DO WE EVEN NEED DERIVED CATEGORIES?

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ABSTRACT. We will discuss the moduli space of stable curves of genus 0 with n marked points and its intersection theory, following [?]. We will give a nice presentation of its Chow ring in terms of boundary divisors.

1. SERRE FORMULA

Recall the following from my March 19 lecture (this is Proposition 7.2.9 in my notes):

Proposition. *Let X be a smooth variety, $V, W \subset X$ be closed subschemes that intersect properly, and Z be an irreducible component of $V \cap W$. Then*

$$V \cdot W = \sum_Z i(Z, V \cdot W; X) \cdot [Z],$$

where $1 \leq i(Z, V \cdot W; X) \leq \ell(\mathscr{O}_{V\cap W, Z})$ is the intersection number and $i(Z, V \cdot W; X) = \ell(\mathscr{O}_{V\cap W,Z})$ if and only if the local ring is Cohen-Macaulay.

However, most rings are **not** Cohen-Macaulay, so we would like a formula to compute the intersection multiplicities in all cases. Serre gives a formula in terms of higher Tor functors (because $\mathscr{O}_{V\cap W,Z} = \mathscr{O}_{V,Z} \otimes \mathscr{O}_{W,Z}$). Before we state the formula, first we will state some results about the higher Tor functors.

First, if X is a locally ringed space and $\mathscr{F}, \mathscr{G}$ are modules on X, then

$$\operatorname{Tor}_i^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x = \operatorname{Tor}_i^{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x).$$

This follows from the construction of the derived tensor product $\otimes^L$ in Stacks, which exposits derived categories much better than I ever could.

Lemma 1.1. *Let X be a locally Noetherian scheme. If $\mathscr{F}, \mathscr{G}$ are coherent, so is $\operatorname{Tor}_i^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$. Also, if $L, K \in D^-_{\mathrm{coh}}(\mathscr{O}_X)$ (this means bounded above complexes of quasicoherent sheaves with coherent homology), then so is $L \otimes^L K$.*

Proof of this fact is pure homological algebra, and again can be found in the Stacks project.

Lemma 1.2. *Let X be a smooth variety and $\mathscr{F}, \mathscr{G}$ be coherent sheaves. Then $\operatorname{Tor}_i^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is supported on $\operatorname{Supp} \mathscr{F} \cap \operatorname{Supp} \mathscr{G}$, and is nonzero only when $0 \leq i \leq \dim X$.*

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Proof. The support condition is clear by looking at the stalks, so we need to consider when the stalks are nonzero. Here, we note that because X is smooth, the local rings $\mathscr{O}_{X,x}$ are regular local rings. By a result of Serre (Theorem 4.4.1 in my commutative algebra notes), $\mathscr{O}_{X,x}$ has finite global dimension $\dim \mathscr{O}_{X,x} \leq \dim X$. Here, the global dimension and the dimension are the same by Theorem 4.3.12 of my commutative algebra notes.[1] $\square$

Now we can compute the intersection multiplicities as (Stacks gives this as the definition of intersection multiplicity)

$$i(Z, V \cdot W; X) = \sum (-1)^i \ell(\mathrm{Tor}_i^{\mathscr{O}_{X,Z}}(\mathscr{O}_{V,Z}, \mathscr{O}_{W,Z})).$$

This formula is due to Serre, and Stacks writes the total intersection as

$$W \cdot V = \sum (-1)^i [\mathrm{Tor}_i^{\mathscr{O}_X}(\mathscr{O}_V, \mathscr{O}_W)].$$

Remark 1.3. Stacks writes the intersection multiplicity as $e(X, V \cdot W, Z)$. I am using the notation in Fulton's book.

Lemma 1.4. *Assume that $\ell(\mathscr{O}_{V \cap W, Z}) = 1$. Then $i(Z, V \cdot W; X) = 1$ and V, W are smooth in a general point of Z.*

Proof. Write $A = \mathscr{O}_{X,Z}$. Then $\dim A = \dim X - \dim Z$. Let I, J be the ideals of V, W. By Proposition 7.2.9 of the notes,[2] $I + J = \mathfrak{m}$. Thus there exists $f_1, \ldots, f_r \in I, g_1, \ldots, f_s \in J$ Forming a basis for $\mathfrak{m}/\mathfrak{m}^2$. But this is a regular sequence and a system of parameters, so $A/(f_1, \ldots, f_r)$ is a regular local ring of dimension $\dim X - \dim V$, so $I = (f_1, \ldots, f_r)$. Similarly, $J = (g_1, \ldots, g_s)$. Now by Corollary 4.4.3 of commutative algebra, the Koszul complex $K(f_1, \ldots, f_r, A)$ resolves A/I, so we obtain

$$\begin{aligned} \mathrm{Tor}_i^A(A/I, A/J) &= H_i(K(f_1, \ldots, f_r, A) \otimes A/J) \\ &= H_i(K(f_1, \ldots, f_r, A/J)). \end{aligned}$$

By Theorem 4.4.2 from commutative algebra, we only have $H_0 = k$. $\square$

Example 1.5. Suppose $V, W \subset X$ are closed subvarieties, $\dim X = 4$, $\widehat{\mathscr{O}}_{X,p} = \mathbb{C}[[x, y, z, w]]$ and $V = (xz, xw, yz, yw), W = (x - z, y - w)$. Then

$$\ell(\mathbb{C}[[x, y, z, w]]/(xz, xw, yz, yw, x - z, x - w)) = 3,$$

but the intersection multiplicity is 2 because V is locally a union $(x = y = 0) \cup (z = w = 0)$.

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[1] Originally there was an argument that the global dimension of a Noetherian local ring is the projective dimension of the residue field, which is Theorem 4.3.10 of the commutative algebra notes, and then by the Auslander-Buchsbaum formula this is the same as the depth, and finally regular implies Cohen-Macaulay, so depth equals dimension.

