CONES: BECAUSE NOT EVERY COHERENT SHEAF IS LOCALLY FREE

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Abstract. We will discuss what a cone is, then define the Segre class of a cone, then define Segre classe of subvarieties and consider their properties, and then discuss deformation to the normal cone following Chapters 4 and 5 of [1]. Classical examples will be used to illustrate the theory.

1. Cones and Segre Classes

Our goal is to define a Segre class $s(X, Y)$ of a subvariety $X \subsetneq Y$ and study its properties.

1.1. Cones.

Definition 1.1. Let $S^\bullet$ be a sheaf of graded $\mathscr{O}_X$-algebras such that $\mathscr{O}_X \to S^0$ is surjective, S^1 is coherent, and $S^\bullet$ is generated by S^1. Then any scheme of the form $C = \operatorname{Spec}_{\mathscr{O}_X}(S^\bullet)$ is called a *cone*.

If C is a cone, then $\mathbb{P}(C \oplus 1) = \operatorname{Proj}(S^\bullet[z])$ is the projective completion with projection $q \colon \mathbb{P}(C \oplus 1 \to X)$. Let $\mathscr{O}(1)$ be the canonical line bundle on $\mathbb{P}(C \oplus 1)$.

Definition 1.2. The *Segre class* $s(C) \in A_* X$ of C is defined as

$$s(C) := q_* \left(\sum_{i \geq 0} c_1(\mathscr{O}(1))^i \cap [\mathbb{P}(C \oplus 1)] \right).$$

Proposition 1.3.

(1) *If E is a vector bundle on X, then $s(E) = c(E)^{-1} \cap [X]$, where $c = 1 + c_1 + \cdots$ is the total Chern class.*

(2) *Let $_1, \ldots, c_t$ by the irreducible components of C with geometric multiplicity m_i. Then*

$$s(C) = \sum_{i=1}^{t} m_i s(C_i).$$

Example 1.4. Let $\mathscr{F}, \mathscr{F}'$ be coherent sheaves and let $\mathscr{E}$ be locally free. Then we may define $C(\mathscr{F}) = \operatorname{Spec}(\operatorname{Sym} \mathscr{F})$. We may define $s(\mathscr{F}) = s(C(\mathscr{F}))$. Now if

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{E} \to 0$$

is exact, then $s(\mathscr{F}') = c(E) \cap s(\mathscr{F})$.

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1.2. **Segre Class of a Subvariety.** Let X be a closed subscheme of Y defined by the ideal sheaf $\mathscr{I}$ and let

$$C = C_X Y = \operatorname{Spec}\left(\sum_{n=0}^{\infty} \mathscr{I}^n / \mathscr{I}^{n+1}\right)$$

be the normal cone. Note that if X is regularly embedded in Y, then $C_X Y$ is a vector bundle.

Definition 1.5. The *Segre class* of X in Y is defined by

$$s(X, Y) := s(C_X Y) \in A_* X.$$

Lemma 1.6. *Let Y be a scheme of pure dimension m and let $Y_1, \ldots, Y_r$ be the irreducible components of Y with multiplicity m_i. If X is a closed subscheme of Y and $X_i = X \cap Y_i$, then*

$$s(X, Y) = \sum m_i s(X_i, Y_i).$$

Proposition 1.7. *Let $f\colon Y' \to Y$ be a morphism of pure-dimensional schemes, $X \subseteq Y$ a closed subscheme, and $g\colon X' = f^{-1}(X) \to X$ be the induced morphism.*

(1) If f is proper, Y is irreducible, and f maps each irreducible component of Y' onto Y, then

$$g_*(s(X', Y')) = \deg(Y'/Y) \cdot s(X, Y).$$

(2) If f is flat, then $g^*(s(X, Y)) = s(X', Y')$.

Remark 1.8. If f is birational, then $f_*(s(X', Y')) = s(X, Y)$. This says that Segre classes are unchanged by pushforward along birational modifications.

