# HIDA THEORY

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Abstract. These are notes from the (ongoing) Student Number Theory Seminar on Hida theory at Columbia University in Fall 2017, which is organized by David Hansen and Samuel Mundy.

Contents



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## 1. Lecture 1 (September 7, 2017): Samuel Mundy
## Ribet's Converse to Herbrand's Theorem

1.1. **Introduction.** Let $p$ be an odd prime, and $\omega : G_{\mathbf{Q}} \to \mu_{p-1} \subseteq \mathbf{Z}_p^\times$ be the Teichmüller character (so that $\sigma\zeta_p = \zeta_p^{\omega(\sigma)}$ for all $\sigma \in G_{\mathbf{Q}}$).

For any $\mathbf{Z}_p[G_{\mathbf{Q}}]$-module $M$ and $\varphi \in \mathrm{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})^\wedge$, write $M^\varphi = \{m \in M \mid \sigma m = \varphi(\sigma)m\}$ to be the $\varphi$-eigenspace of $M$.

Let $B_n$ be the $n$-th Bernoulli number.

Today we are going to prove Ribet's converse to Herbrand's theorem:

**Theorem 1.1** (Ribet)**.** *Let $m$ be an odd integer with $3 \leq m \leq p-2$. If $p \mid B_{p-m}$, then $\mathrm{Cl}_{\mathbf{Q}(\zeta_p)}[p^\infty]^{\omega^m} \neq 0$.*

*Remark.* We have $v_p(B_{p-1}) = -1$ and $\mathrm{Cl}_{\mathbf{Q}(\zeta_p)}[p^\infty]^\omega = 0$.

The idea is to deduce properties of Galois representations attached to automorphic representations of $\mathrm{GL}_1$ using those for $\mathrm{GL}_2$.

1.2. **Modular forms.** Fix embeddings $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}_p} \hookrightarrow \mathbf{C}$. For $N \geq 1$ an integer, $\psi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ a Dirichlet character, and $A \subseteq \mathbf{C}$ a subring containing $\mathbf{Z}[\psi]$, write

$$S_k(N, \psi, A) = \left\{ \text{cusp forms } f = \sum_{n=1}^{\infty} a_n(f) q^n \text{ of weight } k, \text{ level } N, \text{ nebentypus } \psi \mid a_n(f) \in A \right\}$$

where $q = e^{2\pi i z}$. Denote by $\mathbb{T}_k(N, \psi, A)$ the sub-$A$-algebra of $\mathrm{End}_A(S_k(N, \psi, A))$ generated by the Hecke operators $T[n]$ for all $n$.

We define a pairing

$$\begin{aligned} S_k(N, \psi, A) \times \mathbb{T}_k(N, \psi, A) &\to A \\ \langle f, h \rangle &\mapsto a_1(f|h). \end{aligned}$$

In particular, $\langle f, T(n) \rangle = a_n(f)$.

**Theorem 1.2.** *This pairing is perfect.*

**Theorem 1.3.** *Let $B$ be an $A$-algebra with $\phi : A \to B$. Under the isomorphism*

$$\mathrm{Hom}_A(\mathbb{T}_k(N, \psi, A), B) \simeq S_k(N, \psi, A) \otimes_A B,$$

*the $A$-algebra homomorphisms $\mathrm{Hom}_{A\text{-alg}}(\mathbb{T}_k(N, \psi, A), B)$ correspond to the normalized eigenforms, i.e., $f = \sum a_n(f) q^n \in S_k(N, \psi, A) \otimes_A B \subset B[[q]]$ with $a_1(f) = 1$ such that for all $h \in \mathbb{T}_k(N, \psi, A)$, there exists $c \in B$ such that $\phi(f|h) = c\phi(f)$, where the corresponding homomorphism on the LHS sends an operator to its eigenvalue.*

**Theorem 1.4.** *Let $f = \sum_{n=1}^{\infty} a_n(f) q^n \in S_k(N, \psi, A)$ be a normalized eigenform. Then there exists a continuous Galois representation $\rho_f : G_{\mathbf{Q}} \to \mathrm{Aut}(V)$ with $\dim_{\overline{\mathbf{Q}_p}}(V) = 2$ such that*

