ÉTALE FUNDAMENTAL GROUPS

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ABSTRACT. Here are the notes I am taking for Johan de Jong's ongoing course on étale fundamental groups offered at Columbia University in Fall 2015 (MATH G4263: Topics in Algebraic Geometry).

Due to my own lack of understanding of the materials, I have inevitably introduced both mathematical and typographical errors in these notes. Please send corrections and comments to phlee@math.columbia.edu.

WARNING: I am unable to commit to editing these notes outside of lecture time, so they are likely riddled with mistakes and poorly formatted.

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Last updated: December 10, 2015.

1. Lecture 1 (September 8, 2015)

PH: I missed the lecture.

2. Lecture 2 (September 10, 2015)

2.1. References.

- Stacks project, chapter on fundamental groups (tag 0BQ6).
- Lenstra, Galois Theory for Schemes.
- SGA I.
- Murre, Lectures on an introduction to Grothendieck's theory of the fundamental group.

2.2. Galois Categories.

The idea is to consider

$$\text{Topological groups} \leftrightarrow \text{Categories } \mathcal{C} \text{ with } F : \mathcal{C} \to \text{Sets}$$

Those categories which go back to themselves after we go around will be the Galois categories.

Notation. Let G be a topological group. Denote by G-Sets the category with objects (X, a) where X is a set (with the discrete topology) and $a : G \times X \to X$ a continuous action. Because the topology is discrete, this means the stabilizer of any point is open. Morphisms are the obvious ones. Finite-G-Sets is the full subcategory of G-Sets of (X, a) with $\#X < \infty$.

The profinite completion $G^\wedge$ of G is

$$G^\wedge = \lim_{U \triangleleft G \text{ open, finite index}} G/U.$$

It satisfies the universal property: if $G \to H$ is continuous and H is a profinite group, then there exists a unique factorization $G \to G^\wedge \to H$.

The first interesting statement is

Proposition 2.1. *Consider the forgetful functor* $F : \text{Finite-}G\text{-Sets} \to \text{Sets}$. *Then*

$$G^\wedge \cong \mathrm{Aut}(F)$$

as topological groups.

This is not a triviality and I will explain the proof. A basis of the topology on $\mathrm{Aut}(F)$ is the kernels of the maps $\mathrm{Aut}(F) \to \prod_{i=1}^n \mathrm{Aut}(F(X_i))$.

Proof. The steps are

- There exists a map $G \to \mathrm{Aut}(F)$.
- This is continuous.
- We get $G^\wedge \to \mathrm{Aut}(F)$ because $\mathrm{Aut}(F)$ is profinite and universal property.
- $G^\wedge \hookrightarrow \mathrm{Aut}(F)$. To see this, if $U \triangleleft G$ is open and finite index, then $X = G/U$ belongs to Finite-G-Sets. So

$$\begin{array}{ccc} G^\wedge & \longrightarrow & \mathrm{Aut}(F) \\ \downarrow & & \downarrow \\ G/U & \hookrightarrow & \mathrm{Aut}(F(X)) \end{array}$$

  so $\ker(G^\wedge \to \mathrm{Aut}(F)) \subset \bigcap_U U = \{e\}$.
- Enough to show image is dense (by basic topology).

- Pick $X \in$ Finite-G-Sets, and $\gamma \in \mathrm{Aut}(F)$. Enough to find $g \in G$ such that

$$(\gamma_X : X \to X) = (\text{action of } G \text{ on } X)$$

(This uses the fact that the category has finite disjoint unions. Silly.)

Proof. Let $X = \coprod_{i=1,\cdots,n} X_i$, where $X_i \cong G/H_i$ with $H_i \subset G$ open subgroup of finite index. Take $U = \bigcap_{i=1,\cdots,n} \bigcap_{g \in G} gH_ig^{-1} \lhd G$, which is open and finite index. Then $Y = G/U$ maps into X_i for $i = 1, \cdots, n$.

Enough to find $g_0 \in G$ such that γ_Y is the action of g_0. This is because

$$\begin{array}{ccc} Y & \overset{g_0}{\underset{\gamma_Y}{\rightrightarrows}} & Y \\ \downarrow & & \downarrow \\ X & \overset{g_0}{\underset{\gamma_X}{\rightrightarrows}} & X \end{array}$$

Think! ☐

- Say $\gamma_Y(eU) = g_0U$ where $e \in G$ is the neutral element. Then

$$\gamma_Y(gU) \overset{U \lhd G}{\to} \gamma_Y(R_g(eU))$$

where $R_g : G/U \to G/U$ is right multiplication in Arrows(Finite-G-Sets). Since γ is a transformation of functors, this is equal to

$$R_g(\gamma_Y(eU)) = R_g(g_0U) = g_0gU.$$

Done! ☐

Lemma 2.2. *Any* exact *functor F :* Finite-G-Sets $\to$ Sets*, with $F(X)$ finite for all X, is isomorphic to the forgetful functor.*

Proof. Omitted (exercise). ☐

We need to define exactness.

Definition 2.3. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.

(1) If $\mathcal{A}$ has finite limits ($\Leftrightarrow$ $\mathcal{A}$ has a final object $*_{\mathcal{A}}$ and fibre products) and F commutes with them ($\Leftrightarrow$ $F(*_{\mathcal{A}}) = *_{\mathcal{B}}$ and $F(X \times_Y Z) = F(X) \times_{F(Y)} F(Z)$), then we say F is *left exact*.
(2) If $\mathcal{A}$ has finite colimits ($\Leftrightarrow$ $\mathcal{A}$ has an initial object $\emptyset_{\mathcal{A}}$ and pushouts) and F commutes with them ($\Leftrightarrow$ $F(\emptyset_{\mathcal{A}}) = \emptyset_{\mathcal{B}}$ and $F(X \sqcup_Y Z) = F(X) \sqcup_{F(Y)} F(Z)$), then we say F is *right exact*.
(3) F *exact* $\Leftrightarrow$ left and right exact.

In Grothendieck's original exposition, things are more general but it is easier for us to work with exactness properties.