Corollary 1.9. *Let Y be a variety and $X \subseteq Y$ be a proper closed subsecheme. Then let $\widetilde{Y} = \operatorname{Bl}_X Y$ and $\widetilde{X} = \mathbb{P}(C)$ be the exceptional divisor with projection $\eta\colon \widetilde{X} \to X$. Then*

$$s(X, Y) = \sum_{k \geq 1} (-1)^{k-1} \eta_*(\widetilde{X}^k) = \sum_{i \geq 0} \eta_*(c_1(\mathscr{O}(1))^i \cap [\mathbb{P}(C)]).$$

Example 1.10. Let A, B, D be effective Cartier divisors on a surface Y. Then let $A' = A + D, B' = B + D$, and let $X = A' \cap B'$. Suppose that A, B meet transversally at a single smooth point $P \in Y$. Then if $\widetilde{Y} = \operatorname{Bl}_P Y$ and $f\colon \widetilde{Y} \to Y$ is the blowup with exceptional divisor E, we see that $\widetilde{X} = f^{-1}(X) = f^*D + E$, so we have

$$\begin{aligned} s(X, Y) &= f_*[\widetilde{X}] - f_*(\widetilde{X} \cdot [\widetilde{X}]) \\ &= [D] - f_*(f^*D \cdot [f^*D] + 2f^*D \cdot [E] + E \cdot [E]) \\ &= [D] - D \cdot [D] + [P]. \end{aligned}$$

If A, B both have multiplicity m at P and no common tangents at P, then

$$s(X, Y) = [D] + (m^2[P] - D \cdot [D]).$$

In general, the answer is more complicated.

1.3. **Multiplicity.** Let $X \subseteq Y$ be an (irreducible) subvariety. Then the coefficient of $[X]$ in the class $s(X, Y)$ is called the *algebraic multiplicity* of X on Y and is denoted $e_X Y$.

Suppose X has positive codimension n, $p\colon \mathbb{P}(C_X Y) \to X$ and $q\colon \mathbb{P}(C_X Y \oplus 1) \to X$ are the projections to X, and $\widetilde{Y} = \mathrm{Bl}_X Y$ with exceptional divisor $\widetilde{X} = \mathbb{P}(C)$. Then we have

$$\begin{aligned} e_X Y[X] &= q_*(c_1(\mathscr{O}(1))^n \cap [\mathbb{P}[C \oplus 1]]) \\ &= p_*(c_1(\mathscr{O}(1))^{n-1} \cap [\mathbb{P}(C)]) \\ &= (-1)^{n-1} p_*(\widetilde{X}^n). \end{aligned}$$

For example, if X is a point, then we have

$$e_P Y = \int_{\mathbb{P}(C)} c_1(\mathscr{O}(1))^{n-1} \cap [\mathbb{P}(C)] = \deg[P(C)].$$

Example 1.11. Let C be a smooth curve of genus g and $C^{(d)}$ be the d-th symmetric power of C. Then let $P_0 \in C$, $J = J(C)$ be the Jacobian, and $u_d\colon C^{(d)} \to J$ be given by $D \mapsto D - dP_0$. We know that the fibers of u_d are the linear systems $|D| \cong \mathbb{P}^r$; if $d > 2g - 2$, then $u_d\colon C^{(d)} \to J$ is a projective bundle; and if $1 \le d \le g$, then μ_d is birational onto its image W_d. Now if $\deg D = d$ and $\dim |D| = r$, we have

$$s(D, C^{(d)}) = (1 + K)^{g-d+r} \cap [|D|],$$

where $K = c_1(K_{|D|})$. When d is large, this follows from the second bullet, but if d is small, then we may embed

$$C^{(d)} \subset C^{(d+s)} \qquad E \mapsto E + sP_0$$

and then consider the normal bundle to this embedding restricted to $|D|$. Combined with Proposition 4.1.7, this gives us the *Riemann-Kempf formula*, which says that the multiplicity of W_d at $u_d(D)$ is given by $e_{\mu_d(D)} W_d = \binom{g-d+r}{r}$.

Remark 1.12. The previous example can be generalized to the Fano varieties of lines on a cubic threefold X. In particular if F is the Fano variety of lines on X, then there is a morphism of degree 6 from $F \times F$ to the theta divisor, and we can calculate (following Clemens-Griffiths) that

$$\int_F s_2(T_F) = \int_F c_1(T_F)^2 - c_2(T_f) = 45 - 27 = 18,$$

and then the theta divisor has a singular point of multiplicity 3.

1.4. **Linear Systems.** Let L be a line bundle on a variety X (of dimension n) and let $V \subseteq |L|$ be a partial linear system of dimension $r + 1$. Then let B be the base locus of V. Then if $\widetilde{X} = \mathrm{Bl}_B X$, we obtain a morphism $f\colon \widetilde{X} \to \mathbb{P}^r$ resolving the rational map $X \dashrightarrow \mathbb{P}^r$. By definition, we have $f^*\mathscr{O}(1) = \pi^*(L) \otimes \mathscr{O}(-E)$. Define $\deg_f \widetilde{X}$ to be the degree of $f_*[\widetilde{X}] \in A_n \mathbb{P}^r$.