1. *$\rho_f$ is irreducible;*
2. *for all $\ell \nmid pN$, $\rho_f$ is unramified at $\ell$;*
3. *for all $\ell \nmid pN$, $\mathrm{tr}\, \rho_f(\mathrm{Frob}_\ell) = a_\ell(f)$;*
4. *for all $\ell \nmid pN$, $\det \rho_f(\mathrm{Frob}_\ell) = \psi(\ell)\ell^{k-1}$.*

**Definition 1.5.** A normalized eigenform $f \in S_k(1, \overline{\mathbf{Q}_p})$ is *ordinary* if $|a_p(f)|_p = 1$.

**Theorem 1.6** (Mazur–Wiles)**.** *If $f$ is ordinary, then there exists a basis $v_1, v_2$ of $V$ such that*

$$\rho_f|_{G_{\mathbf{Q}_p}} = \begin{pmatrix} \alpha^{-1}\chi^{k-1} & * \\ & \alpha \end{pmatrix},$$

*where $\chi : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ is the cyclotomic character (i.e., $\sigma\zeta = \zeta^{\chi(\sigma)}$ for all $\zeta \in \mu_{p^\infty}$), and $\alpha : G_{\mathbf{Q}_p} \to \overline{\mathbf{Q}_p}^\times$ is the unramified character such that $\alpha(\mathrm{Frob}_p)$ is the unit root of $X^2 - a_p(f)X + p^{k-1}$.*

1.3. **Proof of Ribet's theorem.** The proof consists of the following steps.

**Step 0**: Let $\kappa$ by any finite field of characteristic $p$. Then

$$\begin{aligned} \mathrm{Cl}_{\mathbf{Q}(\zeta_p)}[p]^{\omega^m} \otimes_{\mathbf{F}_p} \kappa &\cong \mathrm{Hom}_{G_{\mathbf{Q}}}(G^{\mathrm{ab,unr}}_{\mathbf{Q}(\zeta_p)}, \kappa(\omega^m)) \\ &\cong H^1_{\mathrm{unr}}(G_{\mathbf{Q}(\zeta_p)}, \kappa(\omega^m))^{G_{\mathbf{Q}}} \\ &\overset{\mathrm{Res}^{-1}}{\cong} H^1_{\mathrm{unr}}(G_{\mathbf{Q}}, \kappa(\omega^m)) \\ &\cong \mathrm{Ext}^{\mathrm{unr}}_{\kappa[G_{\mathbf{Q}}]}(\kappa, \kappa(\omega^m)) \\ &\cong \mathrm{Ext}^{\mathrm{unr}}_{\kappa[G_{\mathbf{Q}}]}(\kappa(\omega^{p-1-m}), \kappa). \end{aligned}$$

**Step 1** (Construction of an eigenform): Let

$$E_k(z) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \left( \sum_{d|n} d^{k-1} \right) q^n$$

be the Eisenstein series of weight $k$, where $k \geq 4$ is even. The following facts are classical:

(1) $\zeta(1-k) = -B_k/k$.
(2) $v_p(B_k) < 0 \Leftrightarrow p-1 \mid k \Leftrightarrow v_p(B_k) = -1$.
(3) $E_k \in M_k(1, \mathbf{C})$. If $p-1 \nmid k$, then $E_k \in M_k(1, \mathbf{Z}_p)$ by (2).
(4) If $m \equiv n \pmod{p-1}$, then $B_m \equiv B_n \pmod{p}$.

Note that

$$\begin{aligned} E_4(z) &= \frac{1}{240} + q + \cdots, \\ E_6(z) &= \frac{-1}{504} + q + \cdots. \end{aligned}$$

Let $k \geq 4$ and $k \equiv p - m \pmod{p-1}$. Write $k = 4a + 6b$ where $a, b \geq 0$. Then define

$$G_k = (240E_4)^a(-504E_6)^b = 1 + \sum_{n=1}^{\infty} a_n q^n$$

where $a_n \in \mathbf{Z}$. Let

$$F_k = E_k - \frac{\zeta(1-k)}{2} G_k = \sum_{n=1}^{\infty} b_n q^n$$

where $b_n = \frac{B_k}{2k} a_n + \sum_{d|n} d^{k-1}$.

