1 Cartier and Weil divisors

Let X be a variety of dimension n over a field k. We want to introduce two notions of divisors, one familiar from the last chapter.

Definition 1.1. A *Weil divisor* of X is an $n-1$-cycle on X, i.e. a finite formal linear combination of codimension 1 subvarieties of X. Thus the Weil divisors form a group $Z_{n-1}X$.

Definition 1.2. A *Cartier divisor* consists of the following data:

- an open cover $\{U_\alpha\}$ of X;
- for each α a nonzero rational function f_α on U_α, defined up to multiplication by a *unit*, i.e. a function without zeros or poles, such that for any α, β we have f_α/f_β a unit on $U_\alpha \cap U_\beta$.

Like the Weil divisors, the Cartier divisors form an abelian group: $(\{U_\alpha, f_\alpha\}) + (\{U_\alpha, g_\alpha\}) = (\{U_\alpha, f_\alpha g_\alpha\})$ (we can assume that the open covers are the same, since if not they refine to $\{U_\alpha \cap V_\beta\}$). We call this abelian group $\operatorname{Div} X$.

Given a Cartier divisor $D = (\{U_\alpha, f_\alpha\})$ and a codimension 1 subvariety V of X, we define

$$\operatorname{ord}_V D = \operatorname{ord}_V(f_\alpha)$$

for α such that $U_\alpha \cap V$ is nonempty; since each f_α is defined up to a unit, this order is well-defined. We define the *associated Weil divisor*

$$[D] = \sum_V \operatorname{ord}_V D \cdot [V].$$

This defines a homomorphism

$$\operatorname{Div} X \to Z_{n-1}X.$$

For any rational function f on X, we get a *principal* Cartier divisor $\operatorname{div}(f)$ by choosing any cover $\{U_\alpha\}$ and defining $f_\alpha = f|_{U_\alpha}$. It is immediate that the image $[\operatorname{div}(f)]$ of this divisor under the map to $Z_{n-1}X$ is the Weil principal divisor. Say that two Cartier divisors D and D' are *linearly equivalent* if $D - D' = \operatorname{div}(f)$ for some f; then we define $\operatorname{Pic} X$ to be the group of Cartier divisors modulo linear equivalence, and the above then shows that the map $\operatorname{Div} X \to Z_{n-1}X$ descends to a map $\operatorname{Pic} X \to A_{n-1}X$. This map is in general neither injective nor surjective.

Notice that the definition of a Cartier divisor yields that of a line bundle on X: given a divisor $D = (\{U_\alpha, f_\alpha\})$, define a line bundle $L = \mathcal{O}(D)$ to be trivialized on each U_α with transition functions f_α/f_β. Two Cartier divisors D and D' are linearly equivalent if and only if $\mathcal{O}(D) = \mathcal{O}(D')$, and so we get the alternate description of $\operatorname{Pic} X$ as the abelian group of line bundles on X with group operation given by the tensor product. Conversely, given a line bundle L, this determines a Cartier divisor $D(L)$ up to some additional data: a nonzero rational section s of L. Therefore we can also think of Cartier divisors as the data of a line bundle together with a nonzero rational section.

We define the *support* $\operatorname{supp} D$ or $|D|$ of a Cartier divisor D to be the union of codimension 1 subvarieties V of X such that f_α is not a unit for U_α nontrivially intersecting V, i.e. $\operatorname{ord}_V D$ is nonzero.

We say that a Cartier divisor $D = (\{U_\alpha, f_\alpha\})$ if all of the f_α are regular, i.e. have no poles.

