Abstract. These are notes from the (ongoing) Student Number Theory Seminar on Hida theory at Columbia University in Fall 2017, which is organized by David Hansen and Samuel Mundy.

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1. Lecture 1 (September 7, 2017): Samuel Mundy
Ribet’s Converse to Herbrand’s Theorem

1.1. Introduction. Let \( p \) be an odd prime, and \( \omega : G_{\mathbb{Q}} \to \mu_{p-1} \subseteq \mathbb{Z}_p^* \) be the Teichmüller character (so that \( \sigma \zeta_p = \zeta_p^{\omega(\sigma)} \) for all \( \sigma \in G_{\mathbb{Q}} \)).

For any \( \mathbb{Z}_p[G_{\mathbb{Q}}] \)-module \( M \) and \( \varphi \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})^\wedge \), write \( M^\varphi = \{ m \in M \mid \sigma m = \varphi(\sigma)m \} \) to be the \( \varphi \)-eigenspace of \( M \).

Let \( B_n \) be the \( n \)-th Bernoulli number.

Today we are going to prove Ribet’s converse to Herbrand’s theorem:

**Theorem 1.1** (Ribet). Let \( m \) be an odd integer with \( 3 \leq m \leq p-2 \). If \( p \mid B_{p-m} \), then \( \text{Cl}_{\mathbb{Q}(\zeta_p)}[p^\infty]_{\omega^m} \neq 0 \).

**Remark.** We have \( v_p(B_{p-1}) = -1 \) and \( \text{Cl}_{\mathbb{Q}(\zeta_p)}[p^\infty]_{\omega} = 0 \).

The idea is to deduce properties of Galois representations attached to automorphic representations of \( GL_1 \) using those for \( GL_2 \).

1.2. Modular forms. Fix embeddings \( \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C} \). For \( N \geq 1 \) an integer, \( \psi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \) a Dirichlet character, and \( A \subseteq \mathbb{C} \) a subring containing \( \mathbb{Z}[\psi] \), write

\[
S_k(N, \psi, A) = \left\{ \text{cusp forms } f = \sum_{n=1}^{\infty} a_n(f)q^n \text{ of weight } k, \text{ level } N, \text{ nebentypus } \psi \mid a_n(f) \in A \right\}
\]

where \( q = e^{2\pi i z} \). Denote by \( \mathbb{T}_k(N, \psi, A) \) the sub-\( A \)-algebra of \( \text{End}_A(S_k(N, \psi, A)) \) generated by the Hecke operators \( T(n) \) for all \( n \).

We define a pairing

\[
\langle f, h \rangle \mapsto a_1(f|h).
\]

In particular, \( \langle f, T(n) \rangle = a_n(f) \).

**Theorem 1.2.** This pairing is perfect.

**Theorem 1.3.** Let \( B \) be an \( A \)-algebra with \( \phi : A \to B \). Under the isomorphism

\[
\text{Hom}_A(\mathbb{T}_k(N, \psi, A), B) \simeq S_k(N, \psi, A) \otimes_A B,
\]

the \( A \)-algebra homomorphisms \( \text{Hom}_{A_{\text{alg}}}(\mathbb{T}_k(N, \psi, A), B) \) correspond to the normalized eigenforms, i.e., \( f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_k(N, \psi, A) \otimes_A B \subseteq B[[q]] \) with \( a_1(f) = 1 \) such that for all \( h \in \mathbb{T}_k(N, \psi, A) \), there exists \( c \in B \) such that \( \phi(f|h) = c\phi(f) \), where the corresponding homomorphism on the LHS sends an operator to its eigenvalue.

**Theorem 1.4.** Let \( f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_k(N, \psi, A) \) be a normalized eigenform. Then there exists a continuous Galois representation \( \rho_f : G_{\mathbb{Q}} \to \text{Aut}(V) \) with \( \dim_{\mathbb{Q}_p}(V) = 2 \) such that

1. \( \rho_f \) is irreducible;
2. for all \( \ell \nmid pN \), \( \rho_f \) is unramified at \( \ell \);
3. for all \( \ell \nmid pN \), \( \text{tr} \rho_f(\text{Frob}_\ell) = a_\ell(f) \);
4. for all \( \ell \nmid pN \), \( \det \rho_f(\text{Frob}_\ell) = \psi(\ell)\ell^{k-1} \).