If $F : \mathcal{C} \to$ Sets is a functor with finite values, we get a functor $\mathcal{C} \to$ Finite-$\mathrm{Aut}(F)$-Sets by sending $X \mapsto F(X)$.

Definition 2.4. Let $F : \mathcal{C} \to$ Sets be a functor. We say $(\mathcal{C}, F)$ is a *Galois category* if

(1) $\mathcal{C}$ has finite limits and colimits;
(2) every object of $\mathcal{C}$ is a finite coproduct of *connected* objects;

(3) F has finite values;
(4) F is exact and *reflects isomorphisms*;

where

- X is *connected* $\Leftrightarrow$ X is not initial and any subobject ($Y \to X$ monomorphism) is isomorphic to either $\emptyset \to X$ or $X \to X$ (this is the definition in the context of Galois categories only);
- F reflects $P \Leftrightarrow$ if ($F(f)$ has $P \Rightarrow f$ has P).

Warning. This definition is not the same as SGA I, but equivalent.

We will do a bunch of lemmas to see that nice things happen for Galois categories.

Lemma 2.5 (Example Fact)**.** *Suppose $a, b : X \to Y$ in $\mathcal{C}$, X connected, $F(a)(x) = F(b)(x)$ for some $x \in F(X)$. Then $a = b$.*

Proof. F commutes with limits, so $\mathrm{Eq}(F(a), F(b)) = F(\mathrm{Eq}(a, b))$ contains x. This implies $\emptyset_{\mathcal{C}} \not\cong \mathrm{Eq}(a, b) \underset{\text{subobject}}{\subset} X$. Since X is connected, $\mathrm{Eq}(a, b) = X$ and $a = b$. $\square$

Corollary 2.6. *$\# \mathrm{Aut}_{\mathcal{C}}(X) \leq \# F(X)$ for X connected.*

Definition 2.7. We say $X \in \mathrm{Ob}(\mathcal{C})$ is Galois if X is connected and equality holds.

Next we need to prove there are enough Galois objects.

Lemma 2.8. *For any connected X, there exists a Galois object Y and $Y \to X$.*

Proof. Say $F(X) = \{x_1, \cdots, x_n\}$. S_n acts on $X^n = \coprod_{t \in T} Z_t$. Then S_n acts in the same way on $F(X)^n = F(X^n) = \coprod F(Z_t)$. Pick $t \in T$ such that $\xi = (x_1, \cdots, x_n) \in F(Z_t)$. Set $G = \{\sigma \in S_n \mid \sigma(Z_t) = Z_t\}$. This is the same as

$$\{\sigma \in S_n \mid F(\sigma)(\xi) = F(Z_t)\}$$

(argument omitted). But $F(\sigma)(\xi) = \sigma(\xi) = (x_{\sigma(1)}, \cdots, x_{\sigma(n)})$, so we win if $(x'_1, \cdots, x'_n) \in F(Z_t)$ implies x'_i pairwise distinct. If not, say $x'_i = x'_j$, then $\mathrm{pr}_i|_{Z_t} = \mathrm{pr}_j|_{Z_t}$ by example fact above. This contradicts $\mathrm{pr}_i(\xi) = \mathrm{pr}_j(\xi)$. $\square$

The same argument can be used to construct Galois extensions of finite separable extensions (everything needs to be dualized, and X^n corresponds to the n-fold tensor product). Also applies to n-fold Galois covering of connected and locally connected spaces.

Lemma 2.9. *Let $(\mathcal{C}, F)$ be a Galois category. The action of $\mathrm{Aut}(F)$ on $F(X)$ is transitive for all $X \in \mathrm{Ob}(\mathcal{C})$ connected.*

Idea of proof. We need to introduce some notations. Let I be the set of isomorphism classes of Galois objects. For $i \in I$, let X_i be a representative. Pick $x_i \in F(X_i)$. We say $i \geq i'$ if there exists a morphism $X_i \to X_{i'}$. We may pick $f_{ii'} : X_i \to X_{i'}$ such that $F(f_{ii'})(x_i) = x_{i'}$ ($\Rightarrow f_{ii'}$ unique) (because Galois objects).

Claim. F is isomorphic to the functor $F' : X \mapsto \mathrm{colim}_{i \in I} \mathrm{Mor}_{\mathcal{C}}(X_i, X)$.

If this is true, then just as in the case of Galois theory, we set

$$H = \lim_{i \in I} \mathrm{Aut}(X_i)$$

and get $H^{\mathrm{opp}} \to \mathrm{Aut}(F') = \mathrm{Aut}(F)$. $\square$

3. Lecture 3 (September 15, 2015)

3.1. **Reminders.** Galois category is a pair $(\mathcal{C}, F)$ where

- $\mathcal{C}$ has all finite limits and colimits;
- $F : \mathcal{C} \to \text{Sets}$ is an exact functor;
- $F(X)$ is finite for all $X \in \text{Ob}(\mathcal{C})$;
- F reflects isomorphisms;
- for all $X \in \text{Ob}(\mathcal{C})$, $X \cong \coprod_{i=1,\cdots,n} X_i$, X_i connected.

I was proving the

Lemma 3.1. *If $X \in \text{Ob}(\mathcal{C})$ is connected, then $\text{Aut}(F)$ acts on $F(X)$ transitively.*

For example, if $\pi_1(T) = S_n$, then the universal cover $\tilde{T}$ has group S_n over T. For the n-to-1 covering T' over T, $\tilde{T}$ has group $\text{Stab}(1)$ over T'.

Definition 3.2. X is *Galois* if X is connected and $\text{Aut}(X)$ acts transitively on $F(X)$.

Sketch of proof.

- Let I be the set of isomorphism classes of Galois objects in $\mathcal{C}$. For each $i \in I$, pick a representative X_i.
- $i \geq i' \Leftrightarrow$ there exists $X_i \to X_{i'}$.
- Pick $\gamma_i \in F(X_i)$, for $i \geq i'$ pick $f_{ii'} : X_i \to X_{i'}$ such that $F(f_{ii'})(\gamma_i) = (\gamma_{i'})$. (This morphism is uniquely determined.)
- $A_i = \text{Aut}(X_i)$ acts transitively on $F(X_i)$.