[2] This may be cheating, and a self-contained argument is given in Stacks

2. Some algebra

Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring. If M is a module and I is an ideal of definition, recall the Hilbert-Samuel polynomial $\varphi_{I,M}(n) = \ell(I^n M / I^{n+1} M)$. Similarly recall the function

$$\chi_{I,M}(n) = \ell(M/I^{n+1}M) = \sum_{i=0}^{n} \varphi_{I,M}(i).$$

Recall that $d(M) := \deg \chi$ is independent of I and equals the dimension of the support of M (from the proof of Theorem 3.2.9 in my commutative algebta notes). Now write $\chi_{I,M}(n) = e_I(M,d)\frac{n^d}{d!} + O(n^{d-1})$.

Definition 2.1. For $d = d(M)$ we write $e_I(M,d)$ as above, and for $d > d(M)$, we set $e_I(M,d) = 0$.

Lemma 2.2. *For all I, M, we have*

$$e_I(M,d) = \sum_{\dim A/\mathfrak{p} = d} \ell_{A_\mathfrak{p}}(M_\mathfrak{p}) e_I(A/\mathfrak{p}, d).$$

Lemma 2.3. *Let P be a polynomial of degree r with leading coefficient a. Then*

$$r!a = \sum_{i=0}^{r} (-1)^i \binom{r}{i} P(t-i)$$

for any t.

Proof. Write Δ for the operator taking a polynomial P to $P(t) - P(t-1)$. Then

$$\begin{aligned}\Delta^{r+1}(P) &= \sum_{i=0}^{r} (-1)^i \binom{r}{i} \Delta(P)(t-i) \\ &= \sum_{i=0}^{r} (-1)^i \binom{r}{i} (P(t-i) - P(t-i-1)).\end{aligned}$$

The desired claim follows from Pascal's identity. □

Theorem 2.4. *Let A be a Noetherian local ring and $I = (f_1, \ldots, f_r)$ be an ideal of definition. Then*

$$e_I(M,r) = \sum (-1)^i \ell(H_i(K(f_1, \ldots, f_r) \otimes M)).$$

There is a very long proof of this statement in Stacks using spectral sequences.

3. Computing intersection multiplicities without derived categories

We give some cases where intersection multiplicities can be computed without using derived categories.

Lemma 3.1. *Suppose $\mathscr{O}_{V,Z}$ and $\mathscr{O}_{W,Z}$ are Cohen-Macaulay. Then $i(Z, V \cdot W; X) = \ell(\mathscr{O}_{V \cap W, Z})$.*

Proof. Write $A = \mathscr{O}_{X,Z}, B = \mathscr{O}_{V,Z}, C = \mathscr{O}_{W,Z}$. Then by Auslander-Buchsbaum (exercise 4d of the final CA homework), we have a resolution $F_\bullet \to B$ of length depth $A -$ depth $B = \dim A - \dim B = \dim C$. Then $F_\bullet \otimes C$ represents $B \otimes^L C$ and is supported in $\{\mathfrak{m}_A\}$, so by Lemma 10.108.2 in Stacks, it has nonzero cohomology only in degree 0. □

Lemma 3.2. *Let A be a Noetherian local ring and $I = (f_1, \ldots, f_r)$ is generated by a regular sequence. If M is a finite A-module with* $\dim \operatorname{Supp} M/IM = 0$*, then*

$$e_I(M,r) = \sum (-1)^i \ell(\operatorname{Tor}_i^A(A/I, M)).$$

In what follows, we will assume V is cut out in $\mathscr{O}_{X,Z}$ by a regular sequence $(f_1, \ldots, f_c)$.

Lemma 3.3. *In this case, we have $i(Z, V \cdot W; X) = c!$. This is the leading coefficient of the "Hilbert polynomial" $n \mapsto \ell(\mathscr{O}_{W,Z}/(f_1, \ldots, f_c)^t)$.*

Proof. By the previous lemma, $e(Z, V \cdot W; X) = e_{(f_1,\ldots,f_c)}(\mathscr{O}_{W,Z}(c))$. Now we need to show that $\dim \mathscr{O}_{W,Z} = c$. But now if $\dim V = r, \dim W = s, \dim X = n, \dim Z = r + s - n$, so $k(Z)$ has transcendence degree $r + s - n$. Because $f_1, \ldots, f_c$ is a regular sequence, $r + c = n$, so $\dim \mathscr{O}_{W,Z} = s - (r + s - n) = s - (n - c + s - n) = c$. □

Lemma 3.4. *Assume $c = 1$ (for example, V is an effeective Cartier divisor). Then $i(Z, V \cdot W; X) = \ell(\mathscr{O}_{W,Z}/(f_1))$.*

Proof. Note that $\mathscr{O}_{W,Z}$ is a Noetherian local domain of dimension 1. Then it is clear that $\ell(\mathscr{O}_{W,Z}/(f_1^t)) = t\ell(\mathscr{O}_{W,Z}/(f_1))$ for all $t \geq 1$. □

Lemma 3.5. *Asssume $\mathscr{O}_{W,Z}$ is Cohen-Macaulay. Then*

$$i(Z, V \cdot W; X) = \ell(\mathscr{O}_{W,Z}/(f_1, \ldots, f_c)).$$

Proof. Because $f_1, \ldots, f_c$ is a regular sequence, it is also quasi-regular by Proposition 3.5.6 of my commutative algebra notes. Then

$$\ell(\mathscr{O}_{W,Z}/(f_1, \ldots, f_c)^t) = \binom{c+t}{c} \ell(\mathscr{O}_{W,Z}/(f_1, \ldots, f_c)).$$

Now take the leading coefficient. □