DOING ITALIAN-STYLE ALGEBRAIC GEOMETRY RIGOROUSLY

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Abstract. We will define intersection multiplicities and then define the Chow ring. For smooth varieties, the Chow ring behaves formally like cohomology in some ways. Finally, we will discuss Bézout's theorem, which has a very short proof in our language and then discuss some classical examples.

1. Intersection multiplicities

Consider a Cartesian square

$$\begin{tikzcd} W \arrow[r, "j"] \arrow[d, "g"] & V \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

where i is a regular embedding of codimension d and V has pure dimension k. Let $C = C_W V$ have components $C_1, \dots, C_r$ with multiplicity m_i. Let Z_i be the support of C_i. We call the Z_i the *distinguished varieties* of the intersection.

Lemma 1.1.

(a) *Every irreducible component of W is distinguished.*

(b) *For any distinguished variety Z, we have $k - d \leq \dim Z \leq k$.*

Definition 1.2. An irreducible component Z of $W = f^{-1}(X)$ is a *proper component* of intersection of V by X if $\dim Z = k - d$. The *intersection multiplicity* of Z in $X \cdot V$, denoted $i(Z, X \cdot V; Y) = i(Z, X \cdot V) = i(Z)$ is the coefficient of Z in the class $X \cdot V \in A_{k-d}(W)$.

If N_Z is the pullback of $N_X Y$ to Z, then $i(Z, X \cdot V; Y)$ is the coefficient of N_Z in $[C]$. Now let $A = \mathscr{O}_{Z,V}$ and $J \subset A$ be the ideal of W. Then A/J has finite length when Z is an irreducible component of W.

Proposition 1.3. *Assume Z is a proper component of W.*

(a) *If $\ell(A/J)$ is the length of A/J, then $1 \leq i(Z, X \cdot V; Y) \leq \ell(A/J)$.*

(b) *If J is generated by a regular sequence of length d, then $i(Z, X \cdot V; Y) = \ell(A/J)$.*

If A is Cohen-Macaulay, then local equations for X in Y give a regular sequence generating J and equality in (b) holds.

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Now suppose Z is a proper component of the intersection of V by X on Y. Let $A = \mathscr{O}_{V,Z}$ and J be the ideal in A generated by the ideal (sheaf) of X in Y, and let $\mathfrak{m}$ be the maximal ideal of A.

Proposition 1.4. *Suppose V is a variety. Then $i(Z, X \cdot V; Y) = 1$ if and only if A is regular and $J = \mathfrak{m}$.*

2. Nonsingular varieties

Let Y be a smooth variety of dimension n. Then $\Delta \subset Y \times Y$ is regularly embedded with codimension n. The global *intersection product* is the map

$$A_k(Y) \otimes A_\ell(Y) \to A_{k+\ell-n}(Y) \qquad x \otimes y \mapsto x \cdot y\Delta^*(x \times y),$$

where δ^* is the Gysin homomorphism.

More generally, let X be a scheme and $f \colon X \to Y$ a morphism to a smooth variety. Then Γ_f is regularly embedded in Y, so we can define a *cap product*

$$A_i(Y) \otimes A_j(X) \to A_{i+j-n}(X) \qquad y \otimes x \mapsto f^*(y) \cap x = \Gamma_f^*(x \times y).$$

If X is smooth, then we write $f^*y = f^*y \cap [X]$.

Remark 2.1. We may also replace the Gysin homomorphisms Δ^*, Γ_f^* with the refined Gysin homomorphisms $\Delta^!, \Gamma_f^!$.

Definition 2.2. Let $f \colon X \to Y$ be a morphism with Y smooth of dimension n. Let $p_X \colon X' \to X, p_Y \colon Y' \to Y$ be morphisms of schemes. Then form the square

$$\begin{tikzcd} X' \times_Y Y' \arrow[r] \arrow[d] & X' \times Y' \arrow[d, "p_X \times p_Y"] \\ X \arrow[r, "\Gamma_f"] & X \times Y. \end{tikzcd}$$

Now define the *refined intersection product* by

$$x \cdot_f y := \Gamma_f^!(x \times y) \in A_{k+\ell-n}(X' \times_Y Y')$$

for $x \in A_k(X'), Y \in A_\ell(Y')$. When $X' = X, Y' = Y$, this is the global product.

