FINE MODULI MEMES FOR 1-CATEGORICAL TEENS

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Abstract. We will discuss the moduli space of stable curves of genus 0 with n marked points and its intersection theory, following [1]. We will give a nice presentation of its Chow ring in terms of boundary divisors.

1. The moduli space

The space $\overline{M}_{0,n}$ parameterizes curves that look like this:

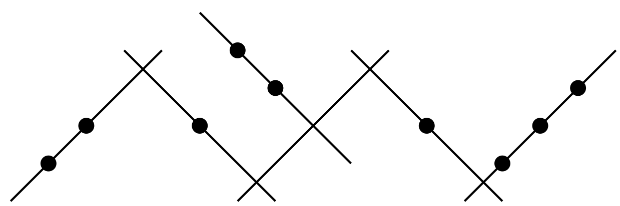

Figure 1. A stable curve

More precisely, these are reduced connected curves that are a tree of $\mathbb{P}^1$s such that each $\mathbb{P}^1$ has at least three marked points or nodes. In addition, at most two components meet at each node and we want $H^1(C, \mathscr{O}_C) = 0$. If all of these conditions are satisfied, we call our curve *stable*. More precisely, we want to represent the functor

$$S \mapsto \left\{ \mathscr{C} \underset{s_1,\ldots,s_n}{\overset{\pi}{\rightleftarrows}} S \ \text{ flat, proper} \ \middle| \ \begin{matrix} s_1, \ldots, s_n \text{ disjoint sections} \\ \text{geometric fibers are stable curves} \end{matrix} \right\}.$$

Theorem 1.1 (Knudsen). *There exists a smooth complete variety $\overline{M}_{0,n}$ and universal curve $U_{0,n} \to \overline{M}_{0,n}$ with universal sections $s_1, \ldots, s_n$ that is a fine moduli space for this functor. $\overline{M}_{0,n}$ also contains the space $M_{0,n} = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta$ as a dense open subset.*

In fact, Knudsen also shows that $U_{0,n} = \overline{M}_{0,n+1}$ and $U_{0,n+1}$ is a blowup of $\overline{M}_{0,n+1} \times_{\overline{M}_{0,n}} \overline{M}_{0,n+1}$ along some subscheme of the diagonal. In order to prove this, Knudsen introduces two operations that we can perform, called contraction and stabilization. Contraction happens when we delete a marked point and stabilization happens when we add a marked point.

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Theorem 1.2 (Knudsen)**.** *Contraction and stabilization are functorial! Moreover, they commute with base change.*

Here are some pictorial depictions of our operations:

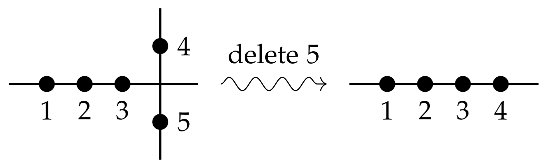


FIGURE 2. Contraction

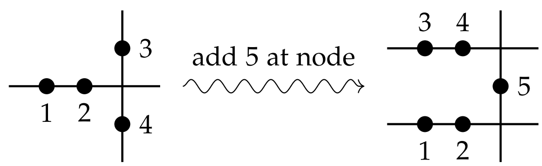


FIGURE 3. Stabilization (1)

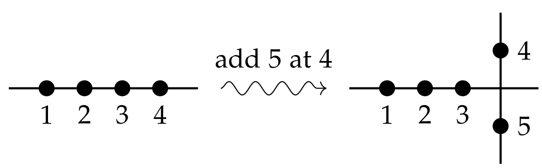
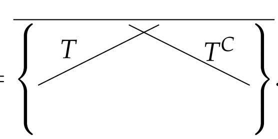


FIGURE 4. Stabilization (2)

2. KEEL CONSTRUCTION OF $\overline{M}_{0,n}$

First, we will describe the boundary divisors of $\overline{M}_{0,n}$. Let $T \subseteq \{1, \ldots, n\} =: [n]$ satisfy $|T|, \left|T^C\right| \geq 2$. Then define

$D^T :=$ .

