Disclaimer

These notes were taken during the seminar using the vimtex package of the editor neovim. Any errors are mine and not the speakers’. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the lecturers. If you find any errors, please contact me at plei@math.columbia.edu.

Seminar Website:
https://math.columbia.edu/~plei/f21-CO.html
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1.1 Review of semisimple Lie algebras

Throughout this lecture, we will work over $\mathbb{C}$.

**Definition 1.1.1.** A Lie algebra $g$ is **semisimple** if any of the following equivalent conditions hold:

1. $g$ is a direct sum of simple Lie algebras (those with no nonzero proper ideals).
2. The Killing form $\kappa(x, y) := \text{tr}(\text{ad}(x) \text{ad}(y))$ is nondegenerate.
3. The radical (maximal solvable ideal) of $g$ is zero.

Some examples of semisimple Lie algebras include $\mathfrak{sl}_n$, $\mathfrak{so}_n$, $\mathfrak{sp}_{2n}$, and in some sense (the classification of simple Lie algebras), these are essentially all semisimple Lie algebras.

Now given a semisimple Lie algebra $g$, we will fix a Cartan subalgebra $h \subset g$, which is just a maximal abelian subalgebra of semisimple elements. This gives us a root decomposition

$$g = h \oplus \bigoplus_{\alpha \in h^* \setminus \{0\}} g_{\alpha},$$

where $g_{\alpha}$ is the subspace of $g$ where $h$ acts with weight $\alpha$. Some important facts about these root systems are the following:

- For all $\alpha$, we have $\dim g_{\alpha} = 1$.
- For all roots $\alpha, \beta$, we have $[g_{\alpha}, g_{\beta}] \subset g_{\alpha + \beta}$.
- If $\alpha$ is a root, so is $-\alpha$.

In addition, the $\alpha$ are required to form a (reduced) **root system** (denoted $\Phi$), the precise definition of which is deliberately omitted. Given a choice of Borel subalgebra containing $h$, we obtain a set $\Phi^+$ of positive roots and a set $\Delta$ of simple roots. In addition, given a root system $\Phi$, there is a **dual root system** $\Phi^\vee$, whose roots are

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}, \alpha \in \Phi.$$
Now suppose that \( g \) is a semisimple Lie algebra with root system \( \Phi \). For every \( \alpha \in \Phi^+ \), we may choose \( x_\alpha \in g_{\alpha} \) and \( y_\alpha \in g_{-\alpha} \), and these determine some \( h_\alpha = [x_\alpha, y_\alpha] \in \mathfrak{h} \). This choice can be made such that \( \alpha(h_\alpha) = 2 \).

Recall that the Lie algebra \( \mathfrak{sl}_2 \) is spanned by the matrices

\[
x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Then the choice of \( x_\alpha, y_\alpha, h_\alpha \) gives an embedding \( \mathfrak{sl}_2 \to g \). These maps, ranging over all \( \alpha \), cover all of \( g \). Now a basis of \( g \) is given by \( x_\alpha, y_\alpha, h_\alpha \) for the simple roots \( \alpha_i \). Therefore, to specify \( g \), we only need to give commutation relations for the basis elements.

Now suppose that \( \Phi \) is some root system. We would like to construct a semisimple Lie algebra \( g \) with root system \( \Phi \). We want to build a semisimple Lie algebra. To do this, choose a set of simple roots \( \alpha_i \), and consider the Lie algebra

\[
\langle x_\alpha_i, y_\alpha_i, h_\alpha_i \rangle / \text{relations},
\]

where the relations are as follows:

- \([h_\alpha_i, h_\alpha_j] = 0\).
- We have \([x_\alpha_i, y_\alpha_j] = h_\alpha_i \) if \( i = j \) and this commutator vanishes otherwise.
- \([h_\alpha_i, x_\alpha_j] = \langle \alpha_j, \alpha_i^\vee \rangle x_\alpha_j\).
- \([h_\alpha_i, y_\alpha_j] = - \langle \alpha_j, \alpha_i^\vee \rangle y_\alpha_j\).
- \( \text{ad}(x_\alpha_i)^{1-\langle \alpha_j, \alpha_i^\vee \rangle}(x_\alpha_j) = 0 \) if \( i \neq j \).
- \( \text{ad}(y_\alpha_i)^{1-\langle \alpha_j, \alpha_i^\vee \rangle}(y_\alpha_j) = 0 \) if \( i \neq j \).

The first four relations are called the \textit{Weyl relations} and the last two are called the \textit{Serre relations}. Given this data, we end up with a semisimple Lie algebra \( g_\Phi \) with root system \( \Phi \). In addition, if \( g \) is any other semisimple Lie algebra with root system \( \Phi \), there is an isomorphism \( g_\Phi \cong g \). Moreover, we have a bijection between semisimple Lie algebras and reduced root systems, which restricts to a bijection between simple Lie algebras and irreducible root systems.

<table>
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<tr>
<th>Irreducible root systems</th>
<th>simple Lie algebras</th>
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<tbody>
<tr>
<td>( A_n )</td>
<td>( \mathfrak{sl}_{n+1} )</td>
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<td>( B_n )</td>
<td>( \mathfrak{so}_{2n+1} )</td>
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<tr>
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<td>( D_n )</td>
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<tr>
<td>( E_6, E_7, E_8, F_4, G_2 )</td>
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We will now discuss the finite-dimensional representation theory of semisimple Lie algebras \( g \).

**Theorem 1.1.2** (Weyl’s complete reducibility theorem). \( g \) is decomposes as a direct sum of simple representations.
Now suppose that \( M \) is a finite-dimensional \( \mathfrak{g} \)-representation. Then \( M \) has a weight decomposition
\[
M = \bigoplus_{\lambda \in h^*} M_{\lambda}.
\]
These \( \lambda \) are integral weights, which simply means that \( \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \) for all roots \( \alpha \). For any root \( \alpha \), \( x_\alpha(M_{\lambda}) \subset M_{\lambda+\alpha} \) and \( y_\alpha(M_{\lambda}) \subset M_{\lambda-\alpha} \). We would like to think that the \( x_\alpha \) raise the weights and \( y_\alpha \) lower the weights, so we introduce a partial order. We say that \( \lambda \succeq \mu \) if \( \lambda - \mu \in \mathbb{Z}_{\geq 0} \Phi^+ \).

By Weyl’s complete reducibility theorem, it remains to classify the irreducible representations of \( \mathfrak{g} \). These are in bijection with the dominant integral weights, which in particular means that \( \langle \lambda, \alpha^\vee \rangle \geq 0 \) for all \( \alpha \in \Phi^+ \). For any dominant weight \( \lambda \), there is a unique highest-weight representation \( L(\lambda) \). Here, \( L(\lambda) \) is generated by a single maximal vector \( v \) of weight \( \lambda \). This means that for all positive roots \( \alpha \), \( x_\alpha v = 0 \).

### 1.2 Introduction to category \( \mathcal{O} \)

We would now like to study infinite dimensional representations of \( \mathfrak{g} \). Of course, this is impossibly complicated in general, so we will impose some finiteness conditions on our representations.

**Definition 1.2.1.** The category \( \mathcal{O} \) is the full subcategory of \( \mathcal{U}(\mathfrak{g}) \)-modules \( M \) satisfying:

1. \( M \) is finitely generated as a \( \mathcal{U}(\mathfrak{g}) \)-module.
2. \( M \) is \( \mathfrak{h} \)-semisimple and has a weight decomposition \( M = \bigoplus_{\lambda \in h^*} M_{\lambda} \).
3. \( M \) is locally \( n \)-finite, where \( n = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g} \). Precisely, this means that the \( \mathcal{U}(n) \) generated by any \( v \in M \) is finite-dimensional.

