Intersection theory

Elmo learns motivic cohomology!

Some motivation

→ **Proposition 1.1** (Fulton, Prop. 1.8). *Let Y be a closed subscheme of a scheme X, and let $U = X - Y$. Let $i : Y \to X$, $j : U \to X$ be the inclusions. Then the sequence*

$$\mathrm{CH}^k Y \xrightarrow{i_*} \mathrm{CH}^k X \xrightarrow{j^*} \mathrm{CH}^k U \to 0$$

is exact for all k.

→ *Want to continue this?*

→ *You need higher Chow groups!*

Spencer Bloch to the rescue!

Recall this definition of rational equivalence

Definition 1.2 (Rat(X)). *Let $Z(X)$ denote the cycles of a scheme X and let Φ be any subvariety of $X \times \mathbb{P}^1$. Then we define* Rat(X) $\subset Z(X)$ *be the subgroup generated by differences of the form*

$$[\Phi \cap (X \times \{0\})] - [\Phi \cap (X \times \{\infty\})].$$

Then rationally equivalent cycles are those which differ by something in Rat(X).

Idea: there is a homotopy between rationally equivalent cycles!

Defining higher Chow groups

- Define algebraic simplex $$\Delta^k = \operatorname{Spec} k[x_0, \ldots, x_n]/(x_0 + \cdots + x_k - 1).$$
- Let $z^i(X, n)$ be the subgroup of $Z^i(X \times \Delta^n)$ that meets all faces properly.
- This gives a simplicial abelian group $z^i(X,\cdot)$ and a chain complex $z^i(X,*)$.
- Define

$$\mathrm{CH}^i(X, m) := \pi_m(z^i(X, \bullet)) = H_m(z^i(X, *)).$$

Properties of higher Chow groups

→ *Homotopy invariance*

The projection $X \times \mathbb{A}^1 \to X$ induces an isomorphism

$$\mathrm{CH}^i(X, m) \cong \mathrm{CH}^i(X \times \mathbb{A}^1, m).$$

→ *Long exact sequence*

There is a distinguished triangle

$$z_p(Y, *) \to z_p(X, *) \to z_p(U, *) \to z_p(Y, *)[1].$$

→ *Isomorphism with rational (algebraic) K-theory*

$$(K_i(X) \otimes \mathbb{Q})^{(q)} \cong \mathrm{CH}^q(X, i) \otimes \mathbb{Q}.$$

Cool! But how are these Chow groups related to motivic cohomology?

$$H^{p,q}(X;A) \;=\; \mathrm{CH}^q(X, 2q-p; A).$$

Correspondences

→ *"Multi-valued functions"*

Definition 2.1. *An* ***elementary correspondence*** *between a smooth connected scheme X/k to a separated scheme Y/k is an irreducible closed subset $W \subset X \times Y$ whose associated integral subscheme is finite and surjective over X.*

If X is not connected, then an elementary correspondence refers to one that is one from a connected component of X to Y.

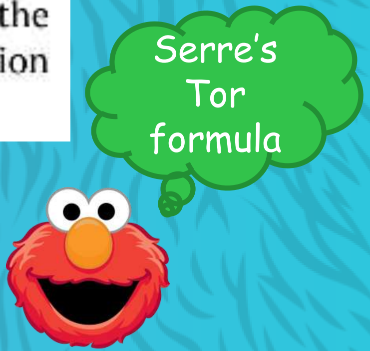
The group **Cor**(X, Y) *of* ***finite correspondences*** *is the free abelian group generated by the elementary correspondences.*

→ *eg* $\Gamma_f \subset X \times Y$

The category of correspondences Cor_k

- *Objects: same as Sm_k: smooth separated schemes of finite type over k*
  - *Given $Z \subset X \times Y$ finite and surjective over X, we set* $[Z] = \sum len(\mathcal{O}_{Z,W_i})[W_i]$
- *Morphisms:* $Cor(X, Y)$

We compose correspondences $V \in \mathbf{Cor}(X, Y)$ and $W \in \mathbf{Cor}(Y, Z)$ as follows. Construct the cycle $[T] = (V \times Z) \cdot (X \times W)$ on $X \times Y \times Z$. Then take its pushforward along the projection $p : X \times Y \times Z \to X \times Z$.