Proposition 2.3. *The refined products satisfy the following formal properties:*

(a) *(Associativity) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ with Y, Z smooth, then*
$$x \cdot_f (y \cdot_g z) = (x \cdot_f y) \cdot_{gf} z \in A_*(X' \times_Y Y' \times_Z Z').$$

(b) *(Commutativity) If $f_i \colon X \to Y_i$ with Y_i smooth, then*
$$(x \cdot_{f_1} y_1) \cdot_{f_2} y_2 = (x \cdot_{f_2} y_2) \cdot_{f_1} y_1 \in A_*(Y_1' \times_{Y_1} X' \times_{Y_2} Y_2').$$

(c) *(Projection) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ with Z smooth. Let $f' \colon X' \to Y'$ be proper with $p_Y f' = f p_X$ and $f'' = f' \times_Z \mathrm{id}_Z$. Then*
$$f''_*(x \times_{gf} z) = f'_*(x) \cdot_g z \in A_*(Y' \times_Z Z').$$

(d) *(Compatibility) Let $f\colon X \to Y$ with Y smooth and $g\colon V' \to Y'$ be a regular embedding. Then*

$$g^!(x \cdot_f y) = x \cdot_f g^! y \in A_*(X' \times_Y V').$$

Corollary 2.4. *Let Y be smooth and $j\colon V \to Y$ be a regular embedding. If x is a cycle on Y, then $x \cdot [V] = j^!(x) \in A_*(|x| \cap V)$.*

Corollary 2.5. *Let $f\colon X \to Y$ with X, Y smooth. Then $x \cdot_f y = (x \times y) \cdot [\Gamma_f] \in A_*(|x| \cap f^{-1}(|y|))$.*

Corollary 2.6. *Let $f\colon X \to Y$ with Y smooth and x a cycle on X. Then $x \cdot_f [Y] = x$.*

Definition 2.7. Let $f\colon X \to Y$ be a morphism with X purely m-dimensional and Y a smooth n-dimensional variety Y. For any morphism $g\colon Y' \to Y$, define a *refined Gysin homomorphism*

$$f^!\colon A_k(Y') \to A_{k+m-n}(X \times_Y Y') \qquad f^!(y) = [X] \cdot_f y.$$

Proposition 2.8.

(a) *If f is flat, then $f^!(y) = f'^*(y)$, where $f'\colon X \times_Y Y' \to Y'$ is the base change.*

(b) *If f is a local complete intersection morphism, then $f^!$ agrees with the morphism constructed in Section 6.6 of Fulton.*

Now let Y be a smooth variety of dimension n. Let V, W be closed subschemes of Y of pure dimension k, ℓ. Now a component $Z \subseteq V \cap W$ is a *proper component* if $\dim Z = k + \ell - n$. If Z is proper, then the coefficient of Z in $V \cdot W \in A_{k+\ell-n}(V \cap W)$ is called the *intersection multiplicity* $i(Z, V \cdot W; Y) = i(Z, \Delta_Y \cdot (V \times W); Y \times Y)$. If every component of $V \cap W$ is proper, then the intersection class is

$$V \cdot W = \sum_Z i(Z, V \cdot W; Y) \cdot [Z].$$

Proposition 2.9. *Assume Z is a proper component of $V \cap W$. Then*

(a) $1 \leq i(Z, V \cdot W; Y) \leq \mathscr{O}_{V \cap W, Z}$.

(b) *If the local ring $\mathscr{O}_{V \cap W, Z}$ is Cohen-Macaulay, then $i(Z, V \cdot W; Y) = \ell(\mathscr{O}_{V \cap W, Z})$.*

(c) *If V, W are varieties, then $i(Z, V \cdot W; Y) = 1$ if and only if the maximal ideal of $\mathscr{O}_{Y,Z}$ is the sum of the prime ideals of V and W. In fact, $\mathscr{O}_{V,Z}, \mathscr{O}_{W,Z}$ are regular.*

Now let Y be a smooth variety of dimension n. Set $A^p(Y) = A_{n-p}(Y)$. Now this indexing, the intersection product is now $A^p(Y) \otimes A^q(Y) \to A^{p+q}(Y)$. In addition, if $f\colon X \to Y$ is a morphism, the cap product is now $A^p(Y) \otimes A_q(X) \xrightarrow{\cap} A_{q-p}(X)$. If X is also smooth, the pullback now reads $f^*\colon A^p(Y) \to A^p(X)$.

Proposition 2.10.

(a) *Suppose Y is a smooth variety. Then the intersection product makes $A^*(Y)$ into a commutative graded ring with unit $A^0 \ni 1 = [Y] \in A_n$. Then $Y \mapsto A^*(Y)$ is a contravariant functor from smooth varieties to rings.*

(b) *If $f\colon X \to Y$ is a morphism from a scheme X to a smooth variety Y, then A_*X is a A^*Y-module with action*

$$A^p(Y) \otimes A_q(X) \xrightarrow{\cap} A_{q-p}(X).$$

(c) *If $f\colon X \to Y$ is a proper morphism of smooth varieties, then*

$$f_*(f^*y \cdot x) = y \cdot f_*(x)$$

for all classes x on X and y on Y.