Here are some facts about category \( \mathcal{O} \), which are stated without proof.

- For all \( M \) in our category and weights \( \lambda \), the weight space \( M_{\lambda} \) is finite-dimensional.
- \( \mathcal{O} \) is a Noetherian (everything satisfies the descending chain condition) abelian category.

We will now describe some infinite-dimensional objects in category \( \mathcal{O} \).

**Definition 1.2.2.** For any weight \( \lambda \), the Verma module \( M(\lambda) \) associated to \( \lambda \) is the module
\[
M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} C_{\lambda},
\]
where \( \mathfrak{b} = \mathfrak{h} + \mathfrak{n} \) is the Borel subalgebra associated to our choice of positive roots and \( C_{\lambda} \) is the \( \mathfrak{b} \)-module associated to the 1-dimensional representation of \( \mathfrak{h} \) with weight \( \lambda \) and the identification \( \mathfrak{b}/\mathfrak{n} = \mathfrak{h} \).
Recall the following fact about semisimple Lie algebras. If we have a decomposition
\[ g = \mathfrak{h} \oplus_{\alpha \in \Phi} g_{\alpha}, \]
where
1. \( \dim g_{\alpha} = 1; \)
2. \( \mathbb{Z}\Phi \subset h^* \) is a lattice of maximal rank;
3. \( \alpha \in \Phi \) implies \( -\alpha \in \Phi; \)
4. \( [[g_{\alpha}, g_{-\alpha}], g_{\alpha}] \neq 0, \)
then \( g \) is a semisimple Lie algebra.

Now recall that the Weyl group \( W \) is the group generated by the reflections \( s_{\alpha} \) in the roots. For any \( w \in W \), we define the \textit{length}
\[ \ell(w) = \# \{ \alpha \in \Phi_+ \mid w(\alpha) \in \Phi_- \}. \]
Next, there is the Bruhat order, where if \( w_2 = sw_1 \) and \( \ell(w_2) > \ell(w_1) \), we say \( w_1 < w_2 \).

Finally, through this lecture, we will denote the weight lattice by \( \Lambda \) and the root lattice by \( Q \). In addition, we will denote the \( \lambda \)-weight space of a module \( M \) by \( M^\lambda \), and \( M_{\lambda} \) will be the Verma associated to \( \lambda \). Also, we will need the notion of the universal enveloping algebra, which we will not write down here.

### 2.1 Definitions

Recall that \( \mathcal{O} \) is the full subcategory of \( \text{Mod}(\mathfrak{l}g) \) of modules \( M \) such that
1. \( M \) is finitely generated over \( \mathfrak{l}g. \)
2. \( M \) is \( \mathfrak{h} \)-semisimple.
3. \( M \) is locally \( n \)-finite.
4. \( \dim M^\lambda < \infty. \)
5. The set of weights of $M$ is contained in some finite union of cones $\lambda - Q_+$.

**Theorem 2.1.1.** The following properties hold for $O$:

1. $O$ is Noetherian.
2. $O$ is closed under submodules, quotients, finite direct sums, and is abelian.
3. $O$ is closed under tensoring with finite-dimensional representations (in fact, if tensoring with $O$ is exact and lands in $O$, then $N$ must be finite-dimensional).
4. $M$ is locally $Zg$-finite.
5. All $M \in O$ are finitely generated over $U\mathfrak{n}^-$.

### 2.2 Highest weight modules

**Definition 2.2.1.** A vector $v_+$ is a maximal vector if $n^+ v_+ = 0$.

**Definition 2.2.2.** A module $M$ is a highest weight module if there exists a maximal $v_+ \in M$ generating $M$.

**Definition 2.2.3.** Let $\lambda \in \mathfrak{h}^*$ and consider the $\mathfrak{b}^+$-module $C_\lambda$. Then the Verma module for $\lambda$ is the module

$$M_\lambda := U\mathfrak{g} \otimes_{U\mathfrak{b}} C_\lambda.$$

Note that we have the standard adjunction $\text{Hom}_{\mathfrak{g}}(M_\lambda, -) = \text{Hom}_{\mathfrak{b}}(C_\lambda, -)$.

**Theorem 2.2.4.** For any highest weight module $M$ with highest weight $\lambda$,

1. $M = \left< f_{\Phi_1}^{n_1} \cdots f_{\Phi_\lambda}^{n_\lambda} \right>$, and in particular $M$ is $\mathfrak{h}$-semisimple.
2. All weights of $M$ are at most $\lambda$.
3. For any $\mu < \lambda$, $\dim M^\mu < \infty$, and $\dim M^\lambda = 1$. In addition, $M \in O$.
4. Any quotient of $M$ is also a highest weight module with highest weight $\lambda$.
5. Any submodule of a highest weight module with weight $\mu < \lambda$ is a proper submodule. if $M$ is simple, all maximal vectors have weight $v^\lambda_\nu$.
6. There exists a unique maximal submodule, and thus $M$ has a unique simple quotient and thus is indecomposable.
7. All simple highest weight modules with highest weight $\lambda$ are isomorphic, so $\dim \text{End} M = 1$.

**Corollary 2.2.5.** Let $M \in O$. Then $M$ admits a filtration whose successive quotients are highest weight modules.
2.3 Verma modules

Let $M_\lambda$ be a verma module and $L_\lambda$ be the unique simple quotient, and $N_\lambda$ be the unique maximal submodule.

**Theorem 2.3.1.** Any simple $L \in \mathcal{O}$ is isomorphic to $L_\lambda$ for some $\lambda$.

**Proposition 2.3.2.** Let $\Sigma$ be the set of simple roots and $\sigma \in \Sigma$. Let $\lambda \in \mathfrak{h}^*$ such that $\sigma^*(\lambda) \in \mathbb{N}$. Choose $v_\lambda^+ \in M_\lambda$ a maximal vector. Then

$$f_{\sigma}^{\sigma^*(\lambda)+1}v_\lambda^+ = v_\lambda^{\lambda-(\sigma^*(\lambda)+1)\sigma}.$$  

In particular, there exists a nonzero morphism $M_{\lambda-(\sigma^*(\lambda)+1)\sigma} \hookrightarrow M_\lambda$.

**Lemma 2.3.3.** We have the commutation relations

$$[e_i, f_j^{k+1}] = 0, \quad [e_i, f_i^{k+1}] = -(k+1)f_i^{k+1}(k-h_i), \quad [h_i, f_j^{k+1}] = -(k+1)\alpha_i(h_i)f_j^{k+1}.$$

2.4 Examples

We will discuss the example of $\mathfrak{sl}_2$. Let $\phi_i$ be the operator that outputs the $i$-th diagonal of a matrix. Then let $\alpha = 2\phi_1 = \phi_1 - \phi_2$ be the root. Let $\alpha^\dagger$ be the matrix such that $\kappa(\alpha^\dagger, -) = \alpha(-)$. In particular, we have $\alpha^\dagger = \frac{1}{2}h$.

Then note that if we choose units so that $\phi_1 = 1$, then $\alpha = 2$. If $\lambda = \nu$, then the Verma module for $\lambda$ has weights $\nu, \nu - 2, \ldots$, and the simple module has weights $\nu, \nu - 2, \ldots, -\nu$. For non-integral weights, we just have an infinite-dimensional representation. To see this, note that hitting any non-integral weight with $n^-$ will not reach another maximal vector.