Now assume $p \mid B_{p-m}$. Then by (4), $p \mid B_k$ and so $b_n \equiv \sum_{d|n} d^{k-1} \pmod{p}$. Thus the map

$$\mathbb{T}_k(1, \mathbf{Z}_p) \to \mathbf{F}_p$$

$$T(n) \mapsto \sum_{d|n} d^{k-1} \pmod{p}$$

is a $\mathbf{Z}_p$-algebra homomorphism, corresponding to $F_k \pmod{p}$.

Let $\mathfrak{m} \subseteq \mathbb{T}_k(1, \mathbf{Z}_p)$ be the kernel, and $\mathfrak{p} \subseteq \mathfrak{m}$ be a minimal prime. Let $f$ be the eigenform corresponding to the homomorphism

$$\mathbb{T}_k(1, \mathbf{Z}_p) \to A := \mathbb{T}_k(1, \mathbf{Z}_p)/\mathfrak{p} \hookrightarrow \mathcal{O}_L,$$

where $L := \operatorname{Frac} A$. Then $f \in S_k(1, \mathcal{O}_L)$ and $f \equiv F_k \pmod{\lambda}$, where $\lambda \in \mathcal{O}_L$ is a uniformizer.

**Step 2** (Analysis of $\rho_f$): Write $f = \sum_{n=1}^{\infty} c_n q^n$. Then

$$c_p \equiv \sum_{d|p} d^{k-1} \equiv 1 + p^{k-1} \equiv 1 \pmod{\lambda},$$

so $f$ is ordinary. Let $v_1, v_2 \in V_{\rho_f}$ such that

$$\rho_f|_{G_{\mathbf{Q}_p}} = \begin{pmatrix} \alpha^{-1}\chi^{k-1} & * \\ & \alpha \end{pmatrix}.$$

Let $\sigma_0 \in G_{\mathbf{Q}_p}$ be an element such that $\chi^{k-1}(\sigma_0) \not\equiv 1 \pmod{\lambda}$ (possible since $k-1$ is odd, so $p - 1 \nmid k - 1$). Write $\beta := \chi^{k-1}(\sigma_0)$. Replacing $v_1$ by $\alpha(\sigma_0)v_1$ and $v_2$ by $\alpha^{-1}(\sigma_0)v_2 + (\text{something}) \cdot v_1$, we may assume

$$\rho_f(\sigma_0) = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}.$$

For $\sigma \in \mathcal{O}_L[G_{\mathbf{Q}}]$, let $\rho_f(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$.

**Lemma 1.7.**

(1) *For all $\sigma, \tau \in \mathcal{O}_L[G_{\mathbf{Q}}]$, we have $a_\sigma, d_\sigma, b_\sigma c_\tau \in \mathcal{O}_L$ and*

$$a_\sigma \equiv \omega^{k-1}(\sigma), \quad d_\sigma \equiv \mathbb{1}(\sigma), \quad b_\sigma c_\tau \equiv 0 \pmod{\lambda},$$

*where $\mathbb{1}$ is the trivial $G_{\mathbf{Q}}$-character, and $\omega^{k-1}, \mathbb{1}$ are extended by linearity to $\mathcal{O}_L[G_{\mathbf{Q}}]$.*

(2) *$C = \{c_\sigma \mid \sigma \in \mathcal{O}_L[G_{\mathbf{Q}}]\}$ is a nonzero fractional ideal.*

(3) *$c_\sigma = 0$ for all $\sigma \in I_\ell$ and primes $\ell$.*

*Proof.* Omitted; play with matrix coefficients using information about the trace. □

**Step 3** (Construction of the lattice): Let $M_1 = \mathcal{O}_L v_1$, $M_2 = Cv_2$ and $M = M_1 \oplus M_2 \subseteq V_{\rho_f}$. This is $G_{\mathbf{Q}}$-stable, generated over $\mathcal{O}_L[G_{\mathbf{Q}}]$ by $v_1$.

Let $\kappa = \mathcal{O}_L/\lambda$ and $\overline{M} = M/\lambda M$.

*Claim.* $\overline{M_2} := M_2/\lambda M_2 \subseteq \overline{M}$ is a $G_{\mathbf{Q}}$-stable line with trivial action.