**Definition 1.5.** A normalized eigenform \( f \in S_k(1, \overline{\mathbb{Q}}_p) \) is ordinary if \( |a_p(f)|_p = 1 \).
Theorem 1.6 (Mazur–Wiles). If $f$ is ordinary, then there exists a basis $v_1, v_2$ of $V$ such that
\[ \rho_f|_{G_{Q_p}} = \left( \begin{array}{cc} \alpha^{-1} & \chi_k - 1 \\ \alpha & \star \end{array} \right), \]
where $\chi : G_{Q} \to \mathbb{Z}_p^\times$ is the cyclotomic character (i.e., $\sigma \zeta = \zeta^{(\sigma)}$ for all $\zeta \in \mu_p$), and $\alpha : G_{Q_p} \to \overline{Q}_p^\times$ is the unramified character such that $\alpha(\text{Frob}_p)$ is the unit root of $X^2 - a_p(f)X + p^{k-1}$.

1.3. Proof of Ribet’s theorem. The proof consists of the following steps.

Step 0: Let $\kappa$ by any finite field of characteristic $p$. Then
\[ \text{Cl}_{Q_{(\wp)}}[p^{\omega_m}] \otimes_{F_\wp} \kappa \cong \text{Hom}_{Q_{(\wp)}}(G_{Q_{(\wp)}}^{ab,unr}, \kappa(\omega^m)) \]
\[ \cong H^1_{unr}(G_{Q_{(\wp)}}, \kappa(\omega^m))^G_Q \]
\[ \cong \text{Res}^{-1} \cong H^1_{unr}(G_{Q}, \kappa(\omega^m)) \]
\[ \cong \text{Ext}_{K_{Q}}(G_{Q}, \kappa(\omega^m)) \]
\[ \cong \text{Ext}_{K_{Q}}(\kappa, \kappa(\omega^{p-1-m})), \kappa). \]

Step 1 (Construction of an eigenform): Let $E_k(z) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \left( \sum_{d|n} d^{k-1} \right) q^n$

be the Eisenstein series of weight $k$, where $k \geq 4$ is even. The following facts are classical:

1. $\zeta(1-k) = -B_k/k$.
2. $v_p(B_k) < 0 \iff p - 1 \mid k \iff v_p(B_k) = -1$.
3. $E_k \in M_k(1, \mathbb{C})$. If $p - 1 \nmid k$, then $E_k \in M_k(1, \mathbb{Z}_p)$ by (2).
4. If $m \equiv n$ (mod $p - 1$), then $B_m \equiv B_n$ (mod $p$).

Note that
\[ E_4(z) = \frac{1}{240} + q + \cdots, \]
\[ E_6(z) = \frac{-1}{504} + q + \cdots. \]

Let $k \geq 4$ and $k \equiv p - m$ (mod $p - 1$). Write $k = 4a + 6b$ where $a, b \geq 0$. Then define
\[ G_k = (240E_4)^a(-504E_6)^b = 1 + \sum_{n=1}^{\infty} a_n q^n \]
where $a_n \in \mathbb{Z}$. Let
\[ F_k = E_k - \frac{\zeta(1-k)}{2} G_k = \sum_{n=1}^{\infty} b_n q^n \]
where $b_n = B_k a_n + \sum_{d|n} d^{k-1}$.

Now assume $p \mid B_{p-m}$. Then by (4), $p \mid B_k$ and so $b_n \equiv \sum_{d|n} d^{k-1}$ (mod $p$). Thus the map
\[ T_k(1, \mathbb{Z}_p) \to F_p \]
\[
T(n) \mapsto \sum_{d|n} d^{k-1} \pmod{p}
\]

is a \(\mathbb{Z}_p\)-algebra homomorphism, corresponding to \(F_k \pmod{p}\).

Let \(m \subset T_k(1, \mathbb{Z}_p)\) be the kernel, and \(p \subset m\) be a minimal prime. Let \(f\) be the eigenform corresponding to the homomorphism

\[
T_k(1, \mathbb{Z}_p) \to A := T_k(1, \mathbb{Z}_p)/p \to \mathcal{O}_L,
\]

where \(L := \text{Frac} \ A\). Then \(f \in S_k(1, \mathcal{O}_L)\) and \(f \equiv F_k \pmod{\lambda}\), where \(\lambda \in \mathcal{O}_L\) is a uniformizer.

**Step 2** (Analysis of \(\sigma\)): Let \(m \in T_k(1, \mathbb{Z}_p)\) be the kernel, and \(p \subset m\) be a minimal prime. Let \(f\) be the eigenform corresponding to the homomorphism

\[
T_k(1, \mathbb{Z}_p) \to A := T_k(1, \mathbb{Z}_p)/p \to \mathcal{O}_L,
\]

where \(L := \text{Frac} \ A\). Then \(f \in S_k(1, \mathcal{O}_L)\) and \(f \equiv F_k \pmod{\lambda}\), where \(\lambda \in \mathcal{O}_L\) is a uniformizer.