2.5 Finite-dimensional modules

**Theorem 2.5.1.** For any weight $\lambda$, $\dim L_\lambda < \infty$ if and only if $\lambda \in \Lambda_+$ is a dominant integral weight. This is equivalent to $\dim \mathcal{L}_\lambda = \dim \mathcal{L}_{\mathcal{W}(\mu)}$ for all $w \in W$.

This result tells us that weights of $L_\lambda$ are actually symmetric under the Weyl group.

**Proof.** First, if the span of $v \in M_\lambda$ is finite-dimensional for $\mathfrak{sl}_2$, then all of $\mathfrak{h}$ stabilizes $\text{Span}_{\mathfrak{sl}_2}v$.

This is because if $v^\mu \in \mathcal{N} := \langle v \rangle$, then

$$h(e_i v^\mu) = e_i h v^\mu + \alpha_i(h)e_i v^\mu,$$

and thus $h(f_i v^\mu) \in \mathcal{C}f_i v^\mu$.

Next, if $\dim L_\lambda < \infty$, then after restriction to $\mathfrak{sl}_2$, we have $\lambda(h_i) = \alpha_i^\dagger(\lambda) \in \mathbb{N}$, and thus $\lambda \in \Lambda_+$.

Now suppose that $\lambda \in \Lambda_+$. Then after restricting $L_\lambda$ to $\mathfrak{sl}_2$, the span of $v_\lambda^+$ is isomorphic to $L_{\alpha_i^\dagger(\lambda) \cdot \phi_i}$, and in particular it is finite-dimensional. Next, we show that $L_\lambda$ is a sum of finitely many $\mathfrak{sl}_2$-summands. To see this, consider the sum $\mathcal{M}$ of all $\mathfrak{sl}_2$-submodules of $L_\lambda$. But then if we denote a summand by $N$, we note that $\mathfrak{g} \otimes N$ is a finite-dimensional representation of $\mathfrak{sl}_2$, so the natural morphism

$$\mathfrak{g} \otimes N \to L_\lambda$$

lands inside $\mathcal{M}$. But then $\mathcal{M} = L_\lambda$ because $\mathcal{M}$ is a nonzero submodule.
Next, recall that for \( M \in \text{Rep}(sl_2) \), then the sets of weights are invariant under reflection across the origin, so we have isomorphisms
\[
f^\alpha(\lambda) : M^\lambda \cong M^{s(\lambda)} ; e^\alpha(\lambda).
\]
Because \( L_\lambda \) is a sum of finite-dimensional representations of \( sl_2 \), for any \( v \in L_\lambda^\mu \), consider the finite-dimensional \( sl_2 \)-module containing it. If we add them up, we know that \( L_\mu^\lambda = L_\lambda^\nu \). But then the \( s_i \) generate \( W \), and thus for all \( w \in W \), \( L_\lambda^\mu \cong L_\lambda^w \).

Finally, for any orbit of \( W \), there exists exactly one representative in the dominant weight lattice, and because there are only finitely many dominant integral weights less than \( \lambda \), there must only be finitely many orbits, so \( L_\lambda \) is finite-dimensional.

### 2.6 Central actions

Here, we will consider the action of \( Zg \) on a module \( M \). Suppose that \( M \) is a highest weight module for weight \( \lambda \). Then we note that
\[
h \cdot z v^\lambda_+ = z h v^\lambda_+ = \lambda(h) z v^\lambda_+,
\]
and therefore \( z v^\lambda_+ = \vartheta_\lambda(z) v^\lambda_+ \). Therefore \( z \) acts by \( \mathcal{V}_\lambda(z) \) on any highest weight module of weight \( \lambda \), and we call the function \( \vartheta_\lambda : Zg \to C \) a central character. In general, all algebra morphisms \( Zg \to C \) arise in this way. Then we have the decomposition
\[
z \in \mathfrak{u}g = \mathfrak{n}^- \otimes \mathfrak{h} \otimes \mathfrak{n}^+,
\]
and write \( \pi_\hbar : \mathfrak{u}g \to \mathfrak{u}h \) for the morphism killing \( n^\pm \). Then \( \vartheta_\lambda(z) = \lambda(\pi_\hbar(z)) \), so \( \pi_\hbar : Zg \to \mathfrak{u}h \) is an algebra homomorphism, and we will call this \( \varphi_{HC} = \vartheta \), the Harish-Chandra morphism. In particular, we obtain a morphism
\[
A^{\dim h} \to \text{Spec } Zg.
\]
Also, we will consider \( \varpi \circ w \cdot w \), where \( \varpi \circ w = w(\lambda + \rho) - \rho \), where
\[
\rho = \frac{1}{2} \sum_{\Phi^+} \alpha.
\]
If we can identify the two morphisms on a Zariski-dense subset, they must agree in general. First, note that \( \varpi(\lambda) = \vartheta_\lambda \), and now it suffices to show that \( \vartheta_\lambda = \vartheta_{w o \lambda} \) for \( \lambda \in \Lambda \).

To prove this, if there exists \( \sigma \in \Sigma \) such that \( \sigma^*(\lambda) \in \mathbb{N} \), then \( M_{s_\sigma o \lambda} \subset M_\lambda \), and thus \( \vartheta_\lambda = \vartheta_{s_\sigma o \lambda} \). In addition, if \( \sigma^*(\lambda) = -1 \), we have \( s_\sigma o \lambda = \lambda \), and if \( \sigma^* \lambda \leq -2 \), we can reverse the roles of \( \lambda, \sigma o \lambda \) because
\[
\sigma^*(s_\sigma o \lambda) = \sigma^* \lambda - 2 \sigma^* \lambda - 2 \geq 0.
\]

### 2.7 More on Harish-Chandra

Consider the twisted Harish-Chandra morphism
\[
Zg \xrightarrow{\psi_{HC}} Sh \xrightarrow{\lambda \mapsto \lambda - \rho} Sh.
\]
This gives us a morphism \( \psi_{HC} \). In particular, we have
\[
\vartheta_\lambda(z) = (\lambda + \rho) \psi_{HC}(z) = \lambda(\varphi_{HC}(z)).
\]
**Theorem 2.7.1.** The image of \( \psi_{HC} \) is contained in \((Sh)^W\).

To see this, note that

\[
\vartheta_{w \circ \lambda} = \psi_{HC}(w \circ \lambda) + \rho = \psi_{HC}(w(\lambda + \rho)) = \psi_{HC}(\lambda + \rho).
\]

**Theorem 2.7.2.**

1. \( \psi_{HC} \) is an isomorphism \( Zg \rightarrow Sh^W \);
2. If \( \lambda, \mu \) are linked, then \( \vartheta_{\lambda} = \vartheta_{\mu} \);
3. Every element of \( \text{Hom}_{\text{Alg}}(Zg, C) \) arises in this way.

There is a simple way to see the last two parts of the theorem if we assume some algebraic geometry.

For a central character \( \vartheta : Zg \rightarrow C \), consider the module

\[
M^\vartheta := \ker^\infty(\ker(\vartheta)) = \{v \in M \mid (z - \vartheta(z))n v = 0 \text{ for all } z\},
\]

where \( n \) depends on \( z \). We have a decomposition

\[
M^\mu = \bigoplus_{\vartheta} M^\mu \cap M^\vartheta,
\]

which gives us

\[
M = \bigoplus M^\vartheta.
\]

Now we may define subcategories of \( \mathcal{O} \) given by

\[
\mathcal{O}^\vartheta := \{M \mid M = M^\vartheta\}.
\]

Some examples are that all highest weight modules of weight \( \lambda \) are contained in \( \mathcal{O}^{\vartheta_{\lambda}} \).