Let $m_2 \in M_2$, so $m_2 = c_\tau v_2$ for some $c_\tau \in C$. If $\sigma \in G_{\mathbf{Q}}$, then

$$\rho_f(\sigma)m_2 = c_\tau b_\sigma v_1 + c_\tau d_\sigma v_2 \equiv d_\sigma m_2 = \mathbb{1}(\sigma)m_2 = m_2 \pmod{\lambda}$$

which proves the claim.

Let $\overline{M_1} = M_1/\lambda M_1$. Then $\overline{M_1} \cong \kappa(\omega^{k-1})$ by a similar argument, and there is a short exact sequence

$$0 \to \kappa \to \overline{M} \to \kappa(\omega^{k-1}) \to 0$$

which is:

- nonsplit over $G_{\mathbf{Q}}$ because, if it were, $\overline{M_2} \cong \kappa$ would be a quotient and hence the image of $v_1$ would belong to the kernel by this quotient. This is because $\sigma_0$ acts as $\beta \not\equiv 1 \pmod{\lambda}$ on $v_1$, but $v_1$ generates $\overline{M}$ over $G_{\mathbf{Q}}$. Hence the kernel is all of $\overline{M}$, a contradiction.
- split over $I_\ell$ for all $\ell$ because $\mathcal{O}_L v_1 \pmod{\lambda}$ is a stable line under $I_\ell$, since

$$\rho(\sigma)v_1 = a_\sigma v_1 + c_\sigma v_2 \equiv a_\sigma v_1 \pmod{\lambda}.$$

Since $\omega^{k-1} = \omega^{p-1-m}$, this extension gives a nontrivial class in $\mathrm{Ext}^{\mathrm{unr}}_{\kappa[G_{\mathbf{Q}}]}(\kappa(\omega^{p-1-m}), \kappa)$. This finishes the proof of Theorem 1.1.

## 2. Lecture 2 (September 14, 2017): Samuel Mundy
## Basic Iwasawa Theory

2.1. **$\mathbf{Z}_p$-extensions and Iwasawa's theorem.** Last time we understood better the structure of $\mathrm{Cl}_{\mathbf{Q}(\zeta_p)}$ as a $G_{\mathbf{Q}}$-module. A natural question is: What about $\mathrm{Cl}_{\mathbf{Q}(\zeta_{p^n})}$?

**Theorem 2.1** (Iwasawa)**.** *There exist integers $n_0$ and $\mu$, $\lambda$, $\kappa$ such that for all $n > n_0$,*

$$\#\mathrm{Cl}_{\mathbf{Q}(\zeta_{p^n})}[p^\infty] = p^{\mu p^n + \lambda n + \kappa}.$$

This statement *a priori* does not have anything to do with the Galois structure of class groups, but the proof relies heavily on that.

Recall that $\mathrm{Gal}(\mathbf{Q}(\zeta_{p^n})/\mathbf{Q}) \simeq (\mathbf{Z}/p^n\mathbf{Z})^\times$. Let $\mathbf{Q}(\zeta_{p^\infty}) = \bigcup_{n\geq 0} \mathbf{Q}(\zeta_{p^n})$. Then

$$\mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q}) \simeq \mathbf{Z}_p^\times,$$
$$\mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q}(\zeta_p)) \simeq \mathbf{Z}_p.$$

The element $1 - \zeta_{p^n} \in \mathcal{O}_{\mathbf{Q}(\zeta_{p^n})} = \mathbf{Z}[\zeta_{p^n}]$ generates the (totally ramified) prime above $p \in \mathbf{Z}$. With these facts we are ready to study basic Iwasawa theory.

**Definition 2.2.** Let $K_0$ be a number field. A $\mathbf{Z}_p$*-extension* of $K_0$ is an algebraic extension $K_\infty/K_0$ such that $\mathrm{Gal}(K_\infty/K_0) \simeq \mathbf{Z}_p$ topologically; equivalently, it is a tower of finite extensions

$$K_0 \subseteq K_1 \subseteq \cdots \subseteq K_\infty = \bigcup_{n\geq 0} K_n$$

such that $\mathrm{Gal}(K_n/K_0) \simeq \mathbf{Z}/p^n\mathbf{Z}$.