**Claim.** \(M_2 := M_2/\lambda M_2 \subset \overline{M}\) is a \(\mathbb{G}_Q\)-stable line with trivial action.

Let \(m_2 \in M_2\), so \(m_2 = c_r v_2\) for some \(c_r \in C\). If \(\sigma \in \mathbb{G}_Q\), then

\[
\rho_f(\sigma) m_2 = c_r b_\sigma v_1 + c_r d_\sigma v_2 \equiv d_\sigma m_2 = 1(\sigma) m_2 = m_2 \pmod{\lambda}
\]

which proves the claim.

**Step 3** (Construction of the lattice): Let \(M_1 = \mathcal{O}_L v_1\), \(M_2 = C v_2\) and \(M = M_1 \oplus M_2 \subset V_{\rho_f}\). This is \(\mathbb{G}_Q\)-stable, generated over \(\mathcal{O}_L[\mathbb{G}_Q]\) by \(v_1\).

Let \(\kappa = \mathcal{O}_L/\lambda\) and \(\overline{M} = M/\lambda M\).

Let \(m_2 \in M_2\), so \(m_2 = c_r v_2\) for some \(c_r \in C\). If \(\sigma \in \mathbb{G}_Q\), then

\[
\rho_f(\sigma) m_2 = c_r b_\sigma v_1 + c_r d_\sigma v_2 \equiv d_\sigma m_2 = 1(\sigma) m_2 = m_2 \pmod{\lambda}
\]

which proves the claim.

**Proof.** Omitted; play with matrix coefficients using information about the trace. \(\square\)

**Lemma 1.7.**

1. For all \(\sigma, \tau \in \mathcal{O}_L[\mathbb{G}_Q]\), we have \(a_\sigma, d_\sigma, b_\sigma c_\tau \in \mathcal{O}_L\) and

\[
\begin{align*}
a_\sigma &\equiv \omega^{k-1}(\sigma), & d_\sigma &\equiv 1(\sigma), & b_\sigma c_\tau &\equiv 0 \pmod{\lambda},
\end{align*}
\]

where \(1\) is the trivial \(\mathbb{G}_Q\)-character, and \(\omega^{k-1}, 1\) are extended by linearity to \(\mathcal{O}_L[\mathbb{G}_Q]\).

2. \(C = \{c_\sigma \mid \sigma \in \mathcal{O}_L[\mathbb{G}_Q]\}\) is a nonzero fractional ideal.

3. \(c_\sigma = 0\) for all \(\sigma \in I_\ell\) and primes \(\ell\).

**Proof.** Omitted; play with matrix coefficients using information about the trace. \(\square\)
• nonsplit over $G_Q$ because, if it were, $\overline{M_2} \cong \kappa$ would be a quotient and hence the image of $v_1$ would belong to the kernel by this quotient. This is because $\sigma_0$ acts as $\beta \not\equiv 1 \pmod{\lambda}$ on $v_1$, but $v_1$ generates $\overline{M}$ over $G_Q$. Hence the kernel is all of $\overline{M}$, a contradiction.
• split over $I_\ell$ for all $\ell$ because $O_Lv_1 \pmod{\lambda}$ is a stable line under $I_\ell$, since $\rho(\sigma)v_1 = a_\sigma v_1 + c_\sigma v_2 \equiv a_\sigma v_1 \pmod{\lambda}$.

Since $\omega^{k-1} = \omega^{p-1-m}$, this extension gives a nontrivial class in $\text{Ext}^{\text{unr}}_{\kappa[G_Q]}(\kappa(\omega^{p-1-m}), \kappa)$. This finishes the proof of Theorem 1.1.

2. Lecture 2 (September 14, 2017): Samuel Mundy
Basic Iwasawa Theory

2.1. $\mathbb{Z}_p$-extensions and Iwasawa’s theorem. Last time we understood better the structure of $\text{Cl}_{Q(\zeta_p)}$ as a $G_Q$-module. A natural question is: What about $\text{Cl}_{Q(\zeta_{p^n})}$?

**Theorem 2.1** (Iwasawa). There exist integers $n_0$ and $\mu$, $\lambda$, $\kappa$ such that for all $n > n_0$,

$$\# \text{Cl}_{Q(\zeta_{p^n})}[p^\infty] = p^{\mu p^n + \lambda n + \kappa}.$$ 

This statement \textit{a priori} does not have anything to do with the Galois structure of class groups, but the proof relies heavily on that.

Recall that $\text{Gal}(Q(\zeta_{p^n})/Q) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times$. Let $Q(\zeta_{p^n}) = \bigcup_{n \geq 0} Q(\zeta_{p^n})$. Then

$$\text{Gal}(Q(\zeta_{p^n})/Q) \cong \mathbb{Z}_p^\times, \quad \text{Gal}(Q(\zeta_{p^n})/Q(\zeta_p)) \cong \mathbb{Z}_p.$$

The element $1 - \zeta_{p^n} \in O_{Q(\zeta_{p^n})} = \mathbb{Z}[\zeta_{p^n}]$ generates the (totally ramified) prime above $p \in \mathbb{Z}$. With these facts we are ready to study basic Iwasawa theory.