**Proposition 2.7.3.** We have a decomposition

\[
\bigoplus_{\vartheta = \vartheta_{\lambda}} \mathcal{O}^\vartheta.
\]

We will now consider blocks of category \( \mathcal{O} \). We say that simple modules \( S_1, S_2 \) are in the same block if there is a nontrivial extension of \( S_2 \) by \( S_1 \). For general \( M \), we know that \( M \) has a finite Jordan-Hölder decomposition because \( \mathcal{O} \) is Artinian, so \( M \) is in some block if all of its Jordan-Hölder quotients are.

**Proposition 2.7.4.** If \( \lambda \in \Lambda \), then \( \mathcal{O}^{\vartheta_{\lambda}} \) is a block of \( \mathcal{O} \).

To prove this, if \( \mu < \lambda \) are linked, then we have the diagram

\[
M_\mu \hookrightarrow N_\lambda \rightarrow M_\lambda
\]

giving us an exact sequence

\[
0 \rightarrow L_\mu \hookrightarrow N_\lambda/\text{Im } N_\mu \rightarrow L_\lambda \rightarrow 0.
\]
3.1 Weyl character and dimension formulas

Today we will see what Category $\mathcal{O}$ tells us about finite-dimensional modules. We will fix a semisimple Lie algebra $g = n_+ \oplus \mathfrak{h} \oplus n_-$. 

Definition 3.1.1. Given $M \in \mathcal{O}$, define the function $\text{ch}_M : \mathfrak{h}^* \to \mathbb{Z}_{\geq 0}$, where $\lambda \mapsto \dim M_{\lambda}$.

Also let $e_\lambda$ be the characteristic function of $\lambda$. Now given $f, g : \mathfrak{h}^* \to \mathbb{Z}_{\geq 0}$, define the convolution product

$$f \ast g(\lambda) = \sum_{\mu + \nu = \lambda} f(\mu)g(\nu).$$

For example, $e_\lambda \ast e_\mu = e_{\lambda + \mu}$. Here, we assume that $f, g$ are supported on a finite union of things of the form $\lambda - \Gamma$, where $\Gamma$ are the non-negative weights. We will call the set of such functions $X$.

Proposition 3.1.2.

1. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then $\text{ch}(M) = \text{ch}(M') + \text{ch}(M'')$.

2. If $M \in \mathcal{O}$ and $L$ is finite-dimensional, then $\text{ch}(L \otimes M) = \text{ch}(M) \ast \text{ch}(L)$.

Proof.

1. Note that $\dim M_\mu = \dim M'_\mu + \dim M''_\mu$ for any such exact sequence.

2. Note that $\dim (L \otimes M)_{\lambda} = \sum_{\mu + \nu = \lambda} \dim L_\mu \dim M_\nu$. \hfill $\square$

Last time we considered central characters for a highest weight module of weight $\lambda$ and highest weight vector $v^+$

$$\chi_{\lambda} : Z(g) \to \mathbb{C}, \quad z \mapsto \frac{z \cdot v^+}{v^+} = \lambda(\text{pr}(z)).$$

If $L(\mu)$ is a subquotient of the Verma module $M(\lambda)$, then, then $\chi_\mu = \chi_\lambda$. Equivalently, $\mu$ and $\lambda$ are linked by some element $w$ of the Weyl group. Because $\mathcal{O}$ is Artinian, for all $M \in \mathcal{O}$ we have a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$
where all $M_{i+1}/M_i$ is simple, so they must be isomorphic to some $L(\lambda_i)$. But then we have

$$\text{ch } M = \sum_i \text{ch } L(\lambda_i) = \sum_{w \in W} a(\lambda, w) L(w \circ \lambda).$$

Our goal is now to compute the $a(\lambda, w)$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be weights linked to lambda arranged such that if $\lambda_i \geq \lambda_j$, then $i \geq j$. Then we should have some identity of the form

$$\begin{pmatrix}
\text{ch } M(\lambda_1) \\
\vdots \\
\text{ch } M(\lambda_n)
\end{pmatrix} = \begin{pmatrix}
1 & * & * \\
0 & \ddots & * \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\text{ch } L(\lambda_1) \\
\vdots \\
\text{ch } L(\lambda_n)
\end{pmatrix}.$$

This implies that $\text{ch } L(\lambda) = \sum_{w \in W} b(\lambda, w) \text{ch } M(w \circ \lambda)$. Note that $b(\lambda, 1) = 1$.

**Definition 3.1.3.** Define the Kostant function

$$p: h^* \to \mathbb{Z}^{\geq 0} \quad \nu \mapsto |\{ (c_\alpha)_{\alpha > 0} \in \mathbb{Z}^{\geq 0} | \sum c_\alpha \alpha = \nu \}|.$$

**Proposition 3.1.4.** $p = \text{ch } M(0)$. More generally, $e_{\lambda} \ast 0 = \text{ch } M(\lambda)$.

**Proof.** By the PBW theorem, we know that $M(0)$ is spanned by $U(n_-)$. This is apparently equivalent to the definition of $p$. \qed

**Definition 3.1.5.** Define the function

$$q = \prod_{\alpha > 0} (e_{\alpha/2} - e_{-\alpha/2}).$$

Also define

$$f_\lambda = e_0 + e_{-\lambda} + \cdots = \begin{cases} 1 & \alpha \neq 0 \\
-k_\lambda, k \in \mathbb{Z}^{\geq 0} \\
0 & \text{otherwise.}
\end{cases}$$

Note that $f_\alpha \ast (1 - e_{-\alpha}) = 1$. Also note that

$$q \ast \prod_{\alpha > 0} f_\alpha = e_\rho \prod_{\alpha > 0} (1 - e_{-\alpha}) \prod_{\alpha > 0} f_\alpha = e_\rho.$$  

Next, $p = \prod_{\alpha > 0} f_\alpha$ and if $\alpha$ is a simple root, then $s_\alpha \cdot q = -q$. The reason for this is that $s_\alpha(\alpha) = -\alpha$ but $s_\alpha$ fixes the other positive roots. This implies that $w \cdot q = (-1)^{\ell(w)} q$.

**Theorem 3.1.6 (Weyl character formula).** If $\lambda \in \Lambda^+$, then

$$q \ast \text{ch } L(\lambda) = \sum_{w \in W} (-1)^{\ell(w)} e_{w_0 \lambda + \rho}.$$  

**Proof.** If we apply $q \ast -$ to the formula

$$\text{ch } L(\lambda) = \sum_{w \in W} b(\lambda, w) \text{ch } M(w \circ \lambda),$$
we obtain

\[ q \ast \text{ch} L(\lambda) = \sum_{w \in W} q \ast \text{ch}(w \circ \lambda) \]

\[ = \sum_{w \in W} b(\lambda, w) e_{w_0\lambda + \rho}. \]

Because \( \lambda \in \Lambda^+ \), we know that \( L(\lambda) \) is finite-dimensional and all weight spaces are symmetric. If \( \alpha \) is a simple root, then we apply \( s_\alpha \) to both sides, and we obtain

\[ -q \text{ch} L(\lambda) = \sum_{w \in W} b(\lambda, w) s_\alpha(w \circ \lambda + \rho) = e_{s_\alpha w_0\lambda + \rho} \]

because \( s_\alpha(w \circ \lambda + \rho) = s_\alpha w(\lambda + \rho) = s_\alpha w \circ \lambda + \rho \). Therefore we see that \( b(\lambda, s_\alpha w) = -b(\lambda, w) \), so \( b(\lambda, w) = (-1)^{\ell(w)} \).