**Proposition 2.3.** *Let $K_\infty/K_0$ be a $\mathbf{Z}_p$-extension. Then:*

1. *Some prime is ramified in $K_\infty$.*
2. *For any ramified prime $\mathfrak{p} \subseteq \mathcal{O}_{K_0}$, there exists $n_0$ such that if $\mathfrak{q} \subseteq \mathcal{O}_{K_{n_0}}$ is such that $\mathfrak{q} \mid \mathfrak{p}$, then $\mathfrak{q}$ is totally ramified in $K_\infty/K_{n_0}$.*
3. *Every ramified prime lies over $p \in \mathbf{Z}$.*

*Proof.*

1. The maximal unramified abelian extension of $K_0$ is finite over $K_0$.
2. Let $\mathfrak{p}$ be ramified in $K_\infty/K_0$, and $I_\mathfrak{p} \subseteq \mathrm{Gal}(K_\infty/K_0) \simeq \mathbf{Z}_p$ be the inertia group at $\mathfrak{p}$. This is nontrivial and closed. Thus $I_\mathfrak{p} \simeq p^{n_0}\mathbf{Z}_p$ for some $n_0$, so $I_\mathfrak{p} = \mathrm{Gal}(K_\infty/K_0)$.

(3) Let $\mathfrak{p} \subseteq \mathcal{O}_{K_0}$ be ramified in $K_\infty$. By local class field theory,

$$\mathrm{Gal}(K_{0,\mathfrak{p}}^{\mathrm{ab}}/K_{0,\mathfrak{p}}) \cong \widehat{K_{0,\mathfrak{p}}^\times} \simeq \pi^{\hat{\mathbf{Z}}} \times \mathcal{O}_{K_{0,\mathfrak{p}}}^\times$$

corresponding to the unramified and ramified parts respectively, so by (2) we get an infinite subgroup of $\mathcal{O}_{K_{0,\mathfrak{p}}}^\times$ isomorphic to $\mathbf{Z}_p$, forcing $K_{0,\mathfrak{p}}$ to have residue characteristic $p$. □

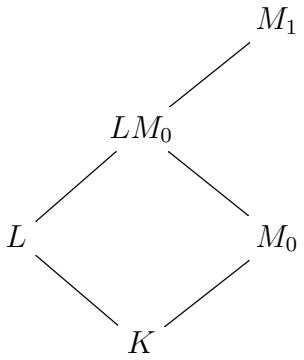
**Proposition 2.4.** *Let $L/K$ be an extension of number fields such that some prime in $K$ is totally ramified in $L$. Then the norm map*

$$\mathrm{Nm} : \mathrm{Cl}_L[p^\infty] \to \mathrm{Cl}_K[p^\infty]$$

*is surjective.*

*Remark.* This is true without taking $p$-primary parts and essentially the same proof will work. It is also true if the ramified prime is at infinity.

*Proof.* Let $M_0/K$ be the maximal unramified abelian $p$-extension of $K$, and $M_1$ that for $L$. Then $M_0$ and $L$ are linearly disjoint over $K$ (i.e., $M_0 \cap L = K$ in any $\overline{K}$).




Since $LM_0/L$ is an unramified abelian $p$-extension, it is contained in $M_1$. Then by class field theory, the diagram

$$\begin{array}{ccc} \mathrm{Cl}_L[p^\infty] & \xrightarrow[\mathrm{Art}]{\sim} & \mathrm{Gal}(M_1/L) \\ {\scriptstyle \mathrm{Nm}}\downarrow & & \downarrow{\scriptstyle \text{restriction to } M_0} \\ \mathrm{Cl}_K[p^\infty] & \xrightarrow[\mathrm{Art}]{\sim} & \mathrm{Gal}(M_0/K) \end{array}$$

commutes. □

This suggests that we should look at

$$C := \varprojlim_{n \geq 0} \mathrm{Cl}_{K_n}[p^\infty],$$

where the inverse limit is taken with respect to Nm. Write $\Gamma_n = \mathrm{Gal}(K_n/K_0)$. Since $\mathrm{Cl}_{K_n}[p^\infty]$ is a $\mathbf{Z}_p[\Gamma_n]$-module, $C$ is a module over

$$\Lambda := \varprojlim_{n \geq 0} \mathbf{Z}_p[\Gamma_n],$$

where the inverse limit is taken with respect to restriction maps $\Gamma_m \twoheadrightarrow \Gamma_n$ if $m \geq n$.

