, dm.$$

By Lemma C.5 we can send $C \to \infty$ to conclude that

$$\lim_{\beta_3 \downarrow \beta_1 - \beta_2} (\beta_2 + \beta_3 - \beta_1) \int \mathfrak{H}_\phi \, \mathrm{LF}_{\mathcal{S}}^{(\beta_i, x_i)}(d\phi) = 2 \int_{-\infty}^{\infty} e^{(\beta_1 - Q)m} \mathbb{E}[\mathfrak{R}_{g+m}] \, dm.$$

Recalling the definitions of H and R in (2.6) and (2.12), this is the desired identity. □

Lemma C.6. *Suppose $\beta_1 \in (\frac{\gamma}{2} \vee (Q-\gamma), Q)$ and $q \in (\frac{Q-\beta_1}{\gamma}, 1 \wedge \frac{2}{\gamma^2})$, then there is a constant $C = C(q, \beta_1)$ such that the following holds. Let $\beta_2, \beta_3 < Q - \gamma q$ satisfy $\beta_2 + \beta_3 > \beta_1$. Let $h \sim P_{\mathcal{S}}$ and define $\widetilde{h}(z) := h(z) - (Q-\beta_1)(0 \vee \Re z) - (Q - \beta_2)(0 \vee (-\Re z)) + \frac{\beta_3}{2} G_{\mathcal{S}}(z, 0)$. Let $M_{\widetilde{h}}$ be the maximum over $t \in \mathbb{R}$ of the average of $\widetilde{h}$ on $\{t\} \times [0, \pi]$. Then*

$$\mathbb{E}[\mathcal{A}_{\widetilde{h}}(\mathcal{S})^q \mid M_{\widetilde{h}}] \le C e^{q\gamma M_{\widetilde{h}}}, \qquad \mathbb{E}[\mathcal{L}_{\widetilde{h}}(\partial \mathcal{S})^{2q} \mid M_{\widetilde{h}}] \le C e^{q\gamma M_{\widetilde{h}}}.$$

Proof. We establish the former conditional inequality; the latter is proved in the same way. In this proof C is a constant that may vary from line to line, but will only depend on p, β_1.

Let X_t be the average of $\widetilde{h}$ on $\{t\} \times [0, \pi]$, and let j^* be the value of $t \in [j, j+1]$ maximizing X_t. An easy modification of [DMS14, Lemma A.5] (where their (p, c) equals our $(1, q\gamma)$) gives

$$\mathbb{E}[\sum_{j \in \mathbb{Z}} e^{q\gamma X_{j^*}} \mid M_{\widetilde{h}}] \le C e^{q\gamma M_{\widetilde{h}}}.$$

Next, let $\widetilde{h}_2$ be the projection of $\widetilde{h}$ to the lateral component and let $A_j = \mathcal{A}_{\widetilde{h}_2}([j, j+1] \times [0, \pi])$, then

$$\mathbb{E}[A_j^q] < C \quad \text{uniformly in } j.$$

Finally,

$$\mathbb{E}[\mathcal{A}_{\widetilde{h}}(\mathcal{S})^q \mid M_{\widetilde{h}}] \le \mathbb{E}[(\sum_j e^{\gamma X_{j^*}} A_j)^q \mid M_{\widetilde{h}}] \le \sum_j \mathbb{E}[e^{q\gamma X_{j^*}} A_j^q \mid M_{\widetilde{h}}] \le C \sum_j \mathbb{E}[e^{q\gamma X_{j^*}} \mid M_{\widetilde{h}}] \le C e^{q\gamma M_{\widetilde{h}}}.$$

□

Proof of Lemma C.5. In this proof, we repeatedly use the notation $\lesssim$ to mean "up to a multiplicative constant not depending on β_3". By Lemma C.6 we have, with q as in Lemma C.6,

$$\mathbb{E}[\mathcal{A}_\phi(\mathcal{S})^q \mid M_\phi] \le C e^{q\gamma M_\phi}, \qquad \mathbb{E}[\mathcal{L}_\phi(\partial \mathcal{S})^{2q} \mid M_\phi] \le C e^{q\gamma M_\phi}.$$

Writing $L = \sum \mu_i L_i$, we prove that $\lim_{C \to \infty} (\beta_2 + \beta_3 - \beta_1) \int |ALe^{-A-L}| \mathrm{LF}_{\mathcal{S}}^{(\beta_i, x_i)}(d\phi) = 0$ uniformly in β_3; the terms Ae^{-A-L} and $L^2 e^{-A-L}$ are similarly bounded.