We would now like to compute \( \dim L(\lambda) \) for dominant integral weights \( \lambda \). We want something like

\[ \text{sum}(q) \cdot \dim L(\lambda) = \sum_{w \in W} (-1)^{\ell(w)}, \]

except that both sides here vanish, so this is too naïve. For example, if we consider \( \text{sl}_2 \), we have

\[ (e_1 - e_{-1}) \text{ch} L(\lambda) = e_{\lambda+1} - e_{\lambda-1}. \]

If we divide, we actually obtain \( \text{ch} L(\lambda) = e_\lambda + e_{\lambda-2} + \cdots + e_{-\lambda} \).

In the general case, let \( \mu \in h^* \) and \( t \in \mathbb{R} \). Define \( F_{\mu,t} : \mathcal{X} \to \mathbb{R} \) by extending \( e_\lambda \mapsto e^{t(\lambda, \mu)} \) linearly. Applying \( F_{\mu,t} \) to the Weyl character formula, we obtain

\[ e^{t(p, \rho)} \prod_{\alpha > 0} \left( 1 - e^{-t(p, \alpha)} \right) F_{\mu,t} \text{ch} L(\lambda) = \sum_{w \in W} (-1)^{\ell(w)} e^{t(p, w(\lambda + \rho))} \]

\[ = \sum_{w \in W} (-1)^{\ell(w)} e^{t(w^{-1}p, \lambda + \rho)} \]

\[ = \sum_{w \in W} (-1)^{\ell(w)} e^{t(wp, \lambda + \rho)} \]

\[ = F_{\lambda+p, t} \sum_{w \in W} (-1)^{\ell(w)} e_{wp} \]

\[ = F_{\lambda+p, t} \left( e_p \prod_{\alpha > 0} (1 - e^{-\alpha}) \right) \]

\[ = e^{t(p, \rho + \lambda)} \prod_{\alpha > 0} (1 - e^{-t(\alpha, \lambda + \rho)}). \]

Note here that \( F_{\rho,t}(e_\lambda \ast e_\mu) = F_{\rho,t}(e_\lambda) \cdot F_{\rho,t}(e_\mu) \). In the \( t \to 0 \) limit, we have \( F_{\rho,t} \text{ch} L(\lambda) \to \dim L(\lambda) \) and \( e^{t(p, \rho)} \to 1 \). Therefore we obtain

\[ \dim L(\lambda) = \lim_{t \to 0} \prod_{\alpha > 0} \frac{1 - e^{-t(\alpha, \lambda + \rho)}}{1 - e^{-t(\alpha, \lambda)}} \]

\[ = \prod_{\alpha > 0} \frac{(\alpha, \lambda + \rho)}{(\alpha, \lambda)} \cdot \]

This is called the Weyl dimension formula.
3.2 Maximal submodules of Verma modules

Theorem 3.2.1. Let \( \lambda \in \Lambda^+ \) and \( \alpha_1, \ldots, \alpha_k \) be simple roots of \( \mathfrak{g} \). Then

\[
\sum M(s_{\alpha_i} \circ \lambda)
\]

is the maximal submodule of \( M(\lambda) \).

Remark 3.2.2. Last time we saw that \( M(s_{\alpha_i} \circ \lambda) \subset M(\lambda) \).

Lemma 3.2.3. Let \( a, b \in U \mathfrak{g} \). Then

\[
[a^k, b] = k[a, b]a^{k-1} + \binom{k}{2} [a[a, b]]a^{k-2} + \cdots + [a, \cdots, [a, [a, b]]].
\]

This is proved by induction, so like a certain Fields medalist, we omit the proof.

If \( x_\alpha, x_\beta \) correspond to roots \( \alpha, \beta \), then note that eventually we will have \([x_\alpha, \cdots, [x_\alpha, [x_\alpha, x_\beta]]] = 0\). In fact, four times is enough.

Lemma 3.2.4. Let \( \alpha \) be a simple root. Then for any \( v \in M(\lambda) \), there exists \( N \gg 0 \) such that \( y_{\alpha, \lambda}^N \cdot v = 0 \) in \( M(\lambda) / \sum M(s_{\alpha_i} \circ \lambda) \).

Proof. We proceed by induction. Suppose that \( v = y_{i_1} y_{i_2} \cdots y_{i_t} v^+ \). When \( t = 0 \), then \( y_{\alpha, \lambda}^{(\alpha, \lambda) + 1} v^+ = 0 \). For \( t > 0 \), we have

\[
y_{\alpha}^N y_{i_1} \cdots y_{i_t} v^+ = y_{i_1} y_{\alpha}^N y_{i_2} \cdots y_{i_t} v^+ + [y_{\alpha}^N, y_{i_1}] y_{i_2} \cdots y_{i_t} v^+.
\]

The first term on the right hand side vanishes by the inductive hypothesis, and the second term becomes \((-1) \cdot y_{\alpha}^{N-3} y_{i_2} \cdots y_{i_t} = 0 \) by the inductive hypothesis.

Proof of Theorem. By the discussion last time and the second lemma, we know that \( M(\lambda) / \sum M(s_{\alpha_i} \circ \lambda) \) is finite-dimensional. This implies that \( M(\lambda) / \sum M(s_{\lambda_i} \circ \lambda) = L(\lambda) \oplus M' \), but we know that the left hand side is a highest weight module, so \( M' = 0 \).

Remark 3.2.5. We have a resolution

\[
\cdots \to \bigoplus_{\ell(w) = 1} M(w \circ \lambda) \to M(\lambda) \to L(\lambda) \to 0.
\]

This is called the BGG resolution.\(^1\)

---

\(^1\)This does not imply that the terms are projective.
Kevin (Oct 20): Duality and projectives in category $O$

4.1 Duality

Recall from the finite-dimensional story that $g$-representation $M$ have duals $M^\vee$ with $g$-action

$$(xf)(v) = -f(xv)$$

for $x \in g, f \in M^\ast, v \in M$. This is not well-behaved for infinite-dimensional representations (for example, $M^{\ast\ast} \not\sim M$), so in this case we would like to construct a better-behaved duality functor.

Note that every semisimple Lie algebra $g$ has a transpose $\tau: g \rightarrow g$ (if $g$ is a matrix Lie algebra, this is literally the transpose) which is an anti-automorphism. Here, we have

$\tau(x_\alpha) = y_\alpha, \quad \tau(y_\alpha) = x_\alpha, \quad \tau(h_\alpha) = h_\alpha.$

This allows us to define\footnote{Kevin is unsure how he is doing on time here.}

**Definition 4.1.1.** Let $M = \bigoplus_\lambda M_\lambda \in O$. Then the dual of $M$ is defined by

$$M^\vee = \bigoplus_\lambda M_\lambda^\vee \quad (xf)(v) = f(\tau(x)v).$$

**Proposition 4.1.2.** $M^\vee \in O$.

**Proof.** To prove finite generation, note that $M^\vee$ has finite length (here, $L(\lambda)^\vee = L(\lambda)$ because duality preserves formal characters and exchanges quotients and submodules). Clearly the weight spaces are finite-dimensional by assumption, and the weights lie in some union $\bigcup \lambda - \Lambda$ because formal characters are preserved, so we have local $n$-finiteness.

Here are some more facts about duality.

- Duality is a contravariant functor. This is obvious because everything is defined on the level of weight spaces.

- There is a natural isomorphism $M^{\vee\vee} \cong M$. This is clear because we are taking double duals of finite-dimensional things and adding them up, so in particular duality is an anti-equivalence of categories.
• We have $L(\lambda)^\vee \cong L(\lambda)$. On the other hand, duality for $M(\lambda)$ is complicated.

