# GEOMETRY OF FUNDAMENTAL LEMMAS

NOTES TAKEN BY PAK-HIN LEE

Abstract. These are notes from the (ongoing) Student Seminar on Geometry of Fundamental Lemmas at Columbia University in Fall 2017, which is organized by Chao Li, Yihang Zhu and myself. A program with more detailed information, written by Chao and Yihang, can be found here.

## Contents



Last updated: September 15, 2017. Please send corrections and comments to phlee@math.columbia.edu.

## 1. Lecture 1 (September 13, 2017): Yihang Zhu Introduction[1]

In this lecture, we will discuss motivations for the seminar. A large part of the lecture is based on Ngô's PCMI lectures (available on his website), which is highly recommended, as well as some of the other expository articles listed in the program.

1.1. **What is a fundamental lemma.** Suppose $G$ and $G'$ are reductive groups over $\mathbf{Q}$. Langlands' principle of functoriality says that: Given a map ${}^LG' \to {}^LG$ (admissible), we expect a relation between automorphic forms (admissible representations in the local case) for $G'$ and $G$.

One of the most powerful methods of proving functoriality is by comparison of trace formulas. In very rough terms, a trace formula for $G$ says that

$$\text{geometric side (GS)} = \text{spectral side (SS)}.$$

The canonical baby example is:

**Example 1.1** (Selberg trace formula for compact quotients)**.** Assume $\Gamma \subset G(\mathbf{R})$ is a congruence subgroup (more generally an arithmetic subgroup, or even more generally a discrete subgroup) such that $\Gamma\backslash G(\mathbf{R})$ is compact. (By reduction theory of Harish-Chandra and Borel, such a congruence subgroup $\Gamma$ exists if and only if $G$ is anisotropic over $\mathbf{Q}$.) Then for any $f \in C_c^\infty(G(\mathbf{R}))$,

$$\sum_{\gamma\in\Gamma/\mathrm{conj}} a_\gamma \cdot O_\gamma(f) = \sum_\pi m_\pi \cdot \mathrm{tr}(f|\pi),$$

where

- $O_\gamma(f) = \displaystyle\int_{G_\gamma(\mathbf{R})\backslash G(\mathbf{R})} f(g^{-1}\gamma g)\, dg$ is the orbital integral;
- $a_\gamma = \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma(\mathbf{R}))$;
- $\pi$ runs over the irreducible representations of $G(\mathbf{R})$ in $L^2(\Gamma\backslash G(\mathbf{R}))$ ("automorphic representations in $L^2(\Gamma\backslash G(\mathbf{R}))$");
- $m_\pi$ is the multiplicity of $\pi$ thereof.

*Fact.* There exists such a $\Gamma$ if and only if we have a discrete decomposition $L^2(\Gamma\backslash G(\mathbf{R})) = \bigoplus_\pi m_\pi \pi$.

Still for $G$ anisotropic over $\mathbf{Q}$, the adelic version of Selberg trace formula is

$$\sum_{\gamma\in G(\mathbf{Q})/\mathrm{conj}} a_\gamma O_\gamma(f) = \sum_\pi m_\pi\, \mathrm{tr}(f|\pi),$$

for $f \in C_c^\infty(G(\mathbf{A}))$, where:

- the orbital integral is $O_\gamma(f) := \displaystyle\prod_v O_{\gamma_v}(f_v)$, if $f = \bigotimes' f_v$. This is local in nature!
- $a_\gamma = \mathrm{vol}(G_\gamma(\mathbf{Q})\backslash G_\gamma(\mathbf{A}))$.
- $\pi$ runs through the $G(\mathbf{A})$-subrepresentations of $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$, and $m_\pi$ is the multiplicity.

[1]We thank Yihang Zhu for proofreading and editting this section.