It is easy to see that $D^T = D^{(T^C)}$. Knudsen proves that D^T is a smooth divisor and that $D^T \cong \overline{M}_{|T|+1} \times \overline{M}_{\left|T^C\right|+1}$.

Now consider the map $\pi \colon \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ coming from the identification $\overline{M}_{0,n+1} = U_{0,n}$. Now Keel proves that we can factor π as

$$\overline{M}_{0,n+1} \xrightarrow{\pi_1 = (\pi, \pi_{1,2,3,n+1})} \overline{M}_{0,n} \times \overline{M}_{0,4} \xrightarrow{p_1} \overline{M}_{0,n},$$

where $\pi_{1,2,3,n+1}\colon \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ forgets all sections besides $1, 2, 3, n+1$. Next, Keel shows that π_1 is a composition of blowups along smooth codimension 2 subvarieties using an inductive construction.

Set $B_1 = \overline{M}_{0,n} \times \overline{M}_{0,4}$. Then the universal sections $s_1, \ldots, s_n\colon \overline{M}_{0,n} \to \overline{M}_{0,n+1}$ induce sections $p \circ s_1, \ldots, p \circ s_n$. In fact, $D^T \cong p \circ s_i(D^T)$ and this is independent of i. We also have

Lemma 2.1 (Keel). *The divisors $D^T \subset \overline{M}_{0,n+1}$ with $T \subset [n]$ are the exceptional divisors of π_1.*

Now we set B_2 to be the blowup of B_1 at $\bigcup_{|T^C|=2} D^T$. Inductively, we set B_{k+1} to be the blowup of B_k at $\bigcup_{|T^C|=k+1} D^T$. Now we summarize the main results as follows:

Theorem 2.2 (Keel). *The map π_1 factors through B_k and $\overline{M}_{0,n+1} = B_{n-2}$.*

3. Intersection theory of $\overline{M}_{0,n}$

There are several major results about the intersection theory of $\overline{M}_{0,n}$. In fact, once we state these results, we will only be seven pages through Keel's paper, and the rest of the paper is dedicated to proving these results.

Theorem 3.1. *We have an isomorphism $A_*(\overline{M}_{0,n}) \to H_*(\overline{M}_{0,n})$. In particular, $\overline{M}_{0,n}$ has no odd homology and $A_*(\overline{M}_{0,n+1})$ is a finitely generated free abelian group. In fact, if a scheme Y satisfies $A^*(Y) = H^*(Y)$, then so does $Y \times \overline{M}_{0,n}$.*

Theorem 3.2. *For any scheme S, there is an isomorphism $A^*(\overline{M}_{0,n} \times S) = A^*(\overline{M}_{0,n}) \otimes A^*(S)$.*

Theorem 3.3. *For all k, we have an isomorphism*

$$A^k(\overline{M}_{0,n+1}) \cong A^k(\overline{M}_{0,n}) \oplus A^{k-1}(\overline{M}_{0,n}) \oplus \bigoplus_{\substack{T\subset[n]\\ |T\cap[3]|\leq 1}} A^{k-1}(D^T)$$

which is induced by the maps

$$A^k(\overline{M}_{0,n}) \xrightarrow{\pi^*} A^k(\overline{M}_{0,n+1})$$
$$A^{k-1}(\overline{M}_{0,n}) \xrightarrow{\pi^*} A^{k-1}(\overline{M}_{0,n+1}) \xrightarrow{\cup \pi^*_{1,2,3,n+1}(c_1(\mathscr{O}(1)))} A^k(\overline{M}_{0,n+1})$$
$$A^{k-1}(D^T) \xrightarrow{g^*} A^{k-1}(D^{T\subset[n+1]}) \xrightarrow{j_*} A^k(\overline{M}_{0,n+1}),$$

where g, j are as in the diagram

$$\begin{array}{ccc} D^{T\subset[n+1]} & \xhookrightarrow{j} & \overline{M}_{0,n+1} \\ \downarrow g & & \downarrow \pi \\ D^T & \xhookrightarrow{i} & \overline{M}_{0,n}. \end{array}$$

Theorem 3.4. *The Chow groups $A^k(\overline{M}_{0,n})$ are free abelian and the ranks $a^k(n) = \mathrm{rk}(A^k(\overline{M}_{0,n}))$ are given by the recursive formula*

$$a^k(n+1) = a^k(n) + a^{k-1}(n) + \frac{1}{2}\sum_{j=2}^{n-2}\binom{n}{j}\sum_{\ell=0}^{k-1} a^\ell(j+1)a^{k-1-\ell}(n-j-1).$$

In particular, we have the Picard rank $a^1(n) = 2^{n-1} - \binom{n}{2} - 1$.