2 Pseudo-divisors

In general, Cartier divisors are not well-behaved under pullbacks (although line bundles are). In particular, given the data of a line bundle L and a nonzero rational section s and a morphism $f : Y \to X$, there is no guarantee that the pullback f^*s is nonzero. Therefore we enlarge the notion to make it behave better: let L be a line bundle on X, $Z \subset X$ be a closed subset, and s be a nowhere vanishing section of L restricted to $X - Z$, or equivalently a trivialization of $L|_{X-Z}$. A *pseudo-divisor* on X consists of the data of such a triple (L, Z, s), up to the following equivalence: two triples (L, Z, s) and (L', Z', s') define the same pseudo-divisor if $Z = Z'$ and there exists an isomorphism $\sigma : L \to L'$ such that restricted to $X - Z$ we have $\sigma \circ s = s'$. Note that this is well-behaved under pullback.

Example 2.1. Let $D = (\{U_\alpha, f_\alpha\})$ be a Cartier divisor, with support $|D|$. Then each f_α away from $|D$ gives a local section of the associated line bundle $\mathcal{O}(D)$, and so these glue to a section s_D of $\mathcal{O}(D)$ on $X - |D|$; this makes $(\mathcal{O}(D), |D|, s_D)$ a pseudo-divisor.

We say that a Cartier divisor D *represents* a pseudo-divisor (L, Z, s) when $|D| \subseteq Z$ and there exists an isomorphism $\sigma : \mathcal{O}(D) \to L$ such that restricted to $X - Z$ we have $\sigma \circ s_D = s$, with notation as above.

Lemma 2.2. *If X is a variety, then every pseudo-divisor (L, Z, s) on X is represented by a Cartier divisor D. If $Z \subsetneq X$, then D is unique; if $Z = X$, then D is unique up to linear equivalence.*

Proof. If $Z = X$, then s is a section on $X - X = \{\}$, and so a pseudo-divisor is just a line bundle; and we saw in the previous section that the group of Cartier divisors up to linear equivalence is isomorphic to the group of line bundles, so L corresponds to a unique linear equivalence class of Cartier divisors.

If $Z \neq X$, let $U = X - Z$. As above, choose a Cartier divisor $D = (\{U_\alpha, f_\alpha\})$ with $\mathcal{O}(D) \simeq L$. The section s consists of a collection of functions s_α on $U \cap U_\alpha$ such that $s_\alpha = f_\alpha / f_\beta \cdot s_\beta$ on $U \cap U_\alpha \cap U_\beta$; thus $s_\alpha / f_\alpha = s_\beta / f_\beta$ on each intersection, i.e. there exists some rational function r such that $s_\alpha / f_\alpha = r$ on each $U \cap U_\alpha$. Then $D' := D + \operatorname{div}(r)$ is the Cartier divisor $(\{U_\alpha, f_\alpha r\})$ and by definition $f_\alpha r = s_\alpha$ on each $U \cap U_\alpha$; therefore using the definition above $s_{D'} = s$. Since D' is linearly equivalent to D, it corresponds to the same line bundle, and since r is regular on each U_α the support of $\operatorname{div}(r)$ is contained in Z; therefore D' represents (L, Z, s).

For uniqueness, suppose that two Cartier divisors $D_1 = (\{U_\alpha, f_\alpha\})$ and $D_2 = (\{V_\beta, g_\beta\})$ both represent (L, Z, s). Then similarly there must exist some rational function r such that

$rf_\alpha = rg_\beta$ on each $U_\alpha \cap V_\beta$. But since $s_{D_1} = s_{D_2} = s$, if $Z \neq X$, i.e. U is nonempty, then s_{D_1} and s_{D_2} must agree on every $U \cap U_\alpha \cap V_\beta$, and so r restricted to U must be 1; since f is rational it follows that $f = 1$ and $D_1 = D_2$. □

For any pseudo-divisor $D = (L, Z, s)$, as for Weil divisors we will write $\mathcal{O}(D) = L$, $|D| = Z$, and $s_D = s$.

If $D = (L, Z, s)$ and $D' = (L', Z', s')$ are two pseudo-divisors, we can define their sum

$$D + D' = (L \otimes L', Z \cup Z', s \otimes s').$$

This agrees with the sum on Cartier divisors, except that the supports may be larger in this case. Similarly defining

$$-D = (L^{-1}, Z, s^{-1})$$

makes the set of pseudo-divisors into an abelian group.