Clearly we have $ALe^{-A-\mu L} \lesssim Ae^{-A} \lesssim A^q$. Let $s = \frac{1}{2} \sum \beta_i - Q$, then by Lemma C.3 and the above moment bounds we have

$$\begin{aligned}(\beta_2 + \beta_3 - \beta_1) \mathrm{LF}_{\mathcal{S}}^{(\beta_i, x_i)}[1_{|M_\phi| > C} ALe^{-A-\mu L}] &\lesssim \int_{-\infty}^{-C} \mathbb{E}[A^q \mid M_\phi = m] e^{sm} \, dm + \int_C^\infty \mathbb{E}[Ae^{-A} \mid M_\phi = m] e^{sm} \, dm \\ &\lesssim \int_{-\infty}^{-C} e^{q\gamma m + sm} \, dm + \int_C^\infty e^{sm} \, dm \lesssim e^{-(\gamma q + s)C} + e^{sC} \xrightarrow{C \to \infty} 0.\end{aligned}$$

□

Appendix D. Two-point quantum disk from mating of trees: proof of Lemma 4.15

The goal of this section is to prove Lemma 4.15. We start by giving the following lemma.

Lemma D.1 ([?, Proposition 5.2])**.**

$$|\mathcal{M}_2^{\mathrm{disk}}(2;\ell,r)| = \frac{(2\pi)^{\frac{4}{\gamma^2}-1}}{(1-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}(\ell+r)^{-\frac{4}{\gamma^2}-1}.$$

The following is a corollary of [AG21, Theorem 1.2]:

$$\mathrm{QD}(x)^\#[e^{-\mu A}] = \frac{2}{\Gamma(\frac{4}{\gamma^2})}\left(\frac{\mu}{4\sin(\frac{\pi\gamma^2}{4})}\right)^{\frac{2}{\gamma^2}} x^{\frac{4}{\gamma^2}} K_{\frac{4}{\gamma^2}}\left(x\sqrt{\frac{\mu}{\sin(\frac{\pi\gamma^2}{4})}}\right). \tag{D.1}$$

Lemma D.2. *Suppose $k\in\mathbb{N}$ satisfies $k+1>\frac{4}{\gamma^2}$. Let $c=\sqrt{1/\sin(\frac{\pi\gamma^2}{4})}$ and $\mu\in(0,c)$. Then for some $C=C(\gamma)$ we have*

$$\mathcal{M}_2^{\mathrm{disk}}(2)[(-L_1)^k e^{-A-\mu L_1}] = C\sum_{i=0}^\infty \frac{(-2/c)^{k+i}}{i!(k+i+1)}\Gamma(\frac{1}{2}(k+i+1-\frac{4}{\gamma^2}))\Gamma(\frac{1}{2}(k+i+1+\frac{4}{\gamma^2}))\mu^i.$$

Proof. Let $C' = \frac{(2\pi)^{\frac{4}{\gamma^2}-1}}{(1-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}\cdot\frac{2}{\Gamma(\frac{4}{\gamma^2})}2^{-\frac{4}{\gamma^2}}$ and suppose $m+1>\frac{4}{\gamma^2}$. Then

$$\mathcal{M}_2^{\mathrm{disk}}(2)[L_1^m e^{-A}] = C'c^{\frac{4}{\gamma^2}}\iint_0^\infty r^m\cdot(\ell+r)^{-1}K_{\frac{4}{\gamma^2}}(c(\ell+r))\,d\ell\,dr \tag{D.2}$$

$$= C'c^{\frac{4}{\gamma^2}}\int_0^\infty\int_0^x r^m\cdot x^{-1}K_{\frac{4}{\gamma^2}}(cx)\,dr\,dx \tag{D.3}$$

$$= C'c^{\frac{4}{\gamma^2}}\int_0^\infty \frac{1}{m+1}\cdot x^m K_{\frac{4}{\gamma^2}}(cx)\,dx \tag{D.4}$$

$$= C'c^{\frac{4}{\gamma^2}-m-1}\frac{2^{m-1}}{m+1}\Gamma(\frac{1}{2}(m+1-\frac{4}{\gamma^2}))\Gamma(\frac{1}{2}(m+1+\frac{4}{\gamma^2})). \tag{D.5}$$