**Definition 2.2.** Let $K_0$ be a number field. A $\mathbb{Z}_p$-extension of $K_0$ is an algebraic extension $K_\infty/K_0$ such that $\text{Gal}(K_\infty/K_0) \cong \mathbb{Z}_p$ topologically; equivalently, it is a tower of finite extensions

$$K_0 \subseteq K_1 \subseteq \cdots \subseteq K_\infty = \bigcup_{n \geq 0} K_n$$

such that $\text{Gal}(K_n/K_0) \cong \mathbb{Z}/p^n\mathbb{Z}$.

**Proposition 2.3.** Let $K_\infty/K_0$ be a $\mathbb{Z}_p$-extension. Then:

1. Some prime is ramified in $K_\infty$.
2. For any ramified prime $p \subseteq O_{K_0}$, there exists $n_0$ such that if $q \subseteq O_{K_{n_0}}$ is such that $q \mid p$, then $q$ is totally ramified in $K_\infty/K_{n_0}$.
3. Every ramified prime lies over $p \in \mathbb{Z}$.

**Proof.**

1. The maximal unramified abelian extension of $K_0$ is finite over $K_0$.
2. Let $p$ be ramified in $K_\infty/K_0$, and $I_p \subseteq \text{Gal}(K_\infty/K_0) \cong \mathbb{Z}_p$ be the inertia group at $p$. This is nontrivial and closed. Thus $I_p \cong p^{n_0}\mathbb{Z}_p$ for some $n_0$, so $I_p = \text{Gal}(K_\infty/K_0)$. 


(3) Let $p \subseteq \mathcal{O}_{K_0}$ be ramified in $K_\infty$. By local class field theory,
\[ \text{Gal}(K_{0,p}^{ab}/K_{0,p}) \cong \hat{K}_{0,p} \simeq \hat{\mathbb{Z}} \times \hat{\mathcal{O}}_{K_{0,p}} \]
corresponding to the unramified and ramified parts respectively, so by (2) we get an infinite subgroup of $\mathcal{O}_{K_{0,p}}^{\times}$ isomorphic to $\mathbb{Z}_p$, forcing $K_{0,p}$ to have residue characteristic $p$. □

**Proposition 2.4.** Let $L/K$ be an extension of number fields such that some prime in $K$ is totally ramified in $L$. Then the norm map
\[ \text{Nm} : \text{Cl}_L[p^\infty] \to \text{Cl}_K[p^\infty] \]
is surjective.

**Remark.** This is true without taking $p$-primary parts and essentially the same proof will work. It is also true if the ramified prime is at infinity.

**Proof.** Let $M_0/K$ be the maximal unramified abelian $p$-extension of $K$, and $M_1$ that for $L$. Then $M_0$ and $L$ are linearly disjoint over $K$ (i.e., $M_0 \cap L = K$ in any $\bar{K}$).

\[
\begin{array}{c}
M_1 \\
\downarrow \\
LM_0 \\
\downarrow \\
L \\
\downarrow \\
M_0 \\
\downarrow \\
K \\
\end{array}
\]

Since $LM_0/L$ is an unramified abelian $p$-extension, it is contained in $M_1$. Then by class field theory, the diagram
\[
\begin{array}{c}
\text{Cl}_L[p^\infty] \xrightarrow{\sim} \text{Gal}(M_1/L) \\
\downarrow \text{Nm} \\
\text{Cl}_K[p^\infty] \xrightarrow{\sim} \text{Gal}(M_0/K) \\
\end{array}
\]
commutes. □

This suggests that we should look at
\[ C := \lim_{\substack{n \geq 0}} \text{Cl}_{K_n}[p^\infty], \]
where the inverse limit is taken with respect to Nm. Write $\Gamma_n = \text{Gal}(K_n/K_0)$. Since $\text{Cl}_{K_n}[p^\infty]$ is a $\mathbb{Z}_p[\Gamma_n]$-module, $C$ is a module over
\[ \Lambda := \lim_{\substack{n \geq 0}} \mathbb{Z}_p[\Gamma_n], \]
where the inverse limit is taken with respect to restriction maps $\Gamma_m \to \Gamma_n$ if $m \geq n$.

**Theorem 2.5** (Iwasawa). $C$ is a finitely generated torsion $\Lambda$-module.
We will see how Theorem 2.1 follows from this.

2.2. Λ-modules. We need to first understand the structure of Λ. Let \( \gamma \in \Gamma := \text{Gal}(K_{\infty}/K_0) \subseteq \Lambda^\times \) be a topological generator. An observation due to Serre is that:

**Proposition 2.6** (Serre). There is an isomorphism

\[ \Lambda \simeq \mathbb{Z}_p[[T]] \]

of topological rings, induced by \( \gamma - 1 \mapsto T \).