Given a pseudo-divisor D on a variety X of dimension X, we can define the Weil class divisor $[D]$ by taking $\tilde{D}$ to be the Cartier divisor which represents D and setting $[D] := [\tilde{D}]$, the associated Weil divisor from the previous section. The above lemma shows that this yields a well-defined element of $A_{n-1}X$; this gives a homomorphism from the group of pseudo-divisors to $A_{n-1}X$.

3 Intersecting with divisors

Let X be a variety of dimension n, D be a pseudo-divisor on X, and V be a subvariety of dimension k. Let $j : V \hookrightarrow X$ be the inclusion of V into X; then the pullback j^*D is a pseudo-divisor on V with support $V \cap |D|$. We define the class $D \cdot [V]$ in $A_{k-1}(V \cap |D|)$ given by the Weil class divisor of j^*D:

$$D \cdot [V] = [j^*D].$$

For any closed subscheme $Y \subset X$ containing $V \cap |D|$, we can also view this as an element of $A_{k-1}Y$; we will also denote this by $D \cdot [V]$.

Let $\alpha = \sum_V n_V \cdot V$ be a k-cycle on X, with support $|\alpha$ the union of the subvarieties V such that n_V is nonzero. For a pseudo-divisor D on X, we define the *intersection class* $D \cdot \alpha$ in $A_{k-1}(V \cap |D|)$ by

$$D \cdot \alpha = \sum_V n_V \cdot (D \cdot [V]).$$

As above, we can also view this as an element of $A_{k-1}Y$ for any Y containing $|\alpha| \cap |D|$.

We will apply this in two main cases. First: f $|D| = X$, then the data of $D = (L, X, s)$ is just that of a line bundle as above; in this case the action of D on a k-cycle α is called that of the first Chern class, written $D \cdot \alpha = c_1(L) \cap \alpha$.

Second: if $i : |D| \hookrightarrow X$ is the inclusion of $|D|$ into X, then $D \cdot \alpha$ is called the Gysin pullback $i^*\alpha$.

Theorem 3.1. *Let X be a scheme, D be a pseudo-divisor on X, and α be a k-cycle on X.*

(*a*) *Let α' be a k-cycle on X. Then*

$$D \cdot (\alpha + \alpha') = D \cdot \alpha + D \cdot \alpha'$$

in $A_{k-1}((|\alpha| \cup |\alpha'|) \cap |D|)$.

(*b*) *Let D' be a pseudo-divisor on X. Then*

$$(D + D') \cdot \alpha = D \cdot \alpha + D' \cdot \alpha$$

in $A_{k-1}(|\alpha| \cap (|D| \cup |D'|))$.

(*c*) *Let $f : Y \to X$ be a proper morphism, β be a k-cycle on Y, and $g : |\beta| \cap f^{-1}(|D|) \to f(|\beta|) \cap |D|$ be the restriction of f to $|\beta| \cap f^{-1}(|D|)$. Then*

$$g_*(f^*D \cdot \beta) = D \cdot f_*\beta$$

in $A_{k-1}(f(|\beta|) \cap |D|)$.

(*d*) *Let $f : Y \to X$ be a flat morphism of relative dimension n and $g : f^{-1}(|\alpha| \cap |D|) \to |\alpha| \cap |D|$ be the restriction of f to $f^{-1}(|\alpha| \cap |D|)$. Then*

$$f^*D \cdot f^*\alpha = g^*(D \cdot \alpha)$$

in $A_{n+k-1}(f^{-1}(|\alpha| \cap |D|))$.

(*e*) *If the line bundle $\mathcal{O}(D)$ is trivial, then*

$$D \cdot \alpha = 0$$

in $A_{k-1}(|\alpha| \cap |D|)$.

Proof. Part (a) is immediate from the definition. Using part (a), then, we can assume by linearity that $\alpha = [V]$ for some k-dimensional subvariety $V \subset X$. Restricting to V, (b) is just the statement that taking the Weil class divisor is compatible with sums.