Theorem 3.5. *The Chow ring $A^*(\overline{M}_{0,n})$ is the quotient of $\mathbb{Z}[D^T \mid T \subset [n], |T|, |T^C| \geq 2]$ by the relations*

(1) $D^T = D^{(T^C)}$;

(2) *For any distinct $i, j, k, \ell \in [n]$, we have the equality*
$$\sum_{\substack{i,j\in T\\ k,\ell\notin T}} D^T = \sum_{\substack{i,k\in T\\ j,\ell\notin T}} D^T = \sum_{\substack{i,\ell\in T\\ j,k\notin T}} D^T.$$

(3) *For $T_1, T_2 \subset [n]$, $D^{T_1}D^{T_2} = 0$ unless one of $T_1 \subset T_2, T_2 \subset T_1, T_1 \subset T_2^C, T_2 \subset T_1^C$ holds.*

Remark 3.6. All of the relations encode geometric content:

(1) As divisors, we already know that $D^T = D^{(T^C)}$.

(2) If we consider the map $\pi_{i,j,k,\ell} \colon \overline{M}_{0,n} \to \overline{M}_{0,4}$, then the three sums are the pullbacks of the three boundary divisors $D^{i,j}, D^{i,k}, D^{i,\ell} \subset \overline{M}_{0,4} = \mathbb{P}^1$.

(3) The final relation encodes the fact that $D^{T_1} \cap D^{T_2} = \emptyset$ unless one of the four inclusions holds. Pictorially, this is encoded in the diagram below:

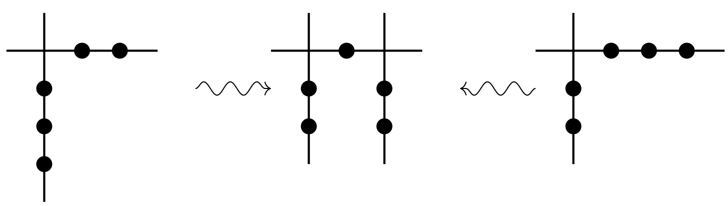


FIGURE 5. Degeneration to a common stable curve

4. INTERSECTION THEORY OF REGULAR BLOWUPS

Let $i \colon X \subset Y$ be a regularly embedded subvariety, $\pi \colon \widetilde{Y} \to Y$ be the blowup along X, and $\widetilde{X}$ be the exceptional divisor. Let $g \colon \widetilde{X} \to X$ and $j \colon \widetilde{X} \to \widetilde{Y}$.

Theorem 4.1. *Suppose i^* is surjective. Then*

$$A^*(\widetilde{Y}) = \frac{A^*(Y)[T]}{(P(T), T \cdot \ker(i^*))},$$

where $P(T)$ has constant term $[X]$ and $i^*P(T) = T^d + T^{d-1}c_1(N_XY) + \cdots + c_d(N_XY)$, where d is the codimension of X in Y. This is induced by $-T = [\widetilde{X}]$.

Theorem 4.2. *A scheme X is HI if $A_*(X) = H_*(X)$. If X, Y are both HI, then so is $\widetilde{Y}$.*

Theorem 4.3. *The map*

$$A_k(Y) \oplus A_{k-1}(X) \xrightarrow{(\pi^*, j_* g^*)} A_k(\widetilde{Y})$$

is an isomorphism.

References

[1] Sean Keel, *Intersection theory of moduli space of stable N-pointed curves of genus zero*, Transactions of the American Mathematical Society, Vol. 300, No. 2 (April 1992), pp. 545-574.