  Suppose

$$\begin{array}{ccc} X_i & \xrightarrow{a_i} & X_i \\ {\scriptstyle f_{ii'}}\downarrow & & \downarrow{\scriptstyle f_{ii'}} \\ X_{i'} & \xrightarrow{\exists !} & X_{i'} \end{array}$$

  This gives a map $\alpha_{ii'} : A_i \to A_{i'}$.
- The collection of A_i and transitive maps $\{\alpha_{ii'} : A_i \to A_{i'}\}$ forms an inverse system of finite groups over $(I, \geq)$.
- $A = \lim A_i$. I claim that $A \twoheadrightarrow A_i$.

  To prove this claim, you show I is a directed partially ordered set (if $i_1, i_2 \in I$, pick a Galois object $Y \to X_{i_1} \times X_{i_2}$, then $Y \cong X_i$ for some i with $i \geq i_1$ and $i \geq i_2$) and you use

  Lemma 3.3 (Set Theory Lemma)**.** *Directed inverse limit of finite nonempty sets is nonempty.*

- There exists $A^{\text{opp}} \to \text{Aut}(F)$ by proving that the functor F' is isomorphic to F where $F'(X) = \text{colim}_I \text{Mor}_{\mathcal{C}}(X_i, X)$.
- $\alpha_{ii'}(a_i)$ is the unique map that makes

$$\begin{array}{ccc} X_i & \xrightarrow{a_i} & X_i \\ \downarrow & & \downarrow \\ X_{i'} & \xrightarrow{\alpha_{ii'}(a_i)} & X_{i'} \end{array}$$

  commute. Details... Details...

- $F' \to F$ is the map

$$(\text{if } f \in F'(X) \text{ is given by } X_i \xrightarrow{f_i} X) \in F'(X) \mapsto F(f_i)(\gamma_i) \in F(X). \qquad \square$$

We have two more statements about general Galois categories.

Proposition 3.4. *Suppose $(\mathcal{C}, F)$ is a Galois category. Then the functor*

$$F : \mathcal{C} \to \text{Finite-}\operatorname{Aut}(F)\text{-Sets}$$

is an equivalence.

Proof.

(1) F is faithful: This is the first result on Galois categories we proved.

(2) F is fully faithful: Let $X, Y \in \operatorname{Ob}(\mathcal{C})$ and $s : F(X) \to F(Y)$ commuting with $\operatorname{Aut}(F)$-action. Then the graph $\Gamma_s \subset F(X) \times F(Y) = F(X \times Y)$ is a union of orbits. This implies by the lemma (on action $\operatorname{Aut}(F)$ transitive on F(connected)) that there exists $Z \subset X \times Y$ which is a coproduct of connected components of $X \times Y$ such that $F(Z) = \Gamma_s$. This implies $\mathrm{pr}_1|_Z : Z \to X$ is an isomorphism because $F(\mathrm{pr}_1|_Z)$ is bijective. Then Z is the graph of a morphism $X \to Y$

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle (\mathrm{pr}_1|_Z)^{-1}} & \searrow \qquad \nearrow & {\scriptstyle \mathrm{pr}_2|_Z} \\ & Z & \end{array}$$

with $F(f) = s$.

(3) Essentially injective:

(a) Enough to construct X with $F(X) \cong \operatorname{Aut}(F)/H$, for $H \subset \operatorname{Aut}(F)$ an open subgroup (automatically of finite index).

(b) Can find Y Galois with $U = \ker(\operatorname{Aut}(F) \to \operatorname{Aut}(F(Y)))$ contained in H.

(c) Then by fully faithfulness

$$\begin{aligned} \operatorname{Aut}(Y) &\cong \operatorname{Aut}_{\operatorname{Aut}(F)\text{-Sets}}(F(Y)) \\ &\cong \operatorname{Aut}_{\operatorname{Aut}(F)\text{-Sets}}(\operatorname{Aut}(F)/U) \\ &\cong (\operatorname{Aut}(F)/U)^{\mathrm{opp}} \end{aligned}$$

which contains $(H/U)^{\mathrm{opp}}$.

(d) Get $H' \leftrightarrow (H/U)^{\mathrm{opp}}$.

(e) Let $X = \operatorname{Coeq}\left(Y \overset{h_1'}{\underset{h_r'}{\rightrightarrows}} Y \right)$, where $H' = \{h_1', \cdots, h_r'\}$.

(f) Since F commutes with colimits,

$$\begin{aligned} F(X) &= \operatorname{Coeq}\left(F(Y) \overset{F(h_1')}{\underset{F(h_r')}{\rightrightarrows}} F(Y) \right) \\ &\cong \operatorname{Coeq}\left(\operatorname{Aut}(F)/U \overset{R_{h_1'}}{\underset{R_{h_r'}}{\rightrightarrows}} \operatorname{Aut}(F)/U \right) \\ &\cong \operatorname{Aut}(F)/H \end{aligned}$$

where $H = h_1 U \sqcup \cdots \sqcup h_r U$. □

Addendum: Let $(\mathcal{C}, F)$ and $(\mathcal{C}', F')$ be two Galois categories, and $H : \mathcal{C} \to \mathcal{C}'$ an exact functor. Then

(1) There exists an isomorphism $t : F \to F' \circ H$.
(2) The choice of t determines $h : G' := \mathrm{Aut}(F') \to G := \mathrm{Aut}(F)$ a continuous homomorphism of groups well-defined up to inner automorphisms of G.
(3)

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \\ \cong\downarrow & & \downarrow\cong \\ \text{Finite-}G\text{-Sets} & \xrightarrow{h} & \text{Finite-}G'\text{-Sets} \end{array}$$

is 2-commutative (via t).

3.2. **Fundamental Groups of Schemes.**

Let $f : X \to Y$ be a morphism of schemes. TFAE:

(1) f is étale.
(2) f is smooth of relative dimension 0.
(3) f is flat, lfp (locally of finite presentation), and fibres étale.
(4) f lfp and infinitesimally lifting criterion:

$$\begin{array}{ccc} X & \longleftarrow & T_0 \\ \downarrow & \overset{\exists!}{\nwarrow} & \downarrow \text{first order thickening} \\ Y & \longleftarrow & T_{\text{affine}} \end{array}$$

(formally étale).
(5) f lfp, flat, unramified ($\Leftrightarrow$ f locally finite type and $\Omega_{X/Y} = 0$.)

