

**ÉTALÉ COHOMOLOGY**

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Abstract. These are notes from the (ongoing) Étale Cohomology Reading Seminar at Columbia University in Fall 2015, which is organized by Remy van Dobben de Bruyn.

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1. Lecture 1 (September 8, 2015)

Remy van Dobben de Bruyn – Introduction: the Weil conjectures

PH: I missed the lecture.

2. Lecture 2 (September 15, 2015)

Remy van Dobben de Bruyn – Étale morphisms; fpqc descent (affine case); Henselisations

Today’s plan is to talk about a few issues of commutative algebra and algebraic geometry that we will be using throughout the semester. We will cover three topics today:

(1) étale morphisms,
(2) fpqc descent,
(3) Henselian rings.

The third topic will be treated minimally, and some good references are:

- Raynaud, Anneaux Locaux Henséliens (ALH).
- Stacks Project.

I will start with the notion of unramified morphisms.

2.1. Unramified morphisms.

Definition 2.1. Consider diagrams

\[
\begin{array}{ccc}
A & \to & C \\
\downarrow f & & \downarrow \\
B & \to & C/I \\
\end{array}
\]

where \( I^2 = 0 \). We say that

- \( f \) is formally unramified if there exists at most one lift,
- \( f \) is formally smooth if there exists at least one lift,
- \( f \) is formally étale if there exists exactly one lift.

This definition is not very practical as we cannot check this condition for all rings \( C \).

Lemma 2.2. \( f : A \to B \) is formally unramified if and only if \( \Omega_{B/A} = 0 \).

Proof. Let \( C_{\text{univ}} = (B \otimes_A B)/J^2 \), where \( J = \ker(B \otimes_A B \to B) \). Let \( I_{\text{univ}} = J/J^2 \). Then

\[
C_{\text{univ}}/I_{\text{univ}} = B.
\]

Let \( \sigma_1, \sigma_2 : B \to C_{\text{univ}} \) be given by \( b \mapsto b \otimes 1 \) and \( b \mapsto 1 \otimes b \) respectively. They both lift the identity \( B \to B \).

Recall that \( I_{\text{univ}} = J/J^2 \cong \Omega_{B/A} \), and the universal derivation is \( B \xrightarrow{\sigma_1-\sigma_2} I_{\text{univ}} \) given by \( b \mapsto b \otimes 1 - 1 \otimes b \).

Now if \( f \) is formally unramified, then \( \sigma_1 = \sigma_2 \), so \( d = 0 \). Thus \( \Omega_{B/A} = 0 \).

Conversely, assume \( \Omega_{B/A} = 0 \). Then if \( \tau_1, \tau_2 : B \to C \) are two lifts, then we get a map \( \varphi : B \otimes_A B \to C \) via \( b_1 \otimes b_2 \mapsto \tau_1(b_1)\tau_2(b_2) \), which factors through \( (B \otimes_A B)/J^2 \), since \( \varphi(J) \subseteq I \). But \( (B \otimes_A B)/J^2 = C_{\text{univ}} \), and \( I_{\text{univ}} = \Omega_{B/A} = 0 \). Thus, \( \sigma_1 = \sigma_2 \), and

\[
\tau_1 = \sigma_1 \varphi = \sigma_2 \varphi = \tau_2.
\]
Thus, there is at most one lift.

**Corollary 2.3.** “Formally unramified” is local on the source and target.

**Definition 2.4.** A morphism $f : X \to Y$ of schemes is **unramified** if $f$ is locally of finite type (l.f.t.) and formally unramified.

**Lemma 2.5.** Let $f : X \to Y$ be l.f.t.. Then $f$ is unramified if and only if the diagonal $\Delta : X \to X \times_Y X$ is an open immersion.

**Proof.** In general, if $W = \bigcup_{V \subseteq X} \text{affine} V \times_Y V$, then $W^\text{open} \subseteq X \times_Y X$, and $\Delta : X \to W$ is a closed immersion. If $I$ is the ideal sheaf, then $I/I^2 = \Omega_{X/Y}$.

Note that $I$ is finitely generated: if $B/A$ is generated by $x_i$, then $I$ is generated by $x_i \otimes 1 - 1 \otimes x_i$. By Nakayama, $V(I)$ is open if and only if $I = I^2$ (exercise).

Thus, $\Delta$ is an open immersion if and only if $\Omega_{X/Y} = 0$. $\Box$

**Exercise.** Let $f : X \to \text{Spec } K$. Then $f$ is unramified if and only if $X = \bigsqcup \text{Spec } L_i$, with $L_i/K$ finite separable.

**Corollary 2.6.** Let $f : X \to Y$ be l.f.t.. Then $f$ is unramified if and only if for all $x \in X$, $y = f(x)$, we have $m_x = m_y \mathcal{O}_{X,x}$ and $k(x)/k(y)$ is finite separable.