Arthur generalized this to the *invariant trace formula* for an arbitrary reductive group $G$ over $\mathbf{Q}$, which is impossible to write down in this talk beyond

$$I(f) = \mathrm{GS} = \mathrm{SS}. \tag{1}$$

For certain $G'$, it is desirable to compare the geometric sides for $G$ and $G'$. Then we get:

- a relationship between the spectral sides for $G$ and $G'$;
- *stabilization*: Deduce from (1) a new trace formula whose both sides are invariant under conjugation over algebraic closures.

(These two applications are often interwoven together.)

By the local nature of orbital integrals, "it suffices to compare orbital integrals locally". The fundamental lemma asserts that this is possible. More precisely, often there are two sets of closely related conjectures:

- *transfer conjectures*: For any $f$ on $G$, there exists an (unspecified) $f'$ on $G'$ such that they have matching orbital integrals.
- *fundamental lemmas*: A particular choice of $(f, f')$ works.

In a lot of cases, fundamental lemmas imply transfer conjectures by global considerations.

Here are some situations where fundamental lemmas can be formulated:

| | $G$ | $G'$ | global TF | orbits | FL |
|---|---|---|---|---|---|
| 1 | reductive over $\mathbf{Q}$ | (elliptic) endoscopic group of $G$ | Arthur's invariant TF | conjugacy classes | Langlands–Shelstad FL |
| $1'$ | reductive over $\mathbf{Q}$ | twisted endoscopic group[2] of $G$ | twisted TF | twisted conjugacy classes | twisted FL |
| 2 | $U_n$ | $\mathrm{GL}_n$ | Jacquet–Rallis TF | adjoint action of $\mathrm{GL}_{n-1}$ on $\mathrm{GL}_n$; adjoint action of $U_{n-1}$ on $U_n$ | Jacquet–Rallis FL |
| 3 | $\mathrm{GL}_n$ over a quadratic field | $\mathrm{GL}_n$ | Jacquet's TF | action[3] of $U \times U$ on $G'$ via $(u_1, u_2) : g \mapsto u_1 g u_2$; action of $U$ on $G$ via $u : g \mapsto {}^t u g \bar{u}$ | Jacquet–Ye FL |

The first part of the seminar will be devoted to Case 2, and the second part will be to Case 1.

The rough form of fundamental lemmas is as follows. Given $G$ and $G'$ over a local field $F$, we have a notion of *matching of orbits*:

$$\{\gamma_i \mid i \in I\} \sim \{\delta_j \mid j \in J\}$$

for a subset $\{\gamma_i \mid i \in I\}$ of $\{\text{orbits in } G'\}$ and a subset $\{\delta_j \mid j \in J\}$ of $\{\text{orbits in } G\}$. Whenever two sets of orbits are matched, there is a notion of *transfer factors*, namely $a_i \in \mathbf{C}$ for all $i \in I$ and $b_j \in \mathbf{C}$ for all $j \in J$, if $\{\gamma_i \mid i \in I\} \sim \{\delta_j \mid j \in J\}$. In general, the definition of these is highly nontrivial (e.g. in Case 1), although they are easier in Case 2.

[2]All classical groups are twisted endoscopic groups of $\mathrm{GL}_n$.

[3]where $U$ is the group of unipotent upper triangular matrices

**Conjecture 1.2** (Fundamental Lemma)**.** *For a certain* $f \in C_c^\infty(G(F))$ *and* $f' \in C_c^\infty(G(F))$ *(usually* $f = \mathbb{1}_{G(\mathcal{O}_F)}$ *and* $f' = \mathbb{1}_{G'(\mathcal{O}_F)}$*), and for all* $\{\gamma_i\}_i \sim \{\delta_j\}_j$*, we have*

$$\sum_i a_i O_{\gamma_i}(f') = \sum_j b_j O_{\delta_j}(f).$$

The known results are:

- (twisted) fundamental lemma: Ngô, Laumon, Waldspurger, Hales, Cluckers, Loeser, Moeglin, Lemaire, *et al.*;
- weighted fundamental lemma: Chaudouard–Laumon;
- Jacquet–Rallis fundamental lemma: Yun, Jacquet (for low ranks);
- Jacquet–Ye fundamental lemma: Jacquet, Ngô.