Moreover,

(6) If Y is locally Noetherian, lfp = lft (locally of finite type)
(7) If $Y = \mathrm{Spec}(k)$, then $X \to Y$ is étale if and only if there exists a set I and $X \cong \coprod_{i \in I} \mathrm{Spec}(k_i)$ with k_i/k finite separable.

Formal properties:

(A) Being étale is preserved under base change:

$$\begin{array}{ccc} Y' \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

(B) Being étale is preserved under composition.
(C)

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle h = g\circ f}\searrow \quad \swarrow{\scriptstyle g} & \\ & Z & \end{array}$$

g, h étale $\Rightarrow$ f étale.

(D) $A \to B$ étale ring map $\Leftrightarrow B \cong A[x_1, \cdots, x_n]/(f_1, \cdots, f_n)$ such that

$$\Delta = \det \left(\frac{\partial f_i}{\partial x_j} \right)$$

is invertible. (There is a structure theorem for étale algebras, which is a stronger statement.)

We will be talking about finite étale morphisms.

Let $f : X \to Y$ be a morphism. TFAE:

(1) f is finite, étale. ($f : X \to Y$ *finite* iff for all $V \subset Y$ affine, $U = f^{-1}V$ is affine and $\mathcal{O}(V) \to \mathcal{O}(U)$ is a finite ring map (module finite).)
(2) For all $V \subset Y$ affine open, $f^{-1}V = U$ is affine and $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ has (*).
(3) Same as (2) for some affine open covering.

(*): $A \to B$ with B finite locally free as A-module and $Q : B \times B \to A$, $(b_1, b_2) \mapsto \mathrm{Tr}_{B/A}(b_1 b_2)$ is nondegenerate.

If $B \cong A^{\oplus n}$ is free and multiplication by $b_1 b_2$ has matrix $(a_{ij}(b_1 b_2))$, then $\mathrm{Tr}_{B/A}(b_1 b_2)$ is the trace of this matrix.

Exercise. If $K \to B$ with (*) and K field, then $B = \prod L_i$ where L_i/K finite separable.

Next time: Let X be a scheme. Then

$$\mathrm{F\acute{E}t}_X = \{Y \to X \text{ finite étale}\} \xrightarrow{F_{\overline{x}}} \mathrm{Sets}$$

(fibre functor) is a Galois category, and $\pi_1(X, \overline{x}) = \mathrm{Aut}(F_{\overline{x}})$.

4. Lecture 4 (September 17, 2015)

Today we will talk about étale fundamental groups.

Notation. Let X be a scheme. $\mathrm{F\acute{E}t}_X$ is the category with

- objects: $Y \xrightarrow{f} X$ finite étale,
- morphisms:

$$\mathrm{Mor}_{\mathrm{F\acute{E}t}_X}(Y \xrightarrow{f} X, Y' \xrightarrow{f'} X) = \mathrm{Mor}_X(Y, Y') = \{g : Y \to Y' \mid f' \circ g = f\}.$$

Example 4.1.

(1) Let $X = \mathrm{Spec}(k)$ where k is a field. Then

$$\mathrm{F\acute{E}t}_X = \{\text{finite separable } k\text{-algebras}\}^{\mathrm{opp}}.$$

(2) Let $X = \mathrm{Spec}(A)$. Then

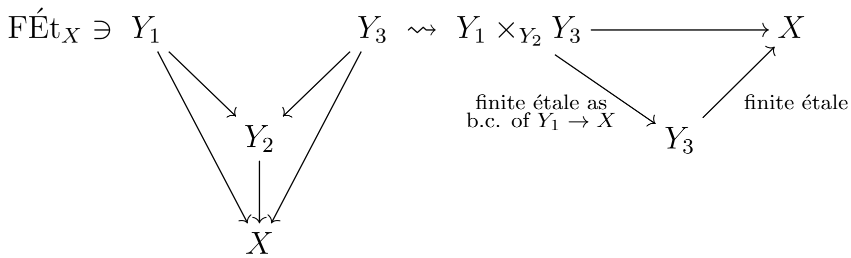
$$\begin{aligned} \mathrm{F\acute{E}t}_X = \{&A \to B \text{ which are finite locally free} \\ &\text{and } Q : B \times B \to A, (b_1, b_2) \mapsto \mathrm{Tr}_{B/A}(b_1 b_2) \text{ is nondegenerate}\}^{\mathrm{opp}} \\ = \{&\text{separable } A\text{-algebras}\}^{\mathrm{opp}}. \end{aligned}$$

$\mathrm{F\acute{E}t}_X$ will be a Galois category only when X is connected. In general,

Lemma 4.2. $\mathrm{F\acute{E}t}_X$ *has all finite limits and finite colimits and for a morphism of schemes* $X' \to X$*, then base change functor* $\mathrm{F\acute{E}t}_X \to \mathrm{F\acute{E}t}_{X'}$, $Y/X \mapsto X' \times_X Y/X'$ *is an exact functor.*

Proof.

- Limits: Check there exists final object and fibre products.
  - Final object: $\mathrm{id} : X \to X$.
  - Fibre products:



- Colimits: enough to check finite coproducts and coequalizers
  - Coproducts: disjoint unions
  - Coequalizers: $Y_1 \overset{a}{\underset{b}{\rightrightarrows}} Y_2$ in $\mathrm{F\acute{E}t}_X$.

  Note $Y_i = \underline{\mathrm{Spec}}_X(f_{i,*}\mathcal{O}_{Y_i})$.

$$f_{1,*}\mathcal{O}_{Y_1} \overset{a^\#}{\underset{b^\#}{\leftleftarrows}} f_{2,*}\mathcal{O}_{Y_2} \longleftarrow \mathcal{A}$$

  where $\mathcal{A}$ is a quasicoherent sheaf of $\mathcal{O}_X$-algebras, and the last map is the equalizer of $a^\#$ and $b^\#$.