The category of correspondences Cor_k

- Sm_k embeds into Cor_k as a subcategory!
  - $(f: X \to Y) \mapsto \Gamma_f \subset X \times Y.$

- Cor_k is a symmetric monoidal category
  - $X \otimes Y = X \times Y$
  - $V \times W \in Cor_k(X \otimes Y, X' \otimes Y')$

Examples

- → $\mathbf{Cor}_k(\operatorname{Spec} k, X)$ is generated by the 0-cycles of X.
- → $\mathbf{Cor}_k(X, \operatorname{Spec} k)$ is generated by the irreducible components of X.
- → Take $W \in \mathbf{Cor}_k(\mathbb{A}^1, X)$ and two k-points $s, t : \operatorname{Spec} k \to \mathbb{A}^1$. Then the zero-cycles $W \circ \Gamma_s$ and $W \circ \Gamma_t$ are rationally equivalent.


Next, we will discuss presheaves with transfers!
Interesting!

Presheaves with transfers

- Definition

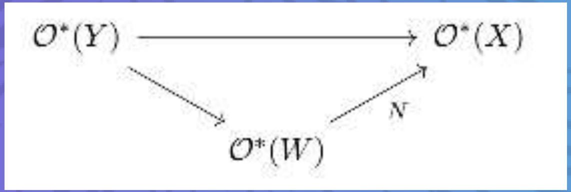
A **presheaf with transfers** is a contravariant additive functor $F : \mathbf{Cor}_k \to \mathrm{Ab}$.

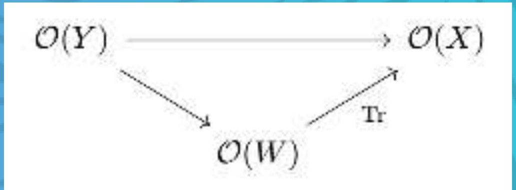
- Additivity means that $F(x \oplus y) = F(x) \oplus F(y)$.
  - We obtain a map $Cor_k(X,Y) \otimes F(Y) \to F(X)$.
- Extra "transfer maps" $F(Y) \to F(X)$ coming from $Cor_k(X,Y)$
- $\boldsymbol{PST}(k)$ is an abelian category with enough injectives and projectives.

Examples

→ *Constant presheaf* A *on* Sm_k. *Extend to* Cor_k *by multiplication by the degree of* W *over* X.

→ $O^*(-)$ *and* $O(-)$, *at least for* X *normal…*





→ $CH^i(-)$, *the Chow groups.*

Representable functors

- All representable functors of Cor_k are presheaves with transfers.
- Denote $$\mathbb{Z}_{tr}(X) := h_X(-).$$
- By Yoneda, $Z_{tr}(X)$ is a projective object.
- Let (X, x) be a pointed scheme and define

$$\mathbb{Z}_{tr}(X, x) := \operatorname{coker}[x_* : \mathbb{Z} \to \mathbb{Z}_{tr}(X)]. \quad \Rightarrow \quad \mathbb{Z}_{tr}(X) \cong \mathbb{Z} \oplus \mathbb{Z}_{tr}(X, x).$$

More definitions

→ DEFINITION 2.12. If (X_i, x_i) are pointed schemes for $i = 1, \dots, n$ we define $\mathbb{Z}_{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))$, or $\mathbb{Z}_{tr}(X_1 \wedge \cdots \wedge X_n)$, to be:

$$\operatorname{coker}\left(\bigoplus_i \mathbb{Z}_{tr}(X_1 \times \cdots \hat{X}_i \cdots \times X_n) \xrightarrow{id \times \cdots \times x_i \times \cdots \times id} \mathbb{Z}_{tr}(X_1 \times \cdots \times X_n) \right).$$

→ LEMMA 2.13. *The presheaf $\mathbb{Z}_{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))$ is a direct summand of $\mathbb{Z}_{tr}(X_1 \times \cdots \times X_n)$. In particular, it is a projective object of* **PST**.