**Proof.** Omitted; see Lang’s *Cyclotomic Fields I and II*, Chapter 5. \( \square \)

**Corollary 2.7** (Nakayama’s lemma). If \( M \) is a compact \( \Lambda \)-module and \( M/TM \equiv M/(\gamma - 1)M \) is finitely generated over \( \mathbb{Z}_p \), then any set of generators in \( M/TM \) lifts to a set of generators in \( M \). In particular, \( M \) is finitely generated.

**Definition 2.8.** A quasi-isomorphism of \( \Lambda \)-modules \( M,N \) is a map \( \varphi : M \to N \) such that there exists an exact sequence

\[ 0 \to A \to M \xrightarrow{\varphi} N \to B \to 0 \]

with \( A,B \) of finite cardinality.

Quasi-isomorphism is an equivalence relation.

**Theorem 2.9** (Weierstrass preparation). Let \( f \in \mathbb{Z}_p[[T]] \) be a nonconstant power series. Then there exist a unique integer \( n \geq 0 \) and unique \( g,u \in \mathbb{Z}_p[[T]] \) such that \( g \) is distinguished and \( u \in \mathbb{Z}_p[[T]]^\times \), and

\[ f = p^nug. \]

(A monic polynomial \( g = T^n + a_{n-1}T^{n-1} + \cdots + a_0 \) is distinguished if \( p \mid a_i \) for all \( i = 0, \ldots, n-1 \).)

**Theorem 2.10.** Let \( M \) be a finitely generated \( \Lambda \)-module. Then there exist integers \( r, n_i \) and distinguished \( f_j \) (where \( i \in I \) and \( j \in J \) with \( I \) and \( J \) finite) such that

\[ M \cong \Lambda^r \oplus \bigoplus_{i \in I} \Lambda/(p^{n_i}) \oplus \bigoplus_{j \in J} \Lambda/(f_j). \]

2.3. **Proof of Iwasawa’s theorem.** We will prove Theorem 2.1 and Theorem 2.5 in the case when there is only one ramified prime \( p \subseteq \mathcal{O}_{K_0} \) above \( p \).

**Proof of Theorem 2.5.** For simplicity, we further assume that \( p \) is totally ramified in \( K_{\infty}/K_0 \). Then we make the main

**Claim.** \( C/(\gamma - 1)C \simeq \text{Cl}_{K_0}. \)

Let \( M_n/K_n \) be the maximal unramified abelian \( p \)-extension, and \( M_{\infty} := \bigcup_{n \geq 0} M_n \), so \( M_{\infty}/K_{\infty} \) is the maximal unramified pro-\( p \) abelian extension. Let \( G_C := \text{Gal}(M_{\infty}/K_{\infty}) \cong C \), \( G := \text{Gal}(M_{\infty}/K_0) \) and \( \Gamma = \text{Gal}(K_{\infty}/K_0) \).

**Claim (Subclaim 1).** If \( I \subset G \) is the inertia at \( p \), then \( G \cong G_C \rtimes I \).
There is an exact sequence

\[ 1 \xrightarrow{=} \text{Gal}(M_\infty/K_\infty) \xrightarrow{=} \text{Gal}(M_\infty/K_0) \xrightarrow{=} \text{Gal}(K_\infty/K_0) \xrightarrow{=} 1. \]

Since \( p \) is totally ramified in \( K_\infty \), \( I \) surjects onto \( \Gamma \) and has empty intersection with \( G_C \). This proves \( \Gamma \cong I \) and subclaim 1.

Let \( \Gamma \cong I \) act on \( G_C \) by conjugation. Then this action is the same as that on \( C \): \( \text{Frob}_{\sigma q} = \sigma \text{Frob}_q \sigma^{-1} \).

**Claim (Subclaim 2).** Let \( G' \) be the commutator subgroup of \( G \). Then \( G' = (\gamma - 1)G \).

Let \( g \in G \). Then \( (\gamma - 1)g = \tilde{\gamma}g\tilde{\gamma}^{-1}g^{-1} \in G' \), where \( \tilde{\gamma} \) is a lift of \( \gamma \) to \( I \). Conversely, \( G/(\gamma - 1)G \) is the largest quotient where \( \Gamma \) acts trivially. So conjugation by elements in \( I \) is trivial, and since \( G_C \) is abelian and \( G = G_C \rtimes I \), we have that \( G/(\gamma - 1)G \) is abelian and hence \( (\gamma - 1)G \subseteq G' \). This proves subclaim 2.