The integral in the last step was evaluated via [DLMF, (10.43.19)], and the condition $m+1>\frac{4}{\gamma^2}$ is necessary for this step. Now, using the Taylor expansion $e^{-\mu L}=\sum_{i=0}^\infty\frac{(-\mu L)^i}{i!}$ gives the result. □

We thus conclude the following:

Lemma D.3. *Suppose $\frac{4}{\gamma^2}\notin\mathbb{Z}$ and set $c=\sqrt{1/\sin(\frac{\pi\gamma^2}{4})}$. Define $f:(-c,c)\to\mathbb{R}$ by $f(\mu)=\cos(\frac{4}{\gamma^2}\arccos(\mu/c))$; that is, if $\sigma\in\mathcal{B}$ satisfies $\mu=c\cos(\pi\gamma(\sigma_j-\frac{Q}{2}))$, then $f(\mu)=\cos(\frac{4\pi}{\gamma}(\sigma-\frac{Q}{2}))$. There is a constant $C(\gamma)$ such that for all $k>\frac{4}{\gamma^2}$ we have*

$$f^{(k)}(\mu) = C\mathcal{M}_2^{\mathrm{disk}}(2)[(\mu(-L_1)^k+k(-L_1)^{k-1})e^{-A-\mu L_1}]\quad \textit{for all } \mu\in(-c,c).$$

Proof. By Lemma D.2, gathering equal powers of μ and simplifying gives

$$\mathcal{M}_2^{\mathrm{disk}}(2)[(\mu(-L_1)^k+k(-L_1)^{k-1})e^{-A-\mu L_1}] = C\sum_{i=0}^\infty\frac{(-2/c)^{k+i-1}}{i!}\Gamma(\frac{1}{2}(k+i-\frac{4}{\gamma^2}))\Gamma(\frac{1}{2}(k+i+\frac{4}{\gamma^2}))\mu^i.$$

Now the result is immediate from the following identity which is recorded in e.g. [ARS21, Lemma 4.15]: for $a \in \mathbb{R}\backslash\mathbb{Z}$ and $x \in (-1,1)$,

$$\cos(a\cos^{-1}(x)) = \frac{a}{2\pi}\sin(\pi a)\sum_{k=0}^{\infty}\frac{(-2)^{n-1}}{n!}\Gamma(\frac{1}{2}(n+a))\Gamma(\frac{1}{2}(n-a))x^n. \qquad \square$$

With this lemma at hand it is now possible to prove Lemma 4.15.

Proof of Lemma 4.15. We have for μ, μ' in $\{z \in \mathbb{C} : \Re z > 0\}$ that $(\mu-\mu')R(\gamma,\mu,\mu') = \hat{g}(\mu) - \hat{g}(\mu')$ for some holomorphic $\hat{g}$, and

$$\hat{g}^{(k)}(\mu_2) = \mu\frac{\partial^k}{\partial\mu^k}R(\gamma,\mu_2,\mu_3) + k\frac{\partial^{k-1}}{\partial\mu^{k-1}}R(\gamma,\mu,\mu').$$

Sending $\mu' \to 0$ gives, since we can analytically continue $\frac{\partial^{k-1}}{\partial\mu^{k-1}}R(\beta,\mu,\mu')$ to $\beta=\gamma$ where it has the probabilistic interpretation via $QD_{0,2}$, by Lemma D.3 we conclude $f^{(k)}(\mu) = \hat{g}^{(k)}(\mu)$, so since $f, \hat{g}$ are both analytic in a suitable domain, we have

$$\hat{g}(\mu) = f(\mu) + \sum_{i=0}^{k_0} a_i\mu^i.$$

Thus,

$$R(\gamma,\sigma,\sigma') = c_\gamma\frac{\cos(\frac{4\pi}{\gamma}(\sigma-\frac{Q}{2})) - \cos(\frac{4\pi}{\gamma}(\sigma'-\frac{Q}{2})) + p(\sigma) - p(\sigma')}{\cos(\gamma\pi(\sigma-\frac{Q}{2})) - \cos(\gamma\pi(\sigma'-\frac{Q}{2}))}$$