• $\tau$ fixes $Z(\mathfrak{g})$ by an exercise in Humphreys. In particular, this means that $(M^X)^\vee = (M^\vee)^X$.

4.2 Projectives

Recall that $P$ is projective if $\text{Hom}(P, -)$ is right exact. Our goal is to prove that $\emptyset$ has enough projectives (which will mean that we can do homological algebra). The first thing we will do is introduce dominance and antidominance.

Recall that for $\lambda \in \Lambda$, $W^\lambda$ contains one dominant weight and one antidominant weight. This gives us two(!) good choices for representatives of $W^\lambda$. Unfortunately, we care about nonintegral weights, and we cannot choose representatives of $W^\lambda$ for general $\lambda \in \mathfrak{h}^*$. 

Remark 4.2.1. From now on we will use the $w \circ -$ action (because this is all we care about), and therefore our new notion of (anti)dominance will not restrict to the old notion of dominance. 

Definition 4.2.2. A weight $\lambda \in \mathfrak{h}^*$ is dominant if $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{<0}$ for all $\alpha \in \Phi^+$. A weight $\lambda \in \mathfrak{h}^*$ is antidominant if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{>0}$ for all $\alpha \in \Phi^+$. 

Note that this is not the same as the undotted definition. For example, $-\rho$ is dominant. Also, the set $W \circ \lambda$ can have multiple dominant and/or antidominant weights.

Definition 4.2.3. We define the subgroup

$$W_{[\lambda]} := \{ w \in W \mid w \circ \lambda - \lambda \in \Lambda_r \}$$

where $\Lambda_r$ is the root lattice. We also define

$$\Phi_{[\lambda]} := \{ \alpha \in \Phi \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \}.$$ 

In fact, $W_{[\lambda]}$ is the Weyl group of $\Phi_{[\lambda]}$. We may similarly define $\Delta_{[\lambda]}$.

Proposition 4.2.4. The following are equivalent:

1. $\lambda$ is dominant.
2. $\langle \lambda + \rho, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta_{[\lambda]}$.
3. $\lambda \geq s_\alpha \circ \lambda$ for all $\alpha \in \Delta_{[\lambda]}$.
4. $\lambda \geq w \circ \lambda$ for all $w \in W_{[\lambda]}$.

Proof. Clearly 1 implies 2, and 2 implies 1 because positive roots are sums of simple roots with nonnegative coefficients. To prove that 2 is equivalent to 3, note that

$$s_\alpha \circ \lambda = \lambda - \langle \lambda + \rho, \alpha^\vee \rangle \alpha.$$ 

---

2 Professor Humphreys, I hope you don’t descend upon us from heaven for not having done this exercise. Also please forgive me (the note taker) for never interacting with you when I was an undergrad.

3 I (note taker) considered not putting this definition in the notes.

4 Apparently Kevin is speaking for us all here.

5 Said old definition has now been Stalined. Unfortunately, Humphreys just ignores the ambiguity.
Finally, to see that 3 is equivalent to 4, note that 4 implies 3 automatically. To prove that 3 implies 4, we induct on \( \ell(w) \). If \( w = w' s_\alpha \) with \( \ell(w') = \ell(w) - 1 \), we see that
\[
\lambda - w \circ \lambda = (\lambda - w' \circ \lambda) + w' \circ (\lambda - s_\alpha \circ \lambda).
\]
It is clear that \( \lambda - w' \circ \lambda \geq 0 \) while \( \lambda - s_\alpha \circ \lambda \) is a nonnegative multiple of \( \alpha \). Because of the length condition, we see that \( w' \circ \alpha \) is positive.

**Corollary 4.2.5.** The orbit \( W(\lambda) \circ \lambda \) has a unique (anti)dominant weight.

**Proof.** This is because 1 is equivalent to 4 in the proposition.

**Theorem 4.2.6.**

1. If \( \lambda \) is dominant, then \( M(\lambda) \) is projective.
2. If \( P \in \mathcal{O} \) is projective and \( L \in \mathcal{O} \) is finite-dimensional, then \( P \otimes L \) is projective.
3. \( \mathcal{O} \) has enough projectives.

**Proof.**

1. Consider \( M \rightarrow N \) and suppose \( v \in N \) is a maximal weight vector with weight \( \lambda \) (coming from a map \( M(\lambda) \rightarrow N \) ). Assume that \( M = M^X, N = N^X \). Our goal is to lift \( v \) to a maximal weight vector in \( M \), but because \( M \rightarrow N \) is surjective, we can lift \( v \) to \( v' \in M_\lambda \). If \( v' \) is maximal, then we are done, so suppose that \( v' \) is not maximal.

   In this case, there exists \( x \in U \) such that \( xv' \) is a maximal vector with weight greater than \( \lambda \). However, this weight must be linked to \( \lambda \), so by dominance of \( \lambda \), it cannot exist.

2. Here, we use the tensor-Hom adjunction
\[
\text{Hom}_\mathcal{O}(P \otimes L, M) \cong \text{Hom}_\mathcal{O}(P, L^* \otimes M).
\]
Because \( L^* \otimes - \) is exact and \( \text{Hom}_\mathcal{O}(P, -) \) is exact, the functor \( \text{Hom}_\mathcal{O}(P \otimes L, -) \) is exact and thus \( P \otimes L \) is projective.\(^6\)

3. The first thing we want to do is to find projectives mapping onto \( L(\lambda) \). For large \( n \), \( \lambda + np \) is dominant. This implies that \( M(\lambda + np) \) is projective, but then \( M(\lambda + rp) \otimes L(np) \) is projective. In fact, there exists a surjection \( M(\lambda + np) \rightarrow L(np) \). To see this, if \( M \) is a \( \mathfrak{u} \mathfrak{g} \)-module and \( L \) is a \( \mathfrak{u} \mathfrak{b} \)-module, then
\[
(\mathfrak{u} \mathfrak{g} \otimes \mathfrak{u} \mathfrak{b} L) \otimes M \cong \mathfrak{u} \mathfrak{g} \otimes \mathfrak{u} \mathfrak{b} (L \otimes M).
\]
This is known as the tensor identity and is apparently not obvious unless you have the arrogance level of a certain Chinese mathematician. Because \( M(\lambda + np) = \mathfrak{u} \mathfrak{g} \otimes \mathfrak{u} \mathfrak{b} C_{\lambda + np} \), we obtain
\[
M(\lambda + np) \otimes L(np) \cong \mathfrak{u} \mathfrak{g} \otimes \mathfrak{u} \mathfrak{b} (C_{\lambda + np} \otimes L(np)) \rightarrow \mathfrak{u} \mathfrak{g} \otimes \mathfrak{u} \mathfrak{b} C_\lambda \cong M(\lambda).
\]
The surjection comes from the fact that the lowest weight of \( L(np) \) is \(-np\), so we can kill all of the higher weights.

---

\(^6\)Apparently Stalinization is an invertible operation, although to be fair it is unclear what the history of the USSR says about this.
The rest of the proof is simply homological algebra. For a general $M \in \mathcal{O}$, because $\mathcal{O}$ is Artinian, we can induct on the length of $M$. First consider a short exact sequence

$$0 \to L(\lambda) \to M \to N \to 0.$$ 

By assumption, there exists a surjection $P \twoheadrightarrow N$, and this morphism lifts to $P \twoheadrightarrow M$. If $P$ does not surject onto $M$, then $\text{Im}(P \to M)$ cannot intersect $L(\lambda)$ (otherwise it would contain all of $L(\lambda)$ and thus surject onto $M$). This implies that $\text{Im}(P \to M) \cong N$, which splits the exact sequence.