3. Bézout's Theorem

We will now use this theory to discuss something classical. It is easy to see that $A_k(\mathbb{P}^n) \cong \mathbb{Z}$ and is generated by $[L^k]$ for a linear subspace $L^k \subset \mathbb{P}^n$. If α is a k-cycle on $\mathbb{P}^n$, we define the *degree* $\deg(\alpha)$ to be the integer satisfying $\alpha = \deg(\alpha) \cdot [L^k]$. Equivalently, we may define

$$\deg(\alpha) = \int_{\mathbb{P}^n} c_1(\mathscr{O}(1))^k \cap \alpha.$$

Theorem 3.1 (Bézout). *Let $\alpha_i \in A^{d_i}(\mathbb{P}^n)$ for $i = 1, \ldots, r$. If $d_1 + \cdots + d_r \leq n$, then*

$$\deg(\alpha_1 \cdots \alpha_r) = \deg(\alpha_1) \cdots \deg(\alpha_r).$$

Proof. We have an isomorphism $A^*(\mathbb{P}^n) = \mathbb{Z}[h]/(h^{n+1})$, where $h = [L^{n-1}]$. Thus $[L^{n-k}] = h^k$, and the desired result follows. □

Now if subschemes $V_1, \ldots, V_r \subseteq \mathbb{P}^n$ representing $\alpha_1, \ldots, \alpha_r$ meet properly, then

$$V_1 \cdots V_r = \sum_j i(Z_j, V_1 \cdots V_r; \mathbb{P}^n) \cdot [Z_j],$$

where Z_j are the components of $\bigcap V_i$. Then Bézout's theorem gives us the identity

$$\sum_j i(Z_j, V_1 \cdots V_r; \mathbb{P}^n) \cdot \deg(Z_j) = \prod \deg(V_i).$$

If $H_1, \ldots, H_n$ are hypersurfaces intersecting properly (so the intersection is a finite number of points), then consider the local ring $\mathscr{O}_{\bigcap H_i, P}$. Then complete intersections are Cohen-Macaulay, so

$$i(P, H_1 \cdots H_n; \mathbb{P}^n) = \ell(\mathscr{O}_{\bigcap H_i, P}).$$

Now $\dim_k \mathscr{O}_{\bigcap H_i, P} = \deg P \ell(\mathscr{O}_{\bigcap H_i, P})$, and thus we obtain

$$\sum_P \dim_k \mathscr{O}_{\bigcap H_i, P} = \prod \deg H_i.$$

This recovers the very classical Bézout's theorem.

Example 3.2. Let s be the hyperplane class on $\mathbb{P}^n$ and t be the hyperplane class on $\mathbb{P}^m$. Then

(1) $A^*(\mathbb{P}^n \times \mathbb{P}^m) = \mathbb{Z}[s, t]/(s^{n+1}, t^{m+1})$.

(2) If $H_1, \dots, H_{n+m}$ are hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^m$ with bidegree (a_i, b_i), then

$$\int [H_1] \cdots [H_{n+m}] = \sum_{\substack{(i_1,\dots,i_n,j_1,\dots,j_m) \\ (n,m)\text{-shuffle}}} a_{i_1} \cdots a_{i_n} b_{j_1} \cdots b_{j_m}.$$

(3) If Δ is the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$, then $[\Delta] = \sum_{i=0}^n s^i t^{n-i} \in A^n(\mathbb{P}^n \times \mathbb{P}^n)$. This formula follows from intersecting Δ with $[L_1 \times L_2]$, where L_1, L_2 are linear subspaces of complementary dimension.

(4) Let $s \colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ be the Segre embedding. It h is the hyperplane class on $\mathbb{P}^{nm+n+m}$, then $s^* u = s + t$. Also, the degree of $s(\mathbb{P}^n \times \mathbb{P}^m)$ is $\binom{n+m}{n}$.

(5) If $v_m \colon \mathbb{P}^n \to \mathbb{P}^{\binom{n+m}{n}-1}$ is the Veronese embedding and s, h are the hyperplane classes on the source and target, then $v_m^* u = m \cdot s$. If V is a k-dimensional subvarity of $\mathbb{P}^n$ of degree d, then $\deg(v_m(V)) = d \cdot m^k$.

References

[1] William Fulton, *Intersection Theory*, 2 ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 2, Springer-Verlag, 1998.