Moreover, the following sequence of presheaves with transfers is split-exact:

$$0 \to \mathbb{Z} \xrightarrow{\{x_i\}} \oplus_i \mathbb{Z}_{tr}(X_i) \to \oplus_{i,j} \mathbb{Z}_{tr}(X_i \times X_j) \to \cdots$$
$$\cdots \to \oplus_{i,j} \mathbb{Z}_{tr}(X_1 \times \cdots \hat{X}_i \cdots \hat{X}_j \cdots \times X_n) \to \oplus_i \mathbb{Z}_{tr}(X \times \cdots \hat{X}_i \cdots \times X_n) \to$$
$$\to \mathbb{Z}_{tr}(X_1 \times \cdots \times X_n) \to \mathbb{Z}_{tr}(X_1 \wedge \cdots \wedge X_n) \to 0.$$

→ *We will be interested in* $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})$

Simplicial objects

- *A simplicial object of* C *is a functor* $F: \Delta^{op} \to C$.
- *Let* F *be a presheaf of abelian groups on* Sm_k. *Then*

$$C_\bullet F : U \mapsto F(U \times \Delta^\bullet)$$

is a simplicial presheaf with transfers.

- *Similarly,* $C_* F(U)$ *gives the complex of abelian groups*

$$\cdots \to F(U \times \Delta^2) \to F(U \times \Delta^1) \to F(U) \to 0.$$

Homotopy invariant presheaves

- F is homotopy invariant if $p^*: F(X) \to F(X \times A^1)$ is an isomorphism.
- These homology presheaves are homotopy invariant.

$$H_n C_* F : X \mapsto H_n C_* F(X)$$

- **Definition 3.8.** *Two finite correspondences from X to Y are $\mathbb{A}^1$-**homotopic** if they are the restrictions along $X \times 0$ and $X \times 1$ of an element of* $\mathbf{Cor}(X \times \mathbb{A}^1, Y)$.

Now we can define motivic cohomology!

Yay!

The motivic complex

- Define the motivic complex (tensor with A to get $A(q)$)

$$\mathbb{Z}(q) := C_* \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})[-q].$$

- These are actually sheaves.
- $q = 0$

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0,$$

- $q = 1$

$$\cdots \to \mathbf{Cor}(Y \times \Delta^2, \mathbb{G}_m) \to \mathbf{Cor}(Y \times \Delta^1, \mathbb{G}_m) \to \mathbf{Cor}(Y, \mathbb{G}_m) \to 0.$$

Motivic cohomology groups

→ DEFINITION 3.4. The **motivic cohomology groups** $H^{p,q}(X,\mathbb{Z})$ are defined to be the hypercohomology of the motivic complexes $\mathbb{Z}(q)$ with respect to the Zariski topology:

$$H^{p,q}(X,\mathbb{Z}) = \mathbb{H}^p_{Zar}(X,\mathbb{Z}(q)).$$

If A is any abelian group, we define:

$$H^{p,q}(X,A) = \mathbb{H}^p_{Zar}(X,A(q)).$$

→ *This satisfies many usual properties of cohomology…*

Weight 1

- There is a quasi-isomorphism $\mathbb{Z}(1) \xrightarrow{\cong} \mathcal{O}^*[-1].$

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$H^{-2,2}$	$H^{-1,2}$	$H^{0,2}$	$H^{1,2}$	$H^{2,2}$	$H^{3,2}$
0	0	0	$\mathcal{O}^*(X)$	$\mathrm{Pic}(X)$	0
0	0	$\mathbb{Z}(X)$	0	0	0
0	0	0	0	0	0

FIGURE 4.1. Weight q motivic cohomology

5 comparison results

(1) $H^{p,q}(X,A) = 0$ for $q < 0$, and for a connected X one has

$$H^{p,0}(X,A) = \begin{cases} A & \text{for } p = 0 \\ 0 & \text{for } p \neq 0; \end{cases}$$

(2) one has

$$H^{p,1}(X,\mathbb{Z}) = \begin{cases} \mathcal{O}^*(X) & \text{for } p = 1 \\ \mathrm{Pic}(X) & \text{for } p = 2 \\ 0 & \text{for } p \neq 1,2; \end{cases}$$