To prove the main claim that \( C/(\gamma - 1)C \cong \text{Cl}_{K_0}[p^\infty] \), it is the same to prove

\[ G_C/(\gamma - 1)G_C \cong \text{Gal}(M_0/K_0). \]

Note \( (\gamma - 1)G_C = (\gamma - 1)G \) (because if \( g \in G_C \), \( t \in I \) and \( \gamma \in \Gamma \), then picking a lift \( \tilde{\gamma} \in I \), we have \( (\gamma - 1)gt = \tilde{\gamma}gt\tilde{\gamma}^{-1}t^{-1}g^{-1} = \tilde{\gamma}gt^{-1}\tilde{\gamma}^{-1}g^{-1} = (\gamma - 1)g \); the reverse inclusion is trivial). Then

\[
G_C/(\gamma - 1)G_C = G_C/(\gamma - 1)G \\
\cong G/I \cdot (\gamma - 1)G \\
= G/I \cdot G' \\
= \text{Gal}(M_0/K_0).
\]

This proves the main claim.

Since \( \text{Cl}_{K_0}[p^\infty] = C/(\gamma - 1)C \) is finite, by Nakayama’s lemma \( C \) is finitely generated. Write

\[ C \cong \bigoplus \Lambda^r \oplus \bigoplus \Lambda/(p^{n_i}) \oplus \bigoplus \Lambda/(f_j). \]

Then since \( \Lambda^r/(\gamma - 1)\Lambda^r \cong \mathbb{Z}_p^r \), we have \( r = 0 \). This proves Theorem 2.5. \( \square \)

**Remark.** Since \( C/(\gamma p^n - 1) \) is finite, we see that \( f_j \) is coprime to \( (1 + T)p^n - 1 \) for all \( j, n \).

Finally, let us prove Theorem 2.1. There exist \( n_0 \) and \( \mu, \lambda, \kappa \) such that for all \( n > n_0 \),

\[ \#\text{Cl}_{K_0}[p^\infty] = p^{\mu p^n + \lambda n + \kappa}. \]

**Exercise.**

1. \( \#\text{Cl}/(p^n, \gamma p^n - 1) = p^{mp^n} \).
2. Let \( f \) be a distinguished polynomial coprime to \( (1 + T)p^n - 1 \) for all \( n \). Then there exists \( n_0 \) such that if \( n > n_0 \), then \( \#\text{Cl}/(f, \gamma p^n - 1) = p^{\deg(f)a + c} \) for some \( c = c(f) \).

**Proof of Theorem 2.7.** Replacing \( K_0 \) by \( K_n \) where \( n \gg 0 \), we may assume the ramified prime is totally ramified. Then setting \( \mu = \sum i_i n_i \) and \( \lambda = \sum j_i \deg f_j \), we get (by the previous exercise and remark) \( \kappa \) such that

\[ \#\text{Cl}_{K_0}[p^\infty] = \#C/(\gamma p^n - 1)C = p^{\mu p^n + \lambda n + \kappa} \]
for sufficiently large \( n \).

\[ \square \]

Remark. For the cyclotomic extension \( \mathbb{Q}(\zeta_{p^{\infty}}) \), it is known that \( \mu = 0 \); this is the Ferrero–Washington theorem.

3. Lecture 3 (September 21, 2017): Samuel Mundy

\( p \)-adic Properties of \( L \)-functions

3.1. Class number formulas. Fix an odd prime \( p \) and embeddings \( \mathbb{Q} \hookrightarrow \mathbb{Q}_p \hookrightarrow \mathbb{C} \). Denote \( K = \mathbb{Q}(\zeta_{p^n}) \) throughout.

Definition 3.1.

1. A character \( \psi : \text{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) \to \mathbb{C}^\times \) is odd if \( \psi(c) = -1 \) for the complex conjugation \( c \), and even otherwise (so \( \psi(c) = 1 \)).
2. Given \( \psi \), \( L(s, \psi) \) is the complex analytic function continuing \( \sum_{n=1}^{\infty} \frac{\psi'(n)}{n^s} \), where \( \psi' \) is the primitive character associated with \( \psi \). (In particular, \( L(s, 1) = \zeta(s) \) and not \( \frac{1}{1-p^{-s}} \zeta(s) \).)
3. Let \( K^+ = \mathbb{Q}(\zeta_{p^n})^+ := \mathbb{Q}(\zeta_{p^n} + \zeta_{p^n}^{-1}) \), so \([K : K^+] = 2\). Let \( h = \#\text{Cl}_K \), \( h^+ = \#\text{Cl}_{K^+} \) and \( h^- = h/h^+ \).

Remark. \( \text{Nm} : \text{Cl}_K \to \text{Cl}_{K^+} \) is surjective, since \( K/K^+ \) is totally ramified at \( \infty \). This implies \( h^- \in \mathbb{Z} \).

We collect the following standard facts.

Theorem 3.2.