By standard homological algebra, because $\mathcal{O}$ is Artinian and has enough projectives, then $\mathcal{O}$ has projective covers (i.e. unique minimal projectives surjecting onto $M$). If we define $P(\lambda)$ to be the projective cover of $L(\lambda)$, the $P(\lambda)$ are precisely the indecomposable projectives. Therefore every projective is a direct sum of $P(\lambda)$.

**Theorem 4.2.7.**

1. $P(\lambda)$ has a standard filtration, which is a filtration with subquotients that are Verma modules.

2. (BGG reciprocity) The multiplicity of $M(\mu)$ in the composition series for $P(\lambda)$ is given by

$$\langle P(\lambda) : M(\mu) \rangle = [M(\mu) : L(\lambda)].$$
We will be discussing the following results during this lecture:

**Theorem 5.0.1.** Let $\lambda \in \mathfrak{h}^*$. Then $M_\lambda = L_\lambda$ if and only if $\lambda$ is $\rho$-antidominant (this is the notion of antidominance that we discussed last time).

**Theorem 5.0.2.** For any $\lambda \in \mathfrak{h}^*$, if $\alpha \in \Phi_+$ has $s_\alpha \circ \lambda \leq \lambda$, then $M_{s_\alpha \circ \lambda} \hookrightarrow M_\lambda$.

### 5.1 Basic facts

Recall that the socle of a module $M$ is defined to be the direct sum of its simple submodules. A fact from ring theory\(^1\) is that if $R$ is left noetherian and has no right zero-divisors, then any two left ideals intersect nontrivially.

**Proposition 5.1.1.** for all $\lambda \in \mathfrak{h}^*$, $M_\lambda$ has a unique simple submodule.

**Proof.** Recall that $M_\lambda \cong \mathfrak{u}n^-$, but then simple submodules of $M_\lambda$ are ideals in $\mathfrak{u}n^-$, which intersect nontrivially. \(\square\)

**Theorem 5.1.2.** For any $\lambda, \mu \in \mathfrak{h}^*$,

1. Any nonzero $\varphi : M_\mu \to M_\lambda$ is an injection.
2. $\dim \text{Hom}_O(M_\mu, M_\lambda) \leq 1$.
3. If $L_\mu \subseteq M_\lambda$ is the unique simple submodule, then $L_\mu = M_\mu$.

**Proof.**

1. $\varphi$ is determined by the image $\varphi(v^\mu_+) = y \cdot v^\lambda_+$ for some $y \in \mathfrak{u}n^-$ of the highest weight vector. But then $\varphi(y'v^\mu_+) = y' \cdot y \cdot v^\lambda_+$, and this follows from the fact that $\mathfrak{u}n^-$ has no zero divisors.

2. If $\varphi_1, \varphi_2$ are morphisms and $L$ is the unique simple submodule of $M_\mu$, then $\varphi_1(L), \varphi_2(L)$ are both simple and thus isomorphic, so there exists $c$ such that $(\varphi_1 - c\varphi_2)(L) = 0$, and so $\varphi - c\varphi_2 = 0$ because it is not injective.

\(\text{\(^1\)}\) that was not in my graduate algebra course as an undergrad
3. Suppose $L_\mu$ is the unique simple submodule of $M_\lambda$. Then we have the sequence

$$M_\mu \rightarrow L_\mu \hookrightarrow M_\lambda,$$

and so $M_\mu$ must inject into $M_\lambda$ and thus $M_\mu = L_\mu$. \hfill \Box

**Proposition 5.1.3.** For $\lambda \in \Lambda - \rho$, if $w = s_{n_1} \cdots s_1$ is a reduced word expression, then there exists a sequence of embeddings

$$M_{\nu \circ \omega} \subseteq M_{s_{n-1} \cdots s_1 \circ \lambda} \subseteq M_{s_1 \circ \lambda} \subseteq M_\lambda.$$

**Proof.** We will induct on $\ell(w)$. Choose some $0 < k < n$. Recall that $\ell(s_{\alpha} w) > \ell(w)$ if and only if $w^{-1} \alpha \in \Phi^+$. Note that $\ell(s_{k+1} \cdots s_1) > \ell(s_k \cdots s_1)$. Thus if $w' = s_k \cdots s_1$, then $w'^{-1} \sigma_{k+1} > 0$. We now compute

$$\sigma_{k+1}^* (s_k \cdots s_1 \circ \lambda + \rho) = \sigma_{k+1}^* (s_k \cdots s_1 (\lambda + \rho)) = (s_1 \cdots s_k \sigma_{k+1})^* (\lambda + \rho) \in \mathbb{N}.$$

Then we know $M_{s_{k+1} \cdots s_1 \circ \lambda} \subseteq M_{s_k \cdots s_1 \circ \lambda}$. \hfill \Box

### 5.2 Proof of first theorem

If $\lambda$ is $\rho$-antidominant, we will prove that $M_\lambda = L_\lambda$. Recall that antidominance is equivalent to $\lambda \leq \nu \circ \omega \lambda$ for all $w \in W_\lambda$. But the Jordan-Holder factors of $M_\lambda$ look like $M_\mu$ for $\mu \leq \lambda$ with $\mu$ is linked to $\lambda$. This is the same as $\mu \in W_\lambda \circ \lambda$. But then the only weight that can appear is $\lambda$, so $M_\lambda = L_\lambda$.

Now we will prove that if $M_\lambda = L_\lambda$, then $\lambda$ is antidominant. This is hard, so we will assume for now that $\lambda \in \Lambda$. Because $M_\lambda$ is simple, then $N_\lambda = 0$. Now assume that for some $\sigma \in \Sigma$, $\sigma^* (\lambda + \rho) > 0$. Then

$$s_{\sigma \circ \lambda} = \lambda - \sigma^* (\lambda + \rho) \sigma < \lambda,$$

and so $M_{s_{\sigma} \circ \lambda} \hookrightarrow M_\lambda$. This gives a contradiction.

We now finish the proof in the general case. Assume there exists $\alpha \in \Phi^+$ such that $\alpha^* (\lambda + \rho) \in \mathbb{Z}_+$. Then $s_{\alpha} \circ \lambda = \lambda - \alpha^* (\lambda + \rho) \alpha < \lambda$. By the second theorem, we have an injection $M_{s_{\alpha} \circ \lambda} \hookrightarrow M_\lambda$, and thus $M_\lambda$ is simple.

### 5.3 More basic facts

**Proposition 5.1.** Let $\lambda, \mu \in h^*$ and $\sigma \in \Sigma$. If $M_{s_{\sigma} \circ \mu} \subseteq M_\mu \subseteq M_\lambda$, then:

- If $\sigma^* (\lambda + \rho) \leq 0$, then $M_\lambda \subseteq M_{s_{\sigma} \circ \lambda}$.
- If $\sigma^* (\lambda + \rho) > 0$, then $M_{s_{\sigma} \circ \mu} \subseteq M_{s_{\sigma} \circ \lambda} \subseteq M_\lambda$.