Essentially most modern advances in number theory make use of some form of fundamental lemmas. More precisely, applications include:

- cases of Arthur's multiplicity conjecture (which is now known for all quasi-split classical groups and some of their inner forms);
- Fermat's last theorem (via Langlands–Tunnell, which is related to Artin's conjecture);
- stabilization of trace formula;
- local Langlands for $\mathrm{GL}_n$ and for classical groups;
- $L$-functions of Shimura varieties;
- construction of Galois representations;
- Gan–Gross–Prasad conjecture;
- Iwasawa main conjecture for $\mathrm{GL}_2$ due to Skinner–Urban;
- etc.

**Example 1.3** (SGA $4\frac{1}{2}$, Exposé 6, § 7)**.** Fix a character $\psi : \mathbf{F}_q \to \mathbf{C}^\times$, an element $a \in \mathbf{F}_q^\times$ and an integer $n \geq 1$. Consider the Kloosterman sum

$$K := \sum_{\substack{x_1,\cdots,x_n \in \mathbf{F}_q \\ \prod_i x_i = a}} \psi(x_1 + \cdots + x_n),$$

and the twisted version

$$K' := \sum_{\substack{x \in \mathbf{F}_{q^n} \\ Nx = a}} \psi(\operatorname{tr} x).$$

In this case, the fundamental lemma says

$$K' = (-1)^{n+1} K,$$

which is a consequence of the classical Hasse–Davenport identities. (When $n = 2$, this falls into Case 3 above.) In SGA $4\frac{1}{2}$, a proof is given by interpreting both sides as traces on $\ell$-adic cohomology and comparing the cohomology groups.

*Exercise.* Read that part of SGA $4\frac{1}{2}$.

The geometric proofs of fundamental lemmas are generalizations of this idea.

1.2. **How to prove a fundamental lemma.** Consider Cases 1 and 2 only. The following reductions can be made:

- Replace the action of $G_1$ on $G_2$ by the action of $G_1$ on $\operatorname{Lie} G_2$.
- Assume $q \gg 0$, and replace $F/\mathbf{Q}_p$ by $F = k((t))$ where $k = \mathbf{F}_q$. This follows from hard work of Waldspurger (in Case 1) or model theory (in both cases). We remark that the geometric proofs of fundamental lemmas, now over $k((t))$, also assumes $q \gg 0$. In Case 1, the assumption that $q \gg 0$ is harmless by work of Hales.

In Case 1, we consider $G(F)$ acting on $\mathfrak{g}(F)$. For $\gamma \in \mathfrak{g}$, the orbital integral is

$$O_\gamma(f) := \int_{G_\gamma(F)\backslash G(F)} f(\operatorname{ad} g^{-1}(\gamma))\, dg.$$

We are interested in $f = \mathbb{1}_{\mathfrak{g}(\mathcal{O}_F)}$. The setup is the same for $G'$.

In Case 2:

- For $G'$, consider the action of $\mathrm{GL}_{n-1}(F)$ on $\mathfrak{gl}_n(F)$. For $\gamma \in \mathfrak{gl}_n(F)$, the orbital integral is
$$O_\gamma := \int_{\mathrm{GL}_{n-1}(F)} \mathbb{1}_{\mathfrak{gl}_n(\mathcal{O}_F)}(g^{-1}\gamma g)\eta(\det(g))\, dg$$
where $\eta : F^\times \to \{\pm 1\}$ is a character (corresponding to an unramified quadratic extension $E/F$).
- For $G$, consider the action of $U_{n-1}(F)$ on $\mathfrak{s}_n(F) = \{\text{Hermitian matrices}\}$ with respect to $E/F$. For $\gamma \in \mathfrak{s}_n(F)$, the orbital integral is
$$O_\gamma := \int_{U_{n-1}(F)} \mathbb{1}_{\mathfrak{s}_n(\mathcal{O}_F)}(g^{-1}\gamma g)\, dg.$$