**Proof.** Use properties of $\Omega$, and Nakayama. $\Box$

**Summary:** TFAE

1. $f$ is l.f.t. and formally unramified.
2. $f$ is l.f.t. and $\Omega_{X/Y} = 0$.
3. $f$ is l.f.t. and $\Delta : X \to X \times_Y X$.
4. $f$ is l.f.t. and for all $x \in X$, $y = f(x)$, we have $m_x = m_y \mathcal{O}_{X,x}$ and $k(x)/k(y)$ is finite separable.

Next we consider étale morphisms.

2.2. **Étale morphisms.**

**Definition 2.7.** $f : X \to Y$ is **étale** if it is l.f.t., flat and unramified.

**Example 2.8.** $X \to \text{Spec } K$ is étale if and only if $X = \bigsqcup \text{Spec } L_i$, for $L_i/K$ finite separable.

**Example 2.9.** A **standard étale morphism** is a ring map $A \to B$ given by $B = (A[x]/(f))(f')$ for some $f \in A[x]$ monic. We compute that $\Omega_{(A[x]/(f))(f')/A} = A \frac{dx}{df} = A \frac{dx}{f'} dx$, so once we invert $f'$, this is 0.

**Example 2.10.** Open immersions are étale.

**Theorem 2.11.** Locally, every étale morphism is given by a standard étale map.

A proof can be found in the Stacks Project or ALH.
Lemma 2.12. Let $f : X \to Y$ and $g : Y \to Z$ be étale. Then $gf$ is étale.

Proof. Clearly $gf$ is l.f.p. and flat, because those properties are preserved under composition. Finally, the “first” exact sequence

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

implies that $\Omega_{X/Z} = 0$. \hfill \Box

Lemma 2.13. “Étale” is stable under base change.

Proof. This is immediate from the properties of $\Omega$. \hfill \Box

Here is a very important proposition.

Proposition 2.14. Let

$$\begin{array}{c}
X \\ h \downarrow \\
S
\end{array} \to
\begin{array}{c}
Y \\ g \downarrow \\
Y \times_S Y
\end{array}$$

such that $g$ is unramified. Then if $h$ is étale, so is $f$.

Proof. The diagram

$$\begin{array}{ccc}
X & \xrightarrow{\Gamma_f} & X \times_S Y \\
\downarrow f & & \downarrow \\
Y & \xrightarrow{\Delta_Y} & Y \times_S Y
\end{array}$$

is a pullback. Thus $\Gamma_f$ is an immersion since $\Delta_Y$ is. But $f = \pi_2 \circ \Gamma_f$, and $\pi_2 : X \times_S Y \to Y$ is the base change of $h$ along $g$.

$$\begin{array}{ccc}
X \times_S Y & \longrightarrow & Y \\
\downarrow g & & \downarrow \\
X & \xrightarrow{h} & S
\end{array}$$

Thus $f$ is étale, because both $\Gamma_f$ are $\pi_2$ are. \hfill \Box

We only used the formal properties of being étale.

2.3. fpqc descent. I hope to give an accessible introduction to fpqc descent. Today we will consider the affine case only, and once we understand the Grothendieck topology we will look at the geometric picture. The reference is EGA IV, part 2, section 2.5–2.7.

(1) “Permanence properties under fpqc base change”,
(2) “Effectiveness of descent data”.

Throughout, $A \to B$ will be a faithfully flat ring homomorphism. The difference between these is that in (1) we start with some object over $A$ and try to descend properties of $B$, whereas in (2), we have an object over $B$ and some descent data, and prove that it descends to $A$.

Faithfully flat means that any of the following equivalent conditions holds:

(1) $B \otimes_A - : A\text{-Mod} \to B\text{-Mod}$ is faithful and exact.
(2) $B$ is flat, and $B \otimes_A M = 0 \Rightarrow M = 0$. 

(3) \( B \) is flat, and \( 0 \to M_1 \to M_2 \to M_3 \to 0 \) is exact if and only if \( 0 \to B \otimes_A M_1 \to B \otimes_A M_2 \to B \otimes_A M_3 \to 0 \) is exact.

(4) \( B \) is flat and \( \text{Spec} \, B \to \text{Spec} \, A \) is surjective.

**Lemma 2.15** ("Permanence properties of modules"). Let \( f : A \to B \) be faithfully flat, \( M \) an \( A \)-module. If \( B \otimes_A M \) is

1. finitely generated,
2. finitely presented,
3. flat,
4. locally free of constant (finite) rank \( n \),
5. etc.,

then so is \( M \). (These are separate statements.)