where $p(\sigma) = \sum_{i=1}^{k_0} a_i(c\cos(\pi\gamma(\sigma-\frac{Q}{2})))^i$. Since f and g are $\frac{\gamma}{2}$-periodic, we conclude that p is $\frac{\gamma}{2}$-periodic. By its definition, it's also clear that p is $\frac{2}{\gamma}$-periodic. Thus, when γ^2 is irrational, then the function has two periods which differ by an irrational factor, hence p must be constant (and thus zero). $\square$

Appendix E. Operator product expansion (OPE)

In this appendix we take a look at the Operator Product Expansion (OPE) lemmas that are used in Section 4. These proofs where first done in... In our case the results are very similar, except that we must handle a functional that does not reduce to a moment of GMC.

Lemma E.1. *(OPE without reflection) Let $\chi = \frac{\gamma}{2}$. Assume that the constraints of (4.8) hold, meaning that $\sigma_1, \sigma_2, \sigma_3, \sigma_1 - \frac{\gamma}{4}, \sigma_2 - \frac{\gamma}{4} \in [-\frac{1}{2\gamma}+\frac{Q}{2}, \frac{1}{2\gamma}+\frac{Q}{2}] \times \mathbb{R}$, $\beta_1, \beta_2, \beta_3 < Q$, $\sum_{i=1}^3 \beta_i > 2Q + \frac{\gamma}{2}$. Assume also that $\beta_1 \in (\frac{\gamma}{2}, \frac{2}{\gamma})$ and $t \in (0,1)$. Then as $t \to 0$, one has*

$$H_{\frac{\gamma}{2}}(t) - H_{\frac{\gamma}{2}}(0) = C_2^+ t^{1-C} + o(t^{1-C}), \tag{E.1}$$

where:

$$C_2^+ = -\frac{\Gamma(-1+\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta_1}{2})}{\Gamma(-\frac{\gamma^2}{4})}\sqrt{\frac{1}{\sin(\pi\frac{\gamma^2}{4})}} \times \left(\cos(\pi\gamma(\sigma_1-\frac{\gamma}{4}-\frac{Q}{2})) - \cos(\pi\gamma(\sigma_2+\frac{\beta_1}{2}-\frac{\gamma}{4}-\frac{Q}{2}))\right) H^{(\beta_1+\frac{\gamma}{2},\beta_2,\beta_3)}_{(\sigma_1-\frac{\gamma}{4},\sigma_2,\sigma_3)}. \tag{E.2}$$

Proof. Using the expression

$$H_{\frac{\gamma}{2}}(t) = \int_{\mathbb{R}} dc\, e^{(\frac{\gamma p}{2} - \frac{\chi}{2})c} \mathbb{E}\Bigg[\exp\Bigg(- e^{\gamma c} \int_{\mathbb{H}} \frac{|x-t|^{\gamma\chi} g(x)^{\frac{\gamma^2}{4}(-p+\frac{\chi}{\gamma}-1)} / |x-\bar{x}|^{\frac{\gamma^2}{2}}}{|x|^{\gamma\beta_1}|x-1|^{\gamma\beta_2}} e^{\gamma X(x)} d^2x \\ - e^{\frac{\gamma c}{2}} \int_{\mathbb{R}} \frac{|r-t|^{\frac{\gamma\chi}{2}} g(r)^{\frac{\gamma^2}{8}(-p+\frac{\chi}{\gamma}-1)}}{|r|^{\frac{\gamma\beta_1}{2}}|r-1|^{\frac{\gamma\beta_2}{2}}} e^{\frac{\gamma}{2}X(r)} d\mu_B^t(r) \Bigg) \Bigg],$$

we derive:

$$H_{\frac{\gamma}{2}}(t) - H_{\frac{\gamma}{2}}(0) \\ = -t^{1-C} \sqrt{\frac{1}{\sin(\pi\frac{\gamma^2}{4})}} \Bigg(\cos(\pi\gamma(\sigma_1 - \frac{\gamma}{4} - \frac{Q}{2})) \int_{\mathbb{R}_+} du \frac{(1+u)^{\frac{\gamma^2}{4}} - u^{\frac{\gamma^2}{4}}}{u^{\frac{\gamma\beta_1}{2}}} \\ - \frac{1}{2} e^{i\pi\gamma(\sigma_2 - \frac{Q}{2}) - i\pi(\frac{\gamma^2}{4} - \frac{\gamma\beta_1}{2})} \int_{\mathbb{R}_+ e^{i\pi}} du \frac{(1+u)^{\frac{\gamma^2}{4}} - u^{\frac{\gamma^2}{4}}}{u^{\frac{\gamma\beta_1}{2}}} - \frac{1}{2} e^{-i\pi\gamma(\sigma_2 - \frac{Q}{2}) + i\pi(\frac{\gamma^2}{4} - \frac{\gamma\beta_1}{2})} \int_{\mathbb{R}_+ e^{-i\pi}} du \frac{(1+u)^{\frac{\gamma^2}{4}} - u^{\frac{\gamma^2}{4}}}{u^{\frac{\gamma\beta_1}{2}}} \Bigg) \\ \times H_{(\sigma_1 - \frac{\gamma}{4}, \sigma_2, \sigma_3)}^{(\beta_1 + \frac{\gamma}{2}, \beta_2, \beta_3)} + o(t^{1-C}).$$

□

Let us now provide a proof of the OPE with reflection. The following result holds for both values of χ.

Lemma E.2. *(OPE with reflection) For $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$, assume again the constraints of (4.8) are satisfied. Consider $t \in (\infty, 1)$. Then there exists a parameter $\beta_0 > 0$ small enough so that under the assumption that $\beta_1 \in (Q - \beta_0, Q)$, for $t \in (0,1)$, as $t \to 0$ the following asymptotic holds*

$$H_{\frac{2}{\gamma}}(t) - H_{\frac{2}{\gamma}}(0) = C_2^+ t^{1 - \frac{2\beta_1}{\gamma} + \frac{4}{\gamma^2}} + o(|t|^{1 - \frac{2\beta_1}{\gamma} + \frac{4}{\gamma^2}}), \tag{E.3}$$

where:

$$C_2^+ = R(\beta_1, \sigma_1 - \frac{1}{\gamma}, \sigma_2 - \frac{1}{\gamma}) H_{(\sigma_1 - \frac{1}{\gamma}, \sigma_2, \sigma_3)}^{(2Q - \beta_1 - \frac{2}{\gamma}, \beta_2, \beta_3)}. \tag{E.4}$$

Similarly, still for $\beta_1 \in (Q - \beta_0, Q)$, by choosing $t \in (-\infty, 0)$, as $t \to 0$ the following asymptotic holds

$$H_{\frac{2}{\gamma}}(t) - H_{\frac{2}{\gamma}}(0) = C_2^- t^{1 - \frac{2\beta_1}{\gamma} + \frac{4}{\gamma^2}} + o(|t|^{1 - \frac{2\beta_1}{\gamma} + \frac{4}{\gamma^2}}), \tag{E.5}$$

with this time:

$$C_2^- = e^{-i\pi(1 - \frac{2\beta_1}{\gamma} + \frac{4}{\gamma^2})} R(\beta_1, \sigma_1, \sigma_2) H_{(\sigma_1 - \frac{1}{\gamma}, \sigma_2, \sigma_3)}^{(2Q - \beta_1 - \frac{2}{\gamma}, \beta_2, \beta_3)}. \tag{E.6}$$

We omit the proof, which is analogue to the one performed in [RZ21].

Appendix F. Cancellation of singular terms for the BPZ equations

At the end of the proof of the BPZ equations, we need to show a list of terms are converging to 0. We prove here the lemma corresponding to the case $\gamma \in (0, \sqrt{2})$.