  Claim. $\underline{\mathrm{Spec}}_X(\mathcal{A}) \to X$ is in $\mathrm{F\acute{E}t}_X$ and is coequalizer of a and b.

  Proof of claim (please find your own).
  (*): Étale locally on X you can write $Y_i = \coprod_{j=1,\cdots,m_i} X$.
  (**): $Y_i = \coprod_{j=1,\cdots,m_i} X$, then any morphism $Y \to Y'$ over X Zariski locally on X comes from a map $\{1,\cdots,m_1\} \to \{1,\cdots,m_2\}$.
  (*) follows from the local structure theorem for finite unramified maps. Suppose we have $g : Y \to X$ finite unramified and $x \in X$. Then there exists $(U,u) \to (X,x)$ étale such that

$$Y \times_X U = \coprod_{j=1,\cdots,m} V_j$$

  where V_j are open and closed subsets, and $V_j \to U$ is a closed immersion for all j.
  Finish the proof:
  (***): Descent theory "says" it is enough to prove claim after étale base change.
  (****): Explicitly compute what happens when Y and Y' are map are as in (**). □

□

4.1. **Fibre functor.**

Definition 4.3. A *geometric point* $\overline{x}$ of a scheme is a morphism $\mathrm{Spec}(K) \xrightarrow{\overline{x}} X$ whre k is an lagebraically closed field.

Abusive notation: $k = \kappa(\overline{x})$, $\overline{x} = \mathrm{Spec}(\kappa(\overline{x}))$, $x = \mathrm{Im}(\overline{x})$.
Base change gives

$$\begin{aligned} F_{\overline{x}} : \text{FÉt}_X &\to \text{FÉt}_{\overline{x}} \cong \text{category of finite sets} \to \text{Sets} \\ Y/X &\mapsto F_{\overline{x}}(Y) = Y_{\overline{x}} \end{aligned}$$

where $Y_{\overline{x}}$ is the set of $\overline{y}$ such that

$$\begin{array}{ccc} & & Y \\ & \overset{\overline{y}}{\nearrow} & \downarrow \\ \overline{x} & & \\ & \underset{\overline{x}}{\searrow} & \\ & & X \end{array}$$

commutes.

Theorem 4.4. *If X is a connected scheme, then $(\text{FÉt}_X, F_{\overline{x}})$ is a Galois category!*

Warning. Not true if X is not connected.

Definition 4.5 (Grothendieck's Fundamental Group)**.** For X connected,

$$\pi_1(X, \overline{x}) := \mathrm{Aut}(F_{\overline{x}}).$$

Proof. Already know

- There exist finite (co)limits.
- $F_{\overline{x}}$ exact.
- Decomposition into connected components
  - Fact: A finite morphism is closed (hence proper and integral).
  - Fact: An étale morphism is open (hence smooth and flat+lfp).
  - If X is Noetherian, then Y is Noetherian and $Y = \coprod_{i=1,\cdots,n} Y_i$ into connected components, so $Y_i \to X$ is finite étale. (Need to check: monos in FÉt_X are open immersions.)
- Lastly: $F_{\overline{x}}$ reflects isomorphisms. Suppose $Y \xrightarrow{g} Y'$ in FÉt_X and $F_{\overline{x}}(g)$ bijective.
  - by above reduce to Y' connected.
  - then g is finite étale (general properties of morphisms)
  - g is finite locally free, so degree of g is locally constant on Y'. Since Y' is connected, degree of g is constant.
  - degree of g is 1 by looking at the fibre over some $\overline{y}' \in F_{\overline{x}}(Y')$.
  - degree 1 implies g isomorphism. □

Lemma 4.6. *Let $f : X' \to X$ be a morphism of connected schemes. Let $\overline{x}'$ be a geometric point of X'. Set $\overline{x} = f(\overline{x}')$. Then we get a canonical continuous homomorphism of profinite groups*

$$f_* : \pi_1(X', \overline{x}') \to \pi_1(X, \overline{x})$$

such that

$$\begin{array}{ccc} \text{FÉt}_{X'} & \xleftarrow{\textit{base change}} & \text{FÉt}_X \\ {\scriptstyle F_{\overline{x}'}} \big\downarrow \cong & & \cong \big\downarrow {\scriptstyle F_{\overline{x}}} \\ \text{Finite-}\pi_1(X', \overline{x}')\text{-Sets} & \xleftarrow{f_*} & \text{Finite-}\pi_1(X, \overline{x})\text{-Sets} \end{array}$$

2-*commutes.*

Proof. $F_{\overline{x}} \cong F_{\overline{x}'} \circ$ (base change) because $\overline{x} = f \circ \overline{x}'$. □

Lemma 4.7 (Change of base point)**.** *If $\overline{x}_1, \overline{x}_2$ are geometric points of connected X, then*

$$\pi_1(X, \overline{x}_1) \cong \pi_1(X, \overline{x}_2)$$

well-defined up to inner automorphism.

Proof. Choose isomorphism $F_{\overline{x}_1} \cong F_{\overline{x}_2}$. □

Example 4.8. Fix $k \subset k^{\mathrm{sep}} \subset \overline{k}$. Then

$$\pi_1(\mathrm{Spec}(k), \mathrm{Spec}(\overline{k})) = \mathrm{Gal}(k^{\mathrm{sep}}/k).$$

Proof. The functor

$$\begin{aligned} \mathrm{F\acute{E}t}_X &\to \text{Finite-}\mathrm{Gal}(k^{\mathrm{sep}}/k)\text{-Sets} \\ Y &\mapsto \mathrm{Mor}_{\mathrm{Spec}(k)}(\mathrm{Spec}(\overline{k}), Y) = \mathrm{Mor}_{\mathrm{Spec}(k)}(\mathrm{Spec}(k^{\mathrm{sep}}), Y) \end{aligned}$$

(the right hand side has a left action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$) is an equivalence. □

Example 4.9. $\pi_1(\mathrm{Spec}(\mathbb{C})) = \{1\}$.

Example 4.10. $\pi_1(\mathrm{Spec}(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$.