**Proof.**

1. Let \( y_i \in B \otimes_A M \) be generators; say \( y_i = \sum_j b_{ij} \otimes m_{ij} \). Let \( M' \subseteq M \) be the submodule generated by the \( m_{ij} \). Then
   - \( B \otimes_A M' \hookrightarrow B \otimes_A M \) (because \( B \) is flat),
   - surjective as well.

   Thus by (3) of f.f., \( M' \hookrightarrow M \) is surjective, i.e., \( M = M' \).

2. Apply (1) twice: \( K \to A^n \to M \to 0 \). Apply it to \( K \).

3. Associativity of \( \otimes \) and (3) of f.f.

4. Locally free of locally constant finite rank \( \Leftrightarrow \) flat and f.p. By (2) and (3), this descends. But if \( M \) has constant rank \( n \) on a component of \( \text{Spec} \, A \), then \( B \otimes_A M \) also has constant rank \( n \) on the induced component.

\( \square \)

**Exercise** (Permanence properties of algebras). If \( C \) is an \( A \)-algebra, and \( B \otimes_A C \) is

1. f.t.,
2. f.p.,
3. finite,

then so is \( C \).

**Lemma 2.16** (Permanence properties of morphisms). Let \( S = \text{Spec} \, A \) and \( T = \text{Spec} \, B \). Let \( f : X \to Y \) be a morphism of \( A \)-schemes, and \( f_T : X_T \to Y_T \) its base change to \( T \). Then if \( f_T \) is

1. l.f.t.,
2. l.f.p.,
3. flat,
4. formally unramified,
5. \( \acute{e} \text{tale}, \)
6. etc.,

then so is \( f \).

**Proof.** First we will do some reductions. Being f.f. is stable under base change, and each of the properties is local on the source and target. Thus, by replacing \( S \) (resp. \( T \)) with \( Y \) (resp. \( Y_T \)), or an affine in it, we may wlog assume \( S = Y \) affine and \( X \) affine; say \( X = \text{Spec} \, C \).

1. and (2) follow from the exercise.
2. (3) follows from the previous lemma.
Since $\Omega_{B \otimes_A C/B} = B \otimes_A \Omega_{C/A}$, we get $\Omega_{B \otimes_A C/B} = 0$ if and only if $\Omega_{C/A} = 0$ by (2) of f.f.

(5) follows from (2), (3) and (4).

These are just a few examples of permanence properties. EGA has a lot more.

Next we will study effectiveness of descent data.

2.4. Descent data.

Definition 2.17. Let $f : A \to B$. A descent datum for $f$ is a pair $(N, \varphi)$, where $N$ is a $B$-module, and

$$\varphi : N \otimes_A B \sim \to B \otimes_A N$$

is a $B \otimes_A B$-linear isomorphism, such that the diagram of $B \otimes_A B \otimes_A B$-isomorphisms

$$
\begin{array}{c}
N \otimes_A B \otimes_A B \\
\varphi_{02} \\
\varphi_{01} = \varphi \otimes B \\
B \otimes_A N \otimes_A B \\
\varphi_{12} = B \otimes \varphi
\end{array}
$$

commutes.

A morphism $f : (N, \varphi) \to (N', \varphi')$ is a $B$-linear $f : N \to N'$ such that

$$
\begin{array}{c}
N \otimes_A B \sim \varphi \\
f \otimes B \\
B \otimes f \\
N \otimes_A B \sim \varphi' \\
N \otimes_A B \otimes_A N'
\end{array}
$$

commutes.

Example 2.18. For $M$ an $A$-module, the canonical descent datum is $(M \otimes_A B, \text{can})$, where

$$\text{can} : (M \otimes_A B) \otimes_A B \sim \to B \otimes_A (M \otimes_A B)$$

$$m \otimes b \otimes c \mapsto b \otimes m \otimes c.$$

This gives a functor

$$F : A\text{-Mod} \to f\text{-Desc}.$$

Theorem 2.19 (fpqc descent). If $f$ is f.f., then $F$ is an equivalence.

Remark. Since $B$ is f.f. over $A$, the functor $F$ is faithful and exact.

Proof. This theorem is fairly involved. A proof can be found in the Stacks Project (Tag 023N), or BLR’s Néron models.

Here are some geometric remarks.

Remark. It says that we can glue modules along “fpqc opens”: given $\mathcal{F}$ on $U \to X$ and an isomorphism $\mathcal{F}|_{U \times_X U} \sim \mathcal{F}|_{U \times_X U}$, with a cocycle condition on $U \times_X U \times_X U$, they glue to a sheaf on $X$. (Next week we will see that a faithfully flat ring homomorphism is an example of a covering in the fpqc topology.)

Unfortunately we ran out of time for Henselian rings. Please read a bit about them, e.g. in ALH, the Stacks project, or Milne’s book on Étale cohomology.