Proposition 1.13. *We have the identity*

$$\deg_f \widetilde{X} = \int_X c_1(L)^n - \int_B c_1(L)^n \cap s(B, X).$$

Example 1.14. Let $B \subset \mathbb{P}^n$ be the rational normal curve. Then let $V \subset |\mathscr{O}(2)|$ be the linear system of quadrics containing B. If $\widetilde{P}^n = \mathrm{Bl}_B \mathbb{P}^n$, we see that

$$\deg_f \widetilde{\mathbb{P}}^n = 2^n - (n^2 - n + 2).$$

If $n = 4$, then $f(\widetilde{\mathbb{P}}^4) = \mathrm{Gr}(2,4) \subset \mathbb{P}^5$.

2. Deformation to the Normal Cone

Let $X \subseteq Y$ be a closed subscheme and $C = C_X Y$ be the normal cone. We will construct a scheme $M = M_X Y$ and a closed embedding $X \times \mathbb{P}^1 \subseteq M$ such that

$$\begin{array}{ccc} X \times \mathbb{P}^1 & \hookrightarrow & M \\ & {\scriptstyle p_2} \searrow \quad \swarrow {\scriptstyle q} & \\ & \mathbb{P}^1 & \end{array}$$

comutes and such that

(1) Away from ∞, we have $q^{-1}(\mathbb{A}^1) = Y \times \mathbb{A}^1$ and the embedding is the trivial embedding $X \times \mathbb{A}^1 \subseteq Y \times \mathbb{A}^1$.

(2) Over ∞, $M_\infty = \mathbb{P}(C \oplus 1) + \widetilde{Y}$ is a sum of two Cartier divisors, where $\widetilde{Y} = \mathrm{Bl}_X Y$. The embedding of X is given by $X \hookrightarrow C \hookrightarrow \mathbb{P}(C \oplus 1)$. We also have $\mathbb{P}(C \oplus 1) \cap \widetilde{Y} = \mathbb{P}(C)$, which is embedded as the hyperplane at ∞ in $\mathbb{P}(C \oplus 1)$ and as the exceptional divisor in $\widetilde{Y}$.

We will now construct this deformation. Let $M = \mathrm{Bl}_{X \times \infty} Y \times \mathbb{P}^1$. Clearly we have $C_{X \times \infty} Y \times \mathbb{P}^1 = C \oplus 1$. But now we can embed $X \times \mathbb{P}^1 \subseteq M$. The first property is obvious by the blowup construction, so now we need to show the second property.

We may assume $Y = \operatorname{Spec} A$ is affine and X is defined by the ideal I. Identify $\mathbb{P}^1 \setminus 0 = \mathbb{A}^1 = \operatorname{Spec} k[t]$. Then if we write $S^n = I, T^n$, then we see that $\mathrm{Bl}_{X \times 0} Y \times \mathbb{A}^1 = \operatorname{Proj} S^\bullet$. But now this is covered by affines

$$\left\{ \operatorname{Spec} S^\bullet_{(a)} \right\}_{a \in (I,T) \text{ generator}}.$$

Now for $a \in I$, we see that $\mathbb{P}(C \oplus 1) \subseteq \operatorname{Spec} S^\bullet_{(a)}$ is defined by the equation $a/1$, while $\widetilde{Y}$ is defined by T/a, and now we see that

$$M_\infty = V(T) = V\left(\frac{a}{1} \cdot \frac{t}{a} \right) = V(a) \cup V(T/a) = \mathbb{P}(C \oplus 1) + \widetilde{Y},$$

as desired.

Now this allows us to define a *specialization* morphism

$$\sigma \colon Z_k Y \to Z_k C \qquad [V] \mapsto [C_{V \cap X} V].$$

Proposition 2.1. *Specialization preserves rational equivalence. Therefore we have a specialization morphism*

$$\sigma \colon A_k Y \to A_k C.$$

Remark 2.2. Supposing that X, Y are smooth, then the embedding of $X \subset \mathbb{P}(N \oplus 1)$ is nicer than $X \subset Y$ in several ways:

(1) There is a retraction $\mathbb{P}(N \oplus 1) \to X$.

(2) There is a vector bundle ξ on $\mathbb{P}(N \oplus 1)$ or rank $\operatorname{codim}_Y X$ and a section $s \in \Gamma(\xi)$ such that $V(s) = X$. Therefore X is represented by the top Chern class of ξ.

References

[1] William Fulton, *Intersection Theory*, 2 ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 2, Springer-Verlag, 1998.