Example 4.11. $\pi_1(\mathrm{Spec}(\mathbb{C}((t)))) = \hat{\mathbb{Z}}$.

Example 4.12. $\pi_1(\mathbb{P}^1_{\mathbb{C}}) = \{1\}$.

Proof. Every object in $\mathrm{F\acute{E}t}_{\mathbb{P}^1_{\mathbb{C}}}$ is isomorphic to a disjoint union of copies of $\mathbb{P}^1_{\mathbb{C}}$. Let $f : Y \to \mathbb{P}^1_{\mathbb{C}}$ be finite étale, with Y connected. Have to show $Y \cong \mathbb{P}^1_{\mathbb{C}}$.

By Hurwitz,

$$2g_Y - 2 = (2g_{\mathbb{P}^1_{\mathbb{C}}} - 2)\deg(f) + \deg(R) = -2\deg(f).$$

Since $g_Y \geq 0$, this implies $\deg(f) = 1$. □

5. Lecture 5 (September 22, 2015)

5.1. Complex Varieties.

Notation. Given a scheme X that is lft (locally of finite type) over $\mathbb{C}$ we denote X^{an} the usual topological space whose underlying set of points is $X(\mathbb{C})$.

Recall $X(\mathbb{C})$ is the set of maps $\mathrm{Spec}(\mathbb{C}) \to X$. For an affine variety $X = V(f_1, \cdots, f_t) \subset \mathbb{A}^n_{\mathbb{C}}$,

$$X(\mathbb{C}) = \{(a_1, \cdots, a_n) \in \mathbb{C}^n \mid f_i(a_1, \cdots, a_n) = 0 \text{ for } i = 1, \cdots, t\}.$$

Properties:

- If $f : X \to Y$ is a morphism of schemes lft over $\mathbb{C}$, then $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is continuous.
- If f is an open/closed immersion, then f^{an} is an open/closed immersion.
- If $X = \mathbb{A}^n_{\mathbb{C}}$, then $X^{\mathrm{an}} = \mathbb{C}^n$ with Euclidean topology.

Question/Remark: The properties of topology on $\mathbb{C}$ we need to do this construction are:

(1) Multivariate polynomials give continuous maps. (This is implied by the fact that $\mathbb{C}$ is a topological field.)
(2) $\{0\} \subset \mathbb{C}$ is closed.

Examples of other fields are $\mathbb{Q}_p$, $\mathbb{R}$, $\mathbb{C}_p$, ...

Lemma 5.1. *If $f : X \to Y$ is a proper morphism of schemes lft over $\mathbb{C}$, then $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is proper ($\Leftrightarrow$ f closed and fibres are compact).*

Proof idea.

- Reduce to projective case by Chow's lemma: given $X \to Y$ proper, there exists a projective surjection $X' \twoheadrightarrow X$ such that the composition $X' \to Y$ is projective.
- Reduce to $\mathbb{P}^n_Y \to Y$ by second axiom.
- $(\mathbb{P}^n)^{\mathrm{an}} = \mathbb{P}^n(\mathbb{C})$ is compact. □

Lemma 5.2. *If $f : X \to Y$ is a morphism of schemes lft over $\mathbb{C}$, and f is* étale *at some $x \in X(\mathbb{C})$, then f^{an} is a local isomorphism at x: there exist $x \in U \underset{open}{\subset} X^{\mathrm{an}}$ and $f(x) \in V \underset{open}{\subset} Y^{\mathrm{an}}$ such that*

$$f^{\mathrm{an}}|_U : U \to V$$

is a homeomorphism.

Proof. We may shrink X and Y and assume they are affine and that f is étale.

First Proof. Reduce to $Y = \mathbb{A}^n_{\mathbb{C}}$; to do this pick

$$\begin{array}{ccc} X & \hookrightarrow & W^{\text{affine}} \\ \downarrow & & \downarrow \text{étale} \\ Y & \hookrightarrow & \mathbb{A}^n_{\mathbb{C}} \end{array}$$

such that $X = Y \times_{\mathbb{A}^n_{\mathbb{C}}} W$.

Lemma 5.3. *Let A be a ring, $I \subset A$ be an ideal and $A/I \to \overline{B}$ étale. Then there exists $A \to B$ étale such that $\overline{B} \cong B/IB$ as A/I-algebras.*

Proof. Write $\overline{B} = A/I[x_1, \cdots, x_n]/(\overline{f_1}, \cdots, \overline{f_n})$ with $\Delta = \det(\frac{\partial \overline{f_i}}{\partial x_j})$ invertible. Set

$$B = A[x_1, \cdots, x_n]/(f_1, \cdots, f_n)\left[\det\left(\frac{\partial f_i}{\partial x_j}\right)\right]^{-1}.$$ □

Then $W \to \mathbb{A}^n_{\mathbb{C}}$ is smooth of relative dimension 0 so we can write

$$W = \operatorname{Spec} \mathbb{C}[x_1, \cdots, x_n, z_1, \cdots, z_m]/(F_1(\underline{z}, \underline{x}), \cdots, F_m(\underline{z}, \underline{x}))$$

and $\det(\frac{\partial F_j}{\partial z_k})_{j,k=1,\cdots,m}$ invertible on W.

Implicit function theorem says

$$(W^{\mathrm{an}}, w) \to (\mathbb{C}^n, f(w))$$

is a local isomorphism. □

Second Proof. Use

Theorem 5.4 (Structure Theorem)**.** *Let $A \to B$ be a ring map étale at a prime $\mathfrak{q} \subset B$. Then there exists a $g \in B$ and $g \notin \mathfrak{q}$ such that $A \to B_g$ is* standard étale*, i.e.,*

$$B_g \cong (A[z]/(P))_Q$$

where $P, Q \in A[z]$, P is monic and $\frac{dP}{dz}$ is invertible in $(A[z]/(P))_Q$.

Suppose $P(z) = z^n + a_1 z^{n-1} + \cdots + a_n \in \mathbb{C}[z]$ and α is a simple root. Then there exists $\epsilon > 0$ such that for $|b_i| < \epsilon$, $i = 1, \cdots, n$, the polynomial

$$P_{b_1,\cdots,b_n}(z) = z^n + (a_1 + b_1)z^{n-1} + \cdots + (a_n + b_n)$$

has a simple root $\alpha(b_1, \cdots, b_n)$ depending continuously on $b_1, \cdots, b_n$ and converging to α as $b_i \mapsto 0$. This can be proved by Newton's method.