For part (c), we can likewise assume that $\beta = [W]$ for some k-dimensional subvariety $W \subset Y$; then $f^*D \cdot \beta$ is the restriction of the Cartier divisor $f^*\tilde{D}$ representing f^*D to W, and so we can assume that $Y = W$. Similarly on the right-hand side $D \cdot f_*\beta = D \cdot \deg(f(W)/W)[f(W)]$ and so concerns only the restriction of D to $f(W)$, and so we can assume that $f(W) = X$. In this case $g = f$ on the support of D and so the statement is

$$f_*(f^*[D]) = \deg(W/f(W))[D]$$

since $D \cdot [X] = [D]$ and $f^*D \cdot [Y] = f^*[D]$. If f is a map of degree d and $D = \mathrm{div}(r)$ for some function r on some open subset of $f(W)$, then from last time we know that locally

$$f_*[\mathrm{div}(f^*r)] = [\mathrm{div}(N(f^*r))] = d[\mathrm{div}(r)]$$

where N is the determinant map from functions on subsets of W to functions on their images, since $N(f^*r) = dr$ since f has degree d. But locally we can always assume that $[D]$ is principal, and so $f_*f^*[D] = d[D]$ as desired.

For (d), we can again assume that $\alpha = [V] = [X]$, so the statement similarly becomes

$$[f^*D] = f^*[D].$$

By linearity, we can assume $D = [W]$ for some subvariety W of $X = V$, at which point the statement is $f^*[W] = [f^{-1}(W)]$, which is true whenever f is flat.

Finally for (e) we can again assume $\alpha = [V] = [X]$, so that the statement is $[D] = 0$ in $A_{n-1}X$ whenever $\mathcal{O}(D)$ is trivial, where n is the dimension of $V = X$. Letting $\tilde{D}$ be the Cartier divisor representing D, we know from section 1 that $\mathcal{O}(D)$ is trivial precisely when $\tilde{D}$ is linearly equivalent to the trivial Cartier divisor $0 = (\{U_\alpha, 1\})$ for which every local function is a unit; and we know that the associated Weil divisor map $\operatorname{Div} X \to Z_{n-1}X$ descends to a map $\operatorname{Pic} X \to A_{n-1}X$, i.e. $[D] = [\tilde{D}] = 0$ whenever $\mathcal{O}(D)$ is trivial. □

4 Commutativity

Suppose that we have two Cartier divisors D, D' on an n-dimensional variety X. Then they both determine associated Weil divisors $[D], [D'] \in Z_{n-1}X$ (and thus in $A_{n-1}X$), and so it is natural to consider the intersections

$$D \cdot [D'], \qquad D' \cdot [D].$$

Theorem 4.1. *In $A_{n-2}(|D| \cap |D'|)$, we have*

$$D \cdot [D'] = D' \cdot [D].$$

Corollary 4.2. *Let D be a pseudo-divisor on a scheme X, and α be a k-cycle on X rationally equivalent to 0. Then*

$$D \cdot \alpha = 0$$

in $A_{k-1}(|D|)$.

Proof. We can assume without loss of generality that $\alpha = [\operatorname{div}(f)]$ for some rational function f on a subvariety V of X. Then letting $\tilde{D}$ be the Cartier divisor representing D we can replace D with $\tilde{D}$ and X with V without changing the result; then we can apply Theorem 4.1 to get

$$D \cdot \alpha = \tilde{D} \cdot [\operatorname{div}(f)] = \operatorname{div}(f) \cdot [\tilde{D}].$$

But by part (e) of Theorem 3.1, we have $\operatorname{div}(f) \cdot [\tilde{D}] = 0$. □

Given a closed subscheme $Y \subset X$ and a k-cycle α on Y, we can construct its intersection $D \cdot \alpha \in A_{k-1}(Y \cap |D|)$ for any pseudo-divisor D on X. This gives a map

$$Z_kY \to A_{k-1}(Y \cap |D|).$$

The above corollary shows that in fact this map descends to a map

$$A_k Y \to A_{k-1}(Y \cap |D|);$$

this is called *intersecting* with D.