**Theorem 2.5** (Iwasawa)**.** *$C$ is a finitely generated torsion $\Lambda$-module.*

We will see how Theorem 2.1 follows from this.

2.2. **$\Lambda$-modules.** We need to first understand the structure of $\Lambda$. Let $\gamma \in \Gamma := \mathrm{Gal}(K_\infty/K_0) \subseteq \Lambda^\times$ be a topological generator. An observation due to Serre is that:

**Proposition 2.6** (Serre)**.** *There is an isomorphism*

$$\Lambda \simeq \mathbf{Z}_p[[T]]$$

*of topological rings, induced by $\gamma - 1 \mapsto T$.*

*Proof.* Omitted; see Lang's *Cyclotomic Fields I and II*, Chapter 5. □

**Corollary 2.7** (Nakayama's lemma)**.** *If $M$ is a compact $\Lambda$-module and $M/TM = M/(\gamma - 1)M$ is finitely generated over $\mathbf{Z}_p$, then any set of generators in $M/TM$ lifts to a set of generators in $M$. In particular, $M$ is finitely generated.*

**Definition 2.8.** A *quasi-isomorphism* of $\Lambda$-modules $M, N$ is a map $\varphi : M \to N$ such that there exists an exact sequence

$$0 \to A \to M \xrightarrow{\varphi} N \to B \to 0$$

with $A, B$ of finite cardinality.

Quasi-isomorphism is an equivalence relation.

**Theorem 2.9** (Weierstrass preparation)**.** *Let $f \in \mathbf{Z}_p[[T]]$ be a nonconstant power series. Then there exist a unique integer $n \geq 0$ and unique $g, u \in \mathbf{Z}_p[[T]]$ such that $g$ is* distinguished *and $u \in \mathbf{Z}_p[[T]]^\times$, and*

$$f = p^n u g.$$

*(A monic polynomial $g = T^n + a_{n-1}T^{n-1} + \cdots + a_0$ is* distinguished *if $p \mid a_i$ for all $i = 0, \cdots, n-1$.)*

**Theorem 2.10.** *Let $M$ be a finitely generated $\Lambda$-module. Then there exist integers $r$, $n_i$ and distinguished $f_j$ (where $i \in I$ and $j \in J$ with $I$ and $J$ finite) such that*

$$M \overset{\text{q-iso}}{\simeq} \Lambda^r \oplus \bigoplus_{i \in I} \Lambda/(p^{n_i}) \oplus \bigoplus_{j \in J} \Lambda/(f_j).$$

2.3. **Proof of Iwasawa's theorem.** We will prove Theorem 2.1 and Theorem 2.5 in the case when there is only one ramified prime $\mathfrak{p} \subseteq \mathcal{O}_{K_0}$ above $p$.

*Proof of Theorem 2.5.* For simplicity, we further assume that $\mathfrak{p}$ is totally ramified in $K_\infty/K_0$. Then we make the main

*Claim.* $C/(\gamma - 1)C \simeq \mathrm{Cl}_{K_0}$.

Let $M_n/K_n$ be the maximal unramified abelian $p$-extension, and $M_\infty := \bigcup_{n \geq 0} M_n$, so $M_\infty/K_\infty$ is the maximal unramified pro-$p$ abelian extension. Let $G_C := \mathrm{Gal}(M_\infty/K_\infty) \cong C$, $G := \mathrm{Gal}(M_\infty/K_0)$ and $\Gamma = \mathrm{Gal}(K_\infty/K_0)$.

*Claim* (Subclaim 1). If $I \subset G$ is the inertia at $\mathfrak{p}$, then $G \simeq G_C \rtimes I$.

There is an exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Gal}(M_\infty/K_\infty) & \longrightarrow & \mathrm{Gal}(M_\infty/K_0) & \longrightarrow & \mathrm{Gal}(K_\infty/K_0) & \longrightarrow & 1. \\ & & \| & & \| & & \| & & \\ & & G_C & & G & & \Gamma & & \end{array}$$

Since $\mathfrak{p}$ is totally ramified in $K_\infty$, $I$ surjects onto $\Gamma$ and has empty intersection with $G_C$. This proves $\Gamma \simeq I$ and subclaim 1.