Returning to the proof, we can shrink the map $X \to Y$ near $x \mapsto y$ (where it is étale) to get a standard étale map. Then we can pick $P_y(z) \in \mathcal{O}_Y(Y)[z]$. □

Corollary 5.5. *Given X lft over $\mathbb{C}$, there is a functor*

$$\mathrm{F\acute{E}t}_X \to \{\textit{finite covering spaces of } X^{\mathrm{an}}\}$$
$$(Y \xrightarrow{f} X) \mapsto (Y^{\mathrm{an}} \xrightarrow{f^{\mathrm{an}}} X^{\mathrm{an}}).$$

Theorem 5.6 (Riemann Existence Theorem, SGA 1, Exp XII)**.** *This functor is an equivalence.*

Corollary 5.7. *Let X lft over $\mathbb{C}$ be connected. Then X^{an} is connected, and for $x \in X(\mathbb{C})$, there is an isomorphism*

$$\pi_1^{\mathrm{top}}(X^{\mathrm{an}}, x)^\wedge \xrightarrow{\cong} \pi_1(X, x),$$

where the left is the profinite completion of the topological fundamental group, and the right is the Grothendieck fundamental group.

Warning. Weird things can happen. For example, the exponential map

$$\exp : (\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} = \mathbb{C} \to \mathbb{G}_{m,\mathbb{C}}^{\mathrm{an}} = \mathbb{C}^\times$$

is not the analytification of any $f : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}$.

Sketch of proof for X smooth and projective.

Claim. For any $X/\mathbb{C}$ smooth there exists a unique structure of a complex manifold on X^{an} such that

- If $f : X \to Y$ is a morphism with X, Y smooth, then $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is holomorphic.
- If f is étale, then f^{an} is a local isomorphism of complex manifolds.
- If $X = \mathbb{A}^n_{\mathbb{C}}$, then get usual complex structure on $\mathbb{C}^n$.

Characterization of smooth morphism: Let $f : X \to S$ be a morphism of schemes. TFAE:

(1) f is smooth.
(2) For all $x \in X$, there exists $x \in U$ such that

$$\begin{array}{ccc} X & \hookleftarrow & U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \text{étale}} \\ S & \longleftarrow & \mathbb{A}^n_S \end{array}$$

commutes.

Let $X/\mathbb{C}$ be a projective and smooth variety. To show

$$\text{FÉt}_X \xrightarrow{\cong} \text{finite topological covers of } X^{\text{an}},$$

let $\pi : M \to X^{\text{an}}$ be a finite topological covering. This implies M has a unique structure of complex manifolds such that π is a local isomorphism of complex manifolds.

Let L be a positive line bundle on X^{an} coming from $X \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$, so $X^{\text{an}} \hookrightarrow \mathbb{P}^n(\mathbb{C})$ and $\mathcal{O}(r)$ with Fubini–Study metric). This means L has a metric whose curvature is positive, so π^*L is positive on M.

By the Kodaira Embedding Theorem, M is algebraic, say $M = Y^{\text{an}}$. (Namely you prove $H^0(M, L^{\otimes k})$ gives $M \hookrightarrow \mathbb{P}^n(\mathbb{C})$. By Chow's theorem, $M \hookrightarrow \mathbb{P}^n(\mathbb{C})$ is cut out by polynomials equations, which gives us Y.)

The problem left over: $Y^{\text{an}} = M \xrightarrow{\pi} X^{\text{an}}$ is equal to f^{an} for some $f : Y \to X$. We can either do Chow's theorem for the graph $\Gamma \hookrightarrow M \times X^{\text{an}}$, or use GAGA. □

Remark. More generally, if X and Y are projective smooth over $\mathbb{C}$, any holomorphic map $X^{\text{an}} \to Y^{\text{an}}$ is f^{an} for some $f : X \to Y$ algebraic.

5.2. **Applications.** Recall the profinite completion

$$\hat{\mathbb{Z}} = \lim_{n \geq 1} \mathbb{Z}/n\mathbb{Z} = \prod_{\ell \text{ prime}} \mathbb{Z}_\ell.$$

- $\pi_1(\mathbb{P}^n_{\mathbb{C}}) = \{1\}$.
- $\pi_1(\mathbb{A}^n_{\mathbb{C}}) = \{1\}$.
- $\pi_1(\mathbb{A}^n_{\mathbb{C}} \backslash \{0\}) = \begin{cases} \hat{\mathbb{Z}} & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$ ($\mathbb{A}^n_{\mathbb{C}} \backslash \{0\}$ is homeomorphic to S^1 and S^{2n-1} in these two cases respectively.)
- $\pi_1(\mathbb{P}^1_{\mathbb{C}} \backslash \{0, 1, \infty\})$ = profinite completion of free group on 2 generators. This is something we don't know how to prove without using topology!

6. Lecture 6 (September 24, 2015)

6.1. **Two more examples.**

- If X is a smooth projective genus g curve over $\mathbb{C}$, then

$$\pi_1(X) \cong \text{profinite completion of the free group on } \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g$$
$$\text{subject to } [\alpha_1, \beta_1][\alpha_2, \beta_2] \cdots [\alpha_g, \beta_g] = 1.$$

- If $A/\mathbb{C}$ is an abelian variety of dimension g, then

$$\pi_1(A) \cong \hat{\mathbb{Z}}^{\oplus 2g}.$$

We don't know how to prove the first one algebraically. Philosophically, it is interesting to prove these without topological or analytic methods not because they are worse than algebraic proofs, but because new proofs give new insights!

Today we will study the fundamental groups of normal schemes and their relationships to Galois groups.

6.2. **Fundamental groups of normal schemes.**

Lemma 6.1. *Let A be a normal* Noetherian *domain with fraction field K. Let L/K be finite separable. Then the integral closure B of A in L is* finite *over A.*

Example 6.2. $A = \mathbb{Z} \subset \mathbb{Q} \subset K$ a number field. Then $\mathcal{O}_K$ is finite over $\mathbb{Z}$.