**Proof.** In the first case, then we know that

$$\sigma^* (s_{\sigma} \circ \lambda + \rho) = \sigma^* (s_{\sigma} (\lambda + \rho)) = (s_{\sigma} \circ \sigma)^* (\lambda + \rho) = -\sigma^* (\lambda + \rho) \geq 0,$$

and so $M_\lambda \hookrightarrow M_{s_{\sigma} \circ \lambda}$.
In the second case, let \( \nu^\lambda_+, \nu^\mu_+ \) be maximal vectors of \( M_\lambda, M_\mu \). Then we know that
\[
f_{\sigma}^{\sigma(\lambda + \rho)} \in M_{\sigma \circ \lambda}
\]
and because \( M_{\sigma \circ \mu} \subseteq M_\mu, f_{\sigma}^{\sigma(\mu + \rho)} \nu^\mu_+ \in M_{\sigma \circ \mu} \) is a maximal vector. Also because \( M_\mu \subseteq M_\lambda \), there exists \( y \in \mathfrak{u} \mathfrak{n}_- \) such that \( y \cdot \nu^\lambda_+ = \nu^\mu_+ \).

It is a fact that if \( n \) is a nilpotent Lie algebra and \( \xi \in n, x \in \mathfrak{u} \mathfrak{n}_+ \) for any \( n \in \mathbb{Z}_+ \) there exists \( k \in \mathbb{Z} \) such that \( \xi^k x \in \mathfrak{u} \mathfrak{n} \langle x^n \rangle \). Applying this fact to \( f_\sigma \in \mathfrak{u} \mathfrak{n}_-, y \in \mathfrak{u} \mathfrak{n}_- \), and \( n = \sigma \lambda + \rho \in \mathbb{Z}_+ \), there exists \( k \) such that
\[
f_{\sigma}^k y \in \mathfrak{u} \mathfrak{n}_- \langle f_\sigma^{\sigma(\lambda + \rho)} \rangle.
\]
But then
\[
f_{\sigma}^k \nu^\mu_+ = f_{\sigma}^k y \nu^\lambda_+ \in \mathfrak{u} \mathfrak{n}_- \langle f_\sigma^{\sigma(\lambda + \rho)} \rangle \subseteq M_{\sigma \circ \lambda}.
\]
If \( k < \sigma^*(\mu + \rho) \), then we know \( M_{\sigma \circ \mu} \subseteq M_{\sigma \circ \lambda} \). On the other hand, if \( k > \sigma^*(\mu + \rho) \), we have
\[
[e_\sigma, f_{\sigma}^k] \cdot \nu^\mu_+ = k f_{\sigma}^{k-1} (h_\sigma - k + 1) \nu^\mu_+
\]
\[
= k f_{\sigma}^{k-1} (\mu(h_\sigma) - k + 1) \nu^\mu_+
\]
\[
= (\sigma^*(\mu + \rho) - k) k f_{\sigma}^{k-1} \nu^\mu_+,
\]
but this is equal to \( e_\sigma f_{\sigma}^k \nu^\mu_+ \), so we are done. \( \square \)

### 5.4 Proof of second theorem

We will prove this result in the integral case. Assume that \( \lambda \in \Lambda \) and \( \mu := s_\alpha \circ \lambda \leq \lambda \). If \( \mu \in \Lambda \), then there exists \( w \) such that \( \mu' = w^{-1} \circ \mu \in \Lambda_+ - \rho \). Thus there exists \( w = s_n \cdots s_1 = s_{n-1} \cdots s_1 \circ \mu' \cdots \), then
\[
M_{\mu_n} \subseteq M_{\mu_{n-1}} \subseteq M_{\mu_1} \subseteq M_{\mu'}.
\]
If we define \( \lambda' = w^{-1} \circ \lambda \) and \( \lambda_n, \ldots, \lambda_1 \) analogously and assume that \( \mu < \lambda \) and thus \( \mu_k \neq \lambda_k \), then we have
\[
\mu_k = s_{\beta_k} \circ \lambda_k,
\]
where \( \beta_k = s_{k+1} \cdots s_n (\alpha) \) so that
\[
\mu_k - \lambda_k = -\beta_k^* (\lambda_k) \beta_k + s_{\beta_k} (\rho) - \rho.
\]
Note that this is the difference of a multiple of \( \beta_k \) and a sum of positive roots.

Continuing, note that \( \beta_k, \mu_k \) are linked to \( \lambda \). These satisfy \( \mu' = \mu_0 \geq \cdots \geq \mu_n = \mu, \mu < \lambda \), and \( \mu' > \lambda' \). Thus there exists \( k \) such that \( \lambda_k < \mu_k \) and \( \lambda_k > \mu_{k+1} \). But then
\[
0 < \mu_k - \lambda_k = -\beta_k^* (\lambda_k) \beta_k - \sum \text{ positives},
\]
and in fact we obtain a multiple of \( \beta_k \). Now there exists \( M_{\mu_{k+1}} \subseteq M_{\lambda_{k+1}} \) and \( M_{\mu_2} \subseteq M_{\lambda_{k+2}} \), so \( M_{\mu_n} = M_\mu \subseteq M_{\lambda_n} = M_\lambda \). To construct this, note that
\[
0 > \mu_{k+1} - \lambda_{k+1} = s_{k+1} \circ \mu_k - s_{k+1} \circ \lambda_k = s_{k+1} (\mu_k - \lambda_k).
\]
This implies that \( s_{k+1} \) flips \( \beta_k \) and thus \( \beta_k = \sigma_{k+1} \). Also, \( \beta_{k+1} = -\sigma_{k+1} \). Therefore,
\[
\mu_{k+1} = s_{-\sigma_{k+1}} \circ \lambda_{k+1} = s_{\sigma_{k+1}} \circ \lambda_{k+1} < \lambda_{k+1}.
\]
But then the coefficient of $\sigma_{k+1}$ in $\lambda_{k+1} - s_{k+1} \circ \lambda$ is positive, and thus $M_{\mu_{k+1}} \leq M_{\lambda_{k+1}}$.

We then have $M_{\mu_{k+2}} \subseteq M_{\mu_{k+1}} \subseteq M_{\lambda_{k+1}}$, and therefore either $M_{\lambda_{k+1}} \subseteq M_{s_{k+2} \circ \lambda_{k+1}} = M_{\lambda_{k+2}}$ or $M_{\mu_{k+2}} \subseteq M_{\lambda_{k+2}} \subseteq M_{\lambda_{k+1}}$, and so we are done.

We sketch the rest of the proof. Fix $\alpha \in \Phi_+$ and $n > 0$ is an integer. Then define

$$X := \{ \lambda \in h^\ast | M_\lambda - n\alpha \hookrightarrow M_\lambda \}.$$  

Then define $H = \{ \lambda | \alpha^\ast (\lambda + \rho) = n \}$. We know that $\Lambda \cap H \subseteq X \subseteq H$. But now $\Lambda \cap H$ is Zariski-dense in $H$, so we need to prove that $X$ is a closed subscheme of $H$. But now $M_{\lambda - \alpha} \hookrightarrow M$ exactly when there exists $y \in \ln^-$ such that $y \in (\ln^-)^{-n\alpha}$. But now $n^+ \cdot y \cdot \nu^+ = 0$, and this is equivalent to

$$e_1 y^\lambda = \cdots = e_r y^\lambda = 0,$$

and this is equivalent to

$$[e_1, y] y^\lambda = \cdots = [e_r, y] y^\lambda = 0.$$

Finally we have $[e_1, y] = y_i + y_i' h_i$. If we write $y = f_1 \cdots f_\ell$, we have a map

$$(\ln^-)^{-n\alpha} \to (\ln^-)^{\oplus \ell} \quad y \mapsto \sum_{i=1}^i y_i + \lambda(h_i) y_i'$$

by an argument about passing from $e_1 f_1 \cdots f_\ell$ to $f_1 \cdots f_\ell e_1$. But now our chain of equivalences continues to $(y_i + \lambda(h_i) y_i') = 0$ for all $i$. But now $X$ is given by the rank of the map being lower than usual, so it is a closed subscheme.