1. Dirichlet’s class number formula: Let \( \zeta_L(s) \) be the Dedekind zeta function for a number field \( L \). Then

\[ \text{Res}_{s=1} \zeta_L(s) = \frac{2^{r_1}(2\pi)^{r_2} \text{Reg}_L h_L}{\#\mu_L \sqrt{|\Delta_L|}}, \]

where the regulator is \( \text{Reg}_L = \det \log |\sigma_i\alpha_j| \) for a \( \mathbb{Z} \)-basis \( \{\alpha_j\} \) of \( \mathcal{O}_L^\times \)/torsion. In particular, we have

\[ \text{Res}_{s=1} \zeta_K(s) = \frac{(2\pi)^{N/2} \text{Reg}_K h}{2p^n \sqrt{|\Delta_K|}} \quad \text{and} \quad \text{Res}_{s=1} \zeta_{K^+}(s) = \frac{2^{N/2} \text{Reg}_{K^+} h^+}{2\sqrt{|\Delta_{K^+}|}}, \]

where \( N = [K : \mathbb{Q}] = (p - 1)p^{n-1} \).
2. Let \( \psi : \text{Gal}(K/\mathbb{Q}) \to \mathbb{C}^\times \) have conductor \( m(\psi) \), and write

\[ \Lambda(s, \psi) := \begin{cases} m(\psi)^{s/2} \pi^{-s/2} \Gamma(\frac{s}{2}) L(s, \psi) & \text{if } \psi \text{ is even,} \\ m(\psi)^{s/2} \pi^{-s/2} \Gamma(\frac{s+1}{2}) L(s, \psi) & \text{if } \psi \text{ is odd.} \end{cases} \]

Then

\[ \Lambda(s, \psi) = \frac{\sqrt{\psi(-1)m(\psi)}}{S(\psi)} \Lambda(1 - s, \psi), \]

where \( S(\psi) \) is the Gauss sum.
3. \( \prod_{\psi \neq 1} S(\psi) = i^{N/2} \sqrt{|\Delta_K|} \) and \( \prod_{\psi \neq 1, \psi \text{ even}} S(\psi) = \sqrt{|\Delta_{K^+}|} \).
4. \( \mathcal{O}_K^\times = \mu_{p^n}\mathcal{O}_{K^+}^\times \).
(5) Let \( B_{n,\psi} \) be the generalized Bernoulli numbers, defined by
\[
\sum_{a=0}^{m(\psi)-1} \psi(a)Te^{aT} - 1 = \sum_{n=0}^{\infty} B_{n,\psi} \frac{T^n}{n!}.
\]

Then for every integer \( n \geq 1 \),
\[
L(1 - n, \psi) = -\frac{B_{n,\psi}}{n}.
\]

We have the following formula for \( h^- \).

**Theorem 3.3.**
\[
h^- = 2p^n \prod_{\psi \text{ odd}} \frac{1}{2} L(0, \psi) \left( \equiv 2p^n \prod_{\psi \text{ odd}} -\frac{1}{2} B_{1,\psi} \right).\]

**Proof.** Note that
\[
\text{Res}_{s=1} \zeta_K(s) = \prod_{\psi \neq \psi} L(1, \psi) \quad \text{and} \quad \text{Res}_{s=1} \zeta_{K^+}(s) = \prod_{\psi \neq 1} L(1, \psi).
\]

Then
\[
h^- = \frac{h}{h^+} \frac{(1) \frac{2^{N/2} \text{Reg}_{K^+}/2\sqrt{|\Delta_K^+|}}{2\pi N/2 \text{Reg}_K/2p^n \sqrt{|\Delta_K|}} \times \prod_{\psi \text{ odd}} L(1, \psi)}{(2) \frac{\text{Reg}_{K^+}}{\text{Reg}_K} \frac{p^n \sqrt{|\Delta_K|}}{\pi N/2 \sqrt{|\Delta_K^+|}} \prod_{\psi \text{ odd}} L(0, \psi) \frac{(1/2) \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(1)} \frac{\text{S}(\psi) \text{m}(\psi)^{1/2} \pi^{-1/2} \Gamma(1)}{\pi^{-1/2} \Gamma(1)}}
\]
\[
= \frac{3) \frac{1}{2} 2^{N/2-1} p^n \pi^{-N/2} \sqrt{|\Delta_K|} \sqrt{|\Delta_K^+|} \left( \prod_{\psi \text{ odd}} L(0, \psi) \right) \left( \prod_{\psi \text{ odd}} \text{S}(\psi)^{-1} \right) \pi^{N/2} \pi^{N/2}}{\text{Res}_{s=1} \zeta_{K^+}(s) \equiv \prod_{\psi \text{ odd}} \frac{1}{2} L(0, \psi).}
\]

**Remark.** The \( p \)-part of \( h^- \) is \( \prod_{\psi \text{ odd}} \#\text{Cl}_K[p^\infty]^\psi \) (left as an exercise). Recall that \( \text{Cl}_{Q(p)}[p^\infty]^\omega = 0 \) and \( v_p(pB_{1,\omega}) = 0 \). Thus the theorem gives
\[
\prod_{\psi \neq \omega} \#\text{Cl}_{Q(p)}[p^\infty]^\psi = \prod_{\psi \neq \omega} \#\mathbb{Z}_p/(L(0, \psi)).
\]

In fact this is true without taking products, as we will see in Corollary 3.8. This strengthens Herbrand–Ribet, because \( v_p(B_{p-m}) = v_p(B_{1,\omega}) \).