Corollary 4.3. *For two pseudo-divisors D, D' on a scheme X and a k-cycle α on X, we have*

$$D \cdot (D' \cdot \alpha) = D' \cdot (D \cdot \alpha)$$

in $A_{k-2}(|\alpha| \cap |D| \cap |D'|)$.

Proof. We can assume without loss of generality that $\alpha = [V]$ for some subvariety $V \subseteq X$ of dimension k. Then we can restrict D and D' to V, so that $D' \cdot [V] = [\mathrm{id}^* D'] = [D']$ and similarly $D \cdot [V] = [D]$; and then applying Theorem 4.1 immediately gives the result. $\square$

For pseudo-divisors $D_1, \dots, D_n$ on X and a k-cycle α on X, we can then define inductively

$$D_1 \cdots D_n \cdot \alpha = D_1 \cdot (D_2 \cdots D_n \cdot \alpha)$$

in $A_{k-n}(|\alpha| \cap (|D_1| \cup \cdots \cup |D_n|))$. Theorem 4.1 implies that the order of the D_i is unimportant, and parts (a) and (b) of Theorem 3.1 implies that the action is linear in each D_i and in α. More generally if $p(t_1, \dots, t_n)$ is a homogeneous polynomial of degree d and Z is a closed subscheme of X containing $|\alpha| \cap (|D_1| \cup \cdots \cup |D_n|)$, then we can define $p(D_1, \dots, D_n) \cdot \alpha$ in $A_{k-d}(Z)$.

Definition 4.4. We say that an algebraic variety Y is *complete* if for any variety Z the projection $Y \times Z \to Y$ is a closed map.

For example, any projective variety is complete.

If $n = k$ and $Y = |\alpha| \cap (|D_1| \cup \cdots \cup |D_k|)$ is complete, then we can define the *intersection number*

$$(D_1 \cdots D_k \cdot \alpha)_X = \int_Y D_1 \cdots D_k \cdot \alpha.$$

Similarly if p is a homogeneous polynomial of degree k in k variables then we can define

$$(p(D_1, \dots, D_k) \cdot \alpha)_X = \int_Y p(D_1, \dots, D_n) \cdot \alpha.$$

For a subvariety V purely of dimension k, we will sometimes write simply V instead of $[V]$; similarly we will sometimes write D instead of $[D]$.

Example 4.5. Let X be the projective completion of the affine surface $X' \subset \mathbb{A}^3$ defined by $z^2 = xy$. Consider the Cartier divisor D on X defined everywhere by the equation x, corresponding to the subvariety cut out by $x = 0$. Define the lines ℓ, ℓ' by $x = z = 0$ and $y = z = 0$ respectively, and let P be the origin $(0, 0, 0)$. Along the subvariety $x = 0$, from the defining equation we also have $z = 0$ (in affine space), and so $[D] = \mathrm{ord}_\ell D \cdot [\ell]$; we have

$$\mathrm{ord}_\ell D = \mathrm{len}_A A/(x),$$

where (in the affine variety) $A = \mathcal{O}_{X,\ell} = K[x,y,z]/(z^2 - xy)$. Thus $A/(x) = K[x,y,z]/(z^2 - xy, x) = K[y,z]/(z^2)$ which has length 2, with maximal proper subsequence of modules given by $0 \subset K[y] = K[y,z]/(z) \subset K[y,z]/(z^2)$. Therefore $[D] = 2[\ell]$. We can compute

$$D \cdot [\ell'] = [j^*D] = [P]$$

where j is the inclusion of ℓ' into X, since restricted to the line $y = z = 0$ the equation $x = 0$ specifies only the point P with multiplicity 1. Therefore there cannot exist any Cartier divisor D' with $[D'] = [\ell']$, since if there were we would have

$$[P] = D \cdot [\ell'] = D \cdot [D'] = D' \cdot [D] = 2D' \cdot [\ell]$$

in either $Z_1 X$ or $A_1 X$, by Theorem 4.1 and the above calculation. This proves our above claim that the maps $\operatorname{Div} X \to Z_{\dim X - 1}$ and $\operatorname{Pic} X \to A_{\dim X - 1} X$ are not in general surjective.