Let $\Gamma \simeq I$ act on $G_C$ by conjugation. Then this action is the same as that on $C$: $\mathrm{Frob}_{\sigma\mathfrak{q}} = \sigma \,\mathrm{Frob}_{\mathfrak{q}}\, \sigma^{-1}$.

*Claim* (Subclaim 2). Let $G'$ be the commutator subgroup of $G$. Then $G' = (\gamma - 1)G$.

Let $g \in G$. Then $(\gamma - 1)g = \tilde{\gamma} g \tilde{\gamma}^{-1} g^{-1} \in G'$, where $\tilde{\gamma}$ is a lift of $\gamma$ to $I$. Conversely, $G/(\gamma - 1)G$ is the largest quotient where $\Gamma$ acts trivially. So conjugation by elements in $I$ is trivial, and since $G_C$ is abelian and $G = G_C \rtimes I$, we have that $G/(\gamma - 1)G$ is abelian and hence $(\gamma - 1)G \subseteq G'$. This proves subclaim 2.

To prove the main claim that $C/(\gamma - 1)C \simeq \mathrm{Cl}_{K_0}[p^\infty]$, it is the same to prove

$$G_C/(\gamma - 1)G_C \cong \mathrm{Gal}(M_0/K_0).$$

Note $(\gamma - 1)G_C = (\gamma - 1)G$ (because if $g \in G_C$, $\iota \in I$ and $\gamma \in \Gamma$, then picking a lift $\tilde{\gamma} \in I$, we have $(\gamma - 1)g\iota = \tilde{\gamma} g \iota \tilde{\gamma}^{-1} \iota^{-1} g^{-1} = \tilde{\gamma} g \iota \iota^{-1} \tilde{\gamma}^{-1} g^{-1} = (\gamma - 1)g$; the reverse inclusion is trivial). Then

$$\begin{aligned} G_C/(\gamma - 1)G_C &= G_C/(\gamma - 1)G \\ &\cong G/I \cdot (\gamma - 1)G \\ &= G/I \cdot G' \\ &= \mathrm{Gal}(M_0/K_0). \end{aligned}$$

This proves the main claim.

Since $\mathrm{Cl}_{K_0}[p^\infty] = C/(\gamma - 1)C$ is finite, by Nakayama's lemma $C$ is finitely generated. Write

$$C \overset{\text{q-iso}}{\simeq} \Lambda^r \oplus \bigoplus \Lambda/(p^{n_i}) \oplus \bigoplus \Lambda/(f_j).$$

Then since $\Lambda^r/(\gamma - 1)\Lambda^r \simeq \mathbf{Z}_p^r$, we have $r = 0$. This proves Theorem 2.5. □

*Remark.* Since $C/(\gamma^{p^n} - 1)C$ is finite, we see that $f_j$ is coprime to $(1 + T)^{p^n} - 1$ for all $j, n$.

Finally, let us prove Theorem 2.1: There exist $n_0$ and $\mu, \lambda, \kappa$ such that for all $n > n_0$,

$$\#\mathrm{Cl}_{K_n}[p^\infty] = p^{\mu p^n + \lambda n + \kappa}.$$

*Exercise.*

(1) $\#\Lambda/(p^m, \gamma^{p^n} - 1) = p^{mp^n}$.
(2) Let $f$ be a distinguished polynomial coprime to $(1 + T)^{p^n} - 1$ for all $n$. Then there exists $n_0$ such that if $n > n_0$, then $\#\Lambda/(f, \gamma^{p^n} - 1) = p^{\deg(f)n + c}$ for some $c = c(f)$.

*Proof of Theorem 2.1.* Replacing $K_0$ by $K_n$ where $n \gg 0$, we may assume the ramified prime is totally ramified. Then setting $\mu = \sum_i n_i$ and $\lambda = \sum_j \deg f_j$, we get (by the previous exercise and remark) $\kappa$ such that

$$\#\mathrm{Cl}_{K_n}[p^\infty] = \#C/(\gamma^{p^n} - 1)C = p^{\mu p^n + \lambda n + \kappa}$$

for sufficiently large $n$. $\square$

*Remark.* For the cyclotomic extension $\mathbf{Q}(\zeta_{p^\infty})$, it is known that $\mu = 0$; this is the Ferrero–Washington theorem.