3.2. \( p \)-adic \( L \)-functions. Fix \( \gamma \in \Gamma \subseteq \Lambda^\times \) a topological generator of \( \Gamma \). Let \( \chi_{\text{cyc}} \) be the cyclotomic character. Let \( \zeta \) be a \( p^n \)-th root of 1 (not necessarily primitive). For \( k \in \mathbb{Z} \), let \( \phi_{k,\zeta} : \Lambda \to \mathbb{Z}_p[\zeta] \) send \( \gamma \) to \( \chi_{\text{cyc}}^k(\gamma) \zeta \).
Theorem 3.4. Let $\psi : \text{Gal}(Q(\zeta_p)/Q) \to C^\times$ be an odd character, and
\[
h_\psi = \begin{cases} 
X_{\text{cyc}}(\gamma)\gamma - 1 & \text{if } \psi = \omega^{-1}, \\
1 & \text{if } \psi \neq \omega^{-1}.
\end{cases}
\]
Then there exists $g_\psi \in \Lambda$ such that
\[
\frac{\phi_{k,\zeta}(g_\psi)}{\phi_{k,\zeta}(h_\psi)} = L(-k, \psi\zeta\omega^{-k}),
\]
where $\psi : \text{Gal}(Q(\zeta_p^m)/Q) \to C^\times$ sends the image of $\gamma$ to $\zeta$.

Definition 3.5. $g_\psi$ is called a $p$-adic $L$-function.

The construction of $g_\psi$ uses Bernoulli measures, whose existence implies the Kummer congruences.

3.3. Main Conjecture.

Definition 3.6. For a finitely generated torsion $\Lambda$-module $M$, write
\[
M \overset{q\text{-iso}}{\cong} \bigoplus_i \Lambda/(p^{n_i}) \oplus \bigoplus_j \Lambda/(f_j)
\]
as in the classification theorem. Then the characteristic ideal of $M$ is
\[
\text{Char}(M) := \left( \prod_i p^{n_i} \cdot \prod_j f_j \right) \Lambda.
\]

Recall that last time we constructed a finitely generated torsion $\Lambda$-module
\[
C := \lim_{\rightarrow} \text{Cl}_{Q(\zeta_{p^n})}[p^\infty].
\]
Let $C^\psi$ be the $\psi$-eigenspace of $C$, where $\psi : \text{Gal}(Q(\zeta_p)/Q) \to C^\times$ is an odd character.

Remark. We have $C^\psi = \lim_{\leftarrow} \left( \text{Cl}_{Q(\zeta_{p^n})}[p^\infty]^\psi \right)$.

The goal of the first part of our seminar is to prove the

Theorem 3.7 (Main Conjecture). $\text{Char}(C^\psi) = (g_\psi)$.

Corollary 3.8. Suppose $\psi \neq \omega$. Then
\[
\#\text{Cl}_{Q(\zeta_p)}[p^\infty]^\psi = \#Z_p/(L(0, \psi)).
\]

To prove this we need the

Proposition 3.9. $C$ has no nonzero finite cardinality $\Lambda$-submodules.

Proof. See Washington P.287–289. \qed

Exercise. If $M$ is a finitely generated torsion $\Lambda$-module with no nonzero finite cardinality $\Lambda$-submodules, and if $M/(\gamma^{p^n} - 1)M$ is finite for all $n$, then
\[
\#M/(\gamma^{p^n} - 1)M = \#\Lambda/(\text{Char}(M), \gamma^{p^n} - 1).
\]
Proof of Corollary 3.8

\[ \#\text{Cl}_{Q(\zeta_p)}[p^{\infty}]^\psi = \#C^\psi / (\gamma - 1)C^\psi \]
\[ = \#\Lambda / (\gamma - 1, \text{Char}(C^\psi)) \quad \text{(by Exercise)} \]
\[ = \#\Lambda / (\gamma - 1, g_\psi) \quad \text{(by MC)} \]
\[ = \#\mathbb{Z}_p / (\phi_0, 1(g_\psi)) \]
\[ = \#\mathbb{Z}_p / (L(0, \psi)). \]