(3) for a field k, one has $H^{p,p}(\mathrm{Spec}(k),A) = K_p^M(k) \otimes A$ where $K_p^M(k)$ is the p-th Milnor K-group of k (see [Mil70]);

(4) for a strictly Hensel local scheme S over k and an integer n prime to $\mathrm{char}(k)$, one has

$$H^{p,q}(S,\mathbb{Z}/n) = \begin{cases} \mu_n^{\otimes q}(S) & \text{for } p = 0 \\ 0 & \text{for } p \neq 0 \end{cases}$$

where $\mu_n(S)$ is the groups of n-th roots of unity in S;

(5) one has $H^{p,q}(X,A) = CH^q(X, 2q-p;A)$. Here $CH^i(X,j;A)$ denotes the higher Chow groups of X introduced by S. Bloch in [Blo86], [Blo94]. In particular,

$$H^{2q,q}(X,A) = CH^q(X) \otimes A,$$

where $CH^q(X)$ is the classical Chow group of cycles of codimension q modulo rational equivalence.

Now let's discuss relations to other fields!

Cool!

Algebraic K-theory

→ *In topology: the Atiyah-Hirzebruch spectral sequence*

$$E_2^{p,q} = H^p(X; K^q(*)) \Rightarrow K^{p+q}(X).$$

→ *Algebraic setting: Friedlander and Suslin (2002), building on work by Bloch and Lichtenbaum*

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) = CH^{-q}(X, -p-q) \quad \Rightarrow \quad K_{-p-q}(X).$$

Motives

In order to express this intuition, of the kinship of these different cohomological theories, I formulated the notion of "motive" associated to an algebraic variety. By this term, I want to suggest that it is the "common motive" (or "common reason") behind this multitude of cohomological invariants attached to an algebraic variety, or indeed, behind all cohomological invariants that are a priori possible.

Chow motives

→ Replace morphisms with correspondences

$$\mathrm{Corr}^r(k)(X,Y) := \bigoplus_i A^{d_i+r}(X_i \times Y)$$

→ Augment objects to make it like an abelian category

→ Take the opposite category

Then cohomology theories factor through this, but the standard conjectures which would realize the power of this approach have been open for > 50 years!

Voevodsky's derived category

- *Using motivic cohomology, Voevodsky constructs a triangulated category $DM(k;R)$ which for all intents and purposes acts as the derived category of the desired category of motives!*

Mixed motives

- *Mixed motives apply to all varieties*
- *Their category is not semisimple: work with* Ext*'s.*
- *Motivic cohomology 'morally' studies these, and leads to Voevodsky's derived category*

Norm residue isomorphism theorem

The norm residue isomorphism theorem (or Bloch–Kato conjecture) states that for a field k and an integer ℓ that is invertible in k, the norm residue map

$$\partial^n : K_n^M(k)/\ell \to H_{\mathrm{et}}^n(k, \mu_\ell^{\otimes n})$$

from Milnor K-theory mod-ℓ to étale cohomology is an isomorphism. The case $\ell = 2$ is the Milnor conjecture, and the case $n = 2$ is the Merkurjev–Suslin theorem.

(wikipedia!)

→ *Proven by Voevodsky (completed ~ 2009)*
- *Develops motivic cohomology, motivic homotopy theory, motivic Steenrod algebra, …*

Special values of L-functions

→ *Another Bloch-Kato conjecture…*

4.14. Conjecture (Soulé, but in more generality [4, p. 271; 24, Conj. 2.2]). *If X is regular and proper over* Spec($\mathbf{Z}$), *then for an integer $n \in \mathbf{Z}$ we have*

$$\operatorname{ord}_{s=n} \zeta_X(s) = -\sum_{i \geq 0} (-1)^i \operatorname{rank}(\mathrm{H}_i(X; \mathbf{Z}(n))),$$

A special case, https://math.mit.edu/~phaine/files/Motivic_Overview.pdf

→ *Implies BSD!*

Any questions?

References

Almost everything is from
“Lectures on motivic cohomology”, a book written by Mazza and Weibel based off of lectures of Voevodsky.