Proof of Lemma 3.11. We start by looking at the term

$$\begin{aligned}
&\frac{\mu\gamma^2}{4\sin(\pi\frac{\gamma^2}{4})}e^{i\pi\frac{\gamma^2}{4}}\int_{\mathbb{R}}du_1\int_0^{\frac{\pi}{2}}d\theta e^{-i\frac{\gamma^2}{2}\theta}(u_1+\frac{1}{2}ie^{i\theta}+i)^{\frac{\gamma^2}{4}-1}(u_1-\frac{1}{2}ie^{i\theta}-i)^{\frac{\gamma^2}{4}-1}\\
=&-\frac{\mu\gamma^2}{4\sin(\pi\frac{\gamma^2}{4})}\int_0^{\frac{\pi}{2}}d\theta\int_{\mathbb{R}}du_1e^{-2i\theta}(iu_1e^{-i\theta}-\frac{1}{2}-e^{-i\theta})^{\frac{\gamma^2}{4}-1}(iu_1e^{-i\theta}+\frac{1}{2}+e^{-i\theta})^{\frac{\gamma^2}{4}-1}\\
=&\frac{\mu\gamma^2}{4\sin(\pi\frac{\gamma^2}{4})}\int_0^{\frac{\pi}{2}}d\theta\int_{(i\mathbb{R}-1)e^{-i\theta}}du_1ie^{-i\theta}(u_1-\frac{1}{2})^{\frac{\gamma^2}{4}-1}(u_1+\frac{1}{2}+2e^{-i\theta})^{\frac{\gamma^2}{4}-1}.
\end{aligned}$$

For the last integral we can deform the contour of u_1 to $i\mathbb{R}$ and will not encounter any singularities. Hence we obtain

$$\begin{aligned}
&-\frac{\mu}{2\sin(\pi\frac{\gamma^2}{4})}\int_{i\mathbb{R}}du_1(u_1-\frac{1}{2})^{\frac{\gamma^2}{4}-1}((u_1+\frac{1}{2}-2i)^{\frac{\gamma^2}{4}}-(u_1+\frac{5}{2})^{\frac{\gamma^2}{4}})\\
=&-\frac{\mu}{2\sin(\pi\frac{\gamma^2}{4})}\int_{i\mathbb{R}-\frac{1}{2}}du_1(u_1+i)^{\frac{\gamma^2}{4}-1}((u_1+1-i)^{\frac{\gamma^2}{4}}-(u_1+3+i)^{\frac{\gamma^2}{4}})\\
=&-\frac{\mu}{2\sin(\pi\frac{\gamma^2}{4})}\int_{\mathbb{R}e^{i\epsilon}-i}du_1(u_1+i)^{\frac{\gamma^2}{4}-1}((u_1+1-i)^{\frac{\gamma^2}{4}}-(u_1+3+i)^{\frac{\gamma^2}{4}}).
\end{aligned}$$

Similarly we obtain from the other singular integral

$$\begin{aligned}
&\frac{\mu\gamma^2}{4\sin(\pi\frac{\gamma^2}{4})}\int_0^{-\frac{\pi}{2}}d\theta\int_{(-i\mathbb{R}+1)e^{-i\theta}}du_1ie^{-i\theta}(u_1+\frac{1}{2})^{\frac{\gamma^2}{4}-1}(u_1-\frac{1}{2}-2e^{-i\theta})^{\frac{\gamma^2}{4}-1}\\
=&\frac{\mu}{2\sin(\pi\frac{\gamma^2}{4})}\int_{-i\mathbb{R}+\frac{1}{2}}du_1(u_1+i)^{\frac{\gamma^2}{4}-1}((u_1-1-i)^{\frac{\gamma^2}{4}}-(u_1-3+i)^{\frac{\gamma^2}{4}})\\
=&\frac{\mu}{2\sin(\pi\frac{\gamma^2}{4})}\int_{\mathbb{R}e^{-i\epsilon}-i}du_1(u_1+i)^{\frac{\gamma^2}{4}-1}((u_1-1-i)^{\frac{\gamma^2}{4}}-(u_1-3+i)^{\frac{\gamma^2}{4}})
\end{aligned}$$