5 The first Chern class

Let X be a scheme, $V \subseteq X$ a subvariety of dimension k, and L a line bundle on X. The restriction of L to V is a line bundle on V and so is isomorphic to $\mathcal{O}(C)$ for some Cartier divisor C on V, determined up to linear equivalence. This in turn defines a well-defined element $[C]$ of $A_{k-1}X$; we write $c_1(L) \cap [V] := [C]$. More generally, if $\alpha = \sum_V n_V \cdot [V]$ is a k-cycle on X then define C_V for each V as above, and write

$$c_1(L) \cap \alpha := \sum_V n_V \cdot [C_V].$$

If $L = \mathcal{O}(D)$ for some pseudo-divisor D, then if $j : V \hookrightarrow X$ is the inclusion then the Cartier divisor $\tilde{D}$ on V representing j^*D satisfies $\mathcal{O}(\tilde{D}) \simeq \mathcal{O}(D)$ by construction; by definition, this means that $[C_V] = [j^*D] = D \cdot [V]$ and so

$$c_1(L) \cap \alpha = D \cdot \alpha$$

in $A_{k-1}X$.

Theorem 5.1. *Let X be a scheme, L be a line bundle on X, and α be a k-cycle on X.*

(*a*) *If α is rationally equivalent to 0, then $c_1(L) \cap \alpha = 0$. Therefore there is an induced homomorphism $c_1(L) \cap - : A_k X \to A_{k-1} X$.*

(*b*) *If L' is a second line bundle on X, then*

$$c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha)$$

in $A_{k-2}X$.

(*c*) *If* $f : Y \to X$ *is a proper morphism and* β *is a* k*-cycle on* Y*, then*

$$f_*(c_1(f^*L) \cap \beta) = c_1(L) \cap f_*\beta$$

in $A_{k-1}X$.

(*d*) *If* $f : Y \to X$ *is a flat morphism of relative dimension* n*, then*

$$c_1(f^*L) \cap f^*\alpha = f^*(c_1(L) \cap \alpha)$$

in $A_{n+k-1}Y$.

(*e*) *If* L' *is a second line bundle on* X*, then*

$$c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L') \cap \alpha$$

and

$$c_1(L^{-1}) \cap \alpha = -c_1(L) \cap \alpha$$

in $A_{k-1}X$.

Proof. A line bundle on X defines a pseudo-divisor with support X, and so the analogous properties from Theorem 3.1 and its corollaries immediately imply these. □

6 The Gysin map

Fix an effective Cartier divisor D on a scheme X, with the inclusion given by $i : |D| \hookrightarrow X$. Then we define the "Gysin homomorphism"

$$i^*\alpha := D \cdot \alpha$$

for k-cycles α on X.

Proposition 6.1. *With notation as above:*

(*a*) *If* α *is rationally equivalent to* 0*, then* $i^*\alpha = 0$*, and so there is an induced homomorphism* $i^* : A_kX \to A_{k-1}(|D|)$.

(*b*) *We have*

$$i_*i^*\alpha = c_1(\mathcal{O}(D)) \cap \alpha.$$

(*c*) *If* β *is a* k*-cycle on* $|D|$*, then*

$$i^*i_*\beta = c_1(i^*\mathcal{O}(D)) \cap \beta.$$

(*d*) *If* X *is purely* n*-dimensional, then*

$$i^*[X] = [D]$$

in $A_{n-1}(|D|)$.

(e) *If L is a line bundle on X, then*

$$i^*(c_1(L) \cap \alpha) = c_1(i^*L) \cap i^*\alpha$$

in $A_{k-2}(|D|)$.

All of these follow immediately from the definitions and the results above.