We call the above three cases Cases 1, $2_{G'}$ and $2_G$. In all these cases, we write the situation as $H$ acting on $\mathfrak{g}$. Recall that $F = k((t))$ with $k = \mathbf{F}_q$. For simplicity, assume $H$ and $\mathfrak{g}$ are defined over $k$. We now describe the proof of the fundamental lemmas.

**Step 1**: Explicitly find invariant functions, i.e., elements $c_1, \cdots, c_k \in k[\mathfrak{g}]^H$, where $\mathfrak{g}$ is considered as an algebraic variety over $k$.

- Case 1, with $\mathfrak{g} = \mathfrak{gl}_n$ and $H = \mathrm{GL}_n$: Take $a_1, \cdots, a_n$ to be the coefficients of the characteristic polynomial. We have $k[\mathfrak{g}]^G = k[a_1, \cdots, a_n]$. In general, it is a theorem of Chevalley that $k[\mathfrak{g}]^G$ is always a polynomial algebra.
- Case $2_{G'}$: Take the same $a_1, \cdots, a_n$, and in addition $b_0, \cdots, b_{n-1}$ which also have an explicit definition.

Let $C := \operatorname{Spec} k[c_1, \cdots, c_k]$. Consider

$$\chi : [\mathfrak{g}/H] \to C$$
$$\gamma \mapsto (c_1(\gamma), \cdots, c_k(\gamma)),$$

where $[\mathfrak{g}/H]$ is the stack quotient (over $k$).

In Case $2_{G'}$, we have the following good news: If $\gamma \in \mathfrak{g}(F)$ such that $(c_1(\gamma), \cdots, c_k(\gamma))$ is in "general position", then the stabilizer $H_\gamma$ is trivial. Define explicitly a nonempty open locus $C^{\mathrm{reg}} \subset C$ smaller than this "general position" condition. In Case 1, the analogue is an open locus in $C$ on which the fibers of $\chi$ assume the minimal dimension.

**Step 2**: Certain spaces known as affine Springer fibers naturally arise in these settings. Given a space $X$ over $k$, we can formally consider "the space of formal arcs in $X$", denoted by $L^+X$, such that

$$L^+X(k) = \{\text{maps } \operatorname{Spec} \mathcal{O}_F = \operatorname{Spec} k[[t]] \to X\}.$$

Then $\chi$ induces

$$L^+\chi : L^+[\mathfrak{g}/H] \to L^+C.$$

For $\alpha \in L^+C(k) = C(\mathcal{O}_F)$ we define $N_\alpha$ to be the fiber of $L^+[\mathfrak{g}/H]$ over $\alpha$.

For the fundamental lemma, we are only interested in orbital integrals along $\gamma \in \mathfrak{g}(\mathcal{O}_F)$ such that its generic fiber, namely the induced element in $\mathfrak{g}(F)$, lies in $\chi^{-1}(C^{\mathrm{reg}}(F))$

Observe: For such $\gamma$, consider $\alpha := (c_i(\gamma)) \in C(\mathcal{O}_F)$ and hence $N_\alpha$. Then in Case $2_{G'}$, we have

$$O_\gamma = \tilde{\#} N_\alpha(k),$$

where $\tilde{\#}$ means a certain weighted count of the cardinality (due to the fact that we have inserted into the integrand the $\eta$ term), or rather, $O_\gamma$ is equal to the trace of Frobenius on $H^*(N_{\alpha,\bar{k}}, L)$ for a local system $L$ on $N_\alpha$. In Case 1, the analogous relation is more complicated due to the non-triviality of the centralizers.