We put them together and deform the contour accordingly, then we have

$$\begin{aligned}
&-\frac{\mu}{2\sin(\pi\frac{\gamma^2}{4})}\int_{-\frac{1}{\epsilon}}^{\frac{1}{\epsilon}}du_1(u_1+i)^{\frac{\gamma^2}{4}-1}((u_1+1-i)^{\frac{\gamma^2}{4}}-(u_1-1-i)^{\frac{\gamma^2}{4}})\\
&+\frac{\mu}{2\sin(\pi\frac{\gamma^2}{4})}\int_{(-\frac{1}{\epsilon},\frac{1}{\epsilon})e^{i\epsilon}}\left((u_1-\frac{3}{2})^{\frac{\gamma^2}{4}-1}(u_1+\frac{3}{2})^{\frac{\gamma^2}{4}}du_1-(\bar{u}_1+\frac{3}{2})^{\frac{\gamma^2}{4}-1}(\bar{u}_1-\frac{3}{2})^{\frac{\gamma^2}{4}}d\bar{u}_1\right)+o_{\epsilon\to 0}(1).
\end{aligned}$$

For the second integral we decompose it into:

$$
\begin{aligned}
&\frac{3\mu}{4\sin(\pi\frac{\gamma^2}{4})}\int_{(-\frac{1}{\epsilon},\frac{1}{\epsilon})e^{i\epsilon}}\left((u_1-\frac{3}{2})^{\frac{\gamma^2}{4}-1}(u_1+\frac{3}{2})^{\frac{\gamma^2}{4}-1}du_1+(\bar{u}_1+\frac{3}{2})^{\frac{\gamma^2}{4}-1}(\bar{u}_1-\frac{3}{2})^{\frac{\gamma^2}{4}-1}d\bar{u}_1\right)\\
&+\frac{\mu}{2\sin(\pi\frac{\gamma^2}{4})}\int_{(-\frac{1}{\epsilon},\frac{1}{\epsilon})e^{i\epsilon}}\left((u_1-\frac{3}{2})^{\frac{\gamma^2}{4}-1}(u_1+\frac{3}{2})^{\frac{\gamma^2}{4}-1}u_1du_1-(\bar{u}_1+\frac{3}{2})^{\frac{\gamma^2}{4}-1}(\bar{u}_1-\frac{3}{2})^{\frac{\gamma^2}{4}}\bar{u}_1d\bar{u}_1\right)\\
=&\frac{3\mu}{4\sin(\pi\frac{\gamma^2}{4})}\int_{i\mathbb{R}}\left((u_1-\frac{3}{2})^{\frac{\gamma^2}{4}-1}(u_1+\frac{3}{2})^{\frac{\gamma^2}{4}-1}du_1+(\bar{u}_1+\frac{3}{2})^{\frac{\gamma^2}{4}-1}(\bar{u}_1-\frac{3}{2})^{\frac{\gamma^2}{4}-1}d\bar{u}_1\right)\\
&-\frac{\mu e^{i\pi\frac{\gamma^2}{4}}}{\sin(\pi\frac{\gamma^2}{4})}\left(-\cos(\pi\frac{\gamma^2}{4})\int_{-\frac{1}{\epsilon}}^{-\frac{3}{2}}du_1|u_1-\frac{3}{2}|^{\frac{\gamma^2}{4}-1}|u_1+\frac{3}{2}|^{\frac{\gamma^2}{4}-1}u_1-\cos(\pi\frac{\gamma^2}{4})\int_{\frac{3}{2}}^{\frac{1}{\epsilon}}du_1|u_1-\frac{3}{2}|^{\frac{\gamma^2}{4}-1}|u_1+\frac{3}{2}|^{\frac{\gamma^2}{4}-1}u_1\right.\\
&\left.+\int_{-\frac{3}{2}}^{\frac{3}{2}}du_1|u_1-\frac{3}{2}|^{\frac{\gamma^2}{4}-1}|u_1+\frac{3}{2}|^{\frac{\gamma^2}{4}-1}u_1\right)+o_{\epsilon\to 0}(1)\\
=&\frac{3\mu}{4\sin(\pi\frac{\gamma^2}{4})}(-ie^{i\pi\frac{\gamma^2}{4}}+ie^{-i\pi\frac{\gamma^2}{4}})\int_{\mathbb{R}}du_1(u_1^2+\frac{9}{4})^{\frac{\gamma^2}{4}-1}+o_{\epsilon\to 0}(1)\\
=&\frac{3\mu}{2}\int_{\mathbb{R}}du_1(u_1^2+\frac{9}{4})^{\frac{\gamma^2}{4}-1}+o_{\epsilon\to 0}(1).
\end{aligned}
$$

□

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