There are counterexamples if A is not assumed to be normal.

Proof. L/K is separable if and only if the trace pairing $Q_{L/K} : L \times L \to K$ given by $(x, y) \mapsto \mathrm{Tr}_{L/K}(xy)$ is nondegenerate. Choose a K-basis $\beta_1, \cdots, \beta_n$ for L. After multiplying by an element of A, we may assume $\beta_i \in B$. Let $\beta_1^\vee, \cdots, \beta_n^\vee \in L$ be the dual basis with respect to $Q_{L/K}$.

Fact. $\mathrm{Tr}_{L/K}(B) \subset A$.

Reason. The minimal polynomial of an element of B has coefficients in A (as A is normal). Then $\mathrm{Tr}_{L/K}(-)$ is expressible in terms of coefficients of the minimal polynomial.

This implies

$$B \subset A\beta_1^\vee + \cdots A\beta_n^\vee$$

(as can be seen by taking dual on $\bigoplus A\beta_i \subset B$, which gives $B^\vee \subset \bigoplus A\beta_i^\vee$). The right hand side is a finite A-module. Since A is Noetherian, B is a finite A-module as well. □

In particular, this lemma applies to geometric rings.

Corollary 6.3. *Let X be a normal Noetherian integral scheme with function field K. Let L/K be finite separable. There exists a finite dominant morphism $Y \to X$ with Y normal and integral such that the function field* f.f.(Y) *is L as extension of $K =$* f.f.(X).

Proof. Lemma over affine opens, and glue. □

Definition 6.4. The Y of Corollary is called the *normalization of X in L.*

More generally, without the Noetherian assumption, we get a normalization that is not necessarily finite.

Definition 6.5. We say X *is unramified in L* if the morphism $Y \to X$ of Corollary is unramified.

Lemma 6.6. *In situation of Corollary, we have $Y \to X$ unramified if and only if $Y \to X$ is étale.*

Proof. See Lenstra or Stacks project. For an easy proof, use the following facts:

- Structure theorem of finite unramified morphisms (earlier in lectures).
- Normalization commutes with smooth (étale) base change.
- Closed immersion is never a normalization. □

Proposition 6.7. *Let X be a Noetherian normal integral scheme with function field K. Then*

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) = \pi_1(\mathrm{Spec}(K), \mathrm{Spec}(\overline{K})) \to \pi_1(X, \mathrm{Spec}(\overline{K}))$$

is surjective and this identifies $\pi_1(X, \mathrm{Spec}(\overline{K}))$ with the quotient

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Gal}(M/K),$$

where M is the compositum of all finite $K \subset L \subset K^{\text{sep}}$ such that X is unramified in L, and M/K is Galois.

The key ingredient to the proof is:

Lemma 6.8. *If $Y \to X$ is finite étale and Y is connected, then Y is normal and integral and is the normalization of X in* f.f.(Y).

Fact (already used in Lemma 6.6). If $f : Y \to X$ is étale, then

$$X \text{ normal} \Rightarrow Y \text{ normal}.$$

(The converse is true if f is surjective.)

Example 6.9. $\pi_1(\text{Spec}(\mathbb{Z})) = \{1\}$, since the maximal unramified extension $\overline{\mathbb{Q}}/M/\mathbb{Q}$ is $\mathbb{Q}$.

Example 6.10. $\pi_1(\mathbb{A}^1_{\mathbb{C}}) = \{1\}$. We saw this last time using topology ($\mathbb{C}$ is a contractible space), but now we can use algebra. Suppose there is an n-to-1 cover. The ramification contribution is at most $n-1$, so

$$2g_C - 2 \leq -2 \cdot n + (n-1).$$

This gives a contradiction if $n > 1$.

Example 6.11. $\pi_1(\mathbb{A}^1_{\overline{\mathbb{F}_p}}) \neq \{1\}$. A map $\mathbb{A}^1_{\overline{\mathbb{F}_p}} \to \mathbb{A}^1_{\overline{\mathbb{F}_p}}$ given by $x \mapsto f(x)$ is unramified if $f'(x) \neq 0$ for all x. An example is $f(x) = x^p - x$. Look up Artin–Schreier coverings. In fact,

$$\text{Hom}\left(\pi_1(\mathbb{A}^1_{\overline{\mathbb{F}_p}}), \mathbb{Z}/p\mathbb{Z}\right) \cong \Gamma(\mathbb{A}^1_{\overline{\mathbb{F}_p}}, \mathcal{O})/(\text{subgroup of } f^p - f).$$

6.3. **Action of Galois groups on fundamental groups.**

Proposition 6.12. *Let k be a perfect field. Let X be a quasicompact and quasiseparated scheme over k. Assume $X_{\overline{k}}$ is connected (i.e., X is geometrically connected over k). Pick $\overline{\xi} \in X_{\overline{k}}$. Then there exists a short exact sequence*

$$1 \to \pi_1(X_{\overline{k}}, \overline{\xi}) \to \pi_1(X, \overline{\xi}) \to \pi_1(\text{Spec}(k), \overline{\xi}) \to 1.$$

The first term is the geometric fundamental group of X, and the last term is $\text{Gal}(k^{\text{sep}}/k)$. The middle term is sometimes called the arithmetic fundamental group.

Example 6.13. $X = \mathbb{P}^1_{\mathbb{Q}} \backslash \{0, 1, \infty\}$. Then

$$1 \to \hat{F}_2 \to \pi_1(X) \to \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1,$$

where F_2 is the free group on 2 generators. Here we have used:

Fact. $\pi_1(\mathbb{P}^1_{\mathbb{C}} \backslash \{0, 1, \infty\}) \xrightarrow{\cong} \pi_1(\mathbb{P}_{\overline{\mathbb{Q}}} \backslash \{0, 1, \infty\})$.

The exact sequence gives a continuous group homomorphism

$$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \text{Out}(\hat{F}_2) = \frac{\text{Aut}(\hat{F}_2)}{\text{Inn}(\hat{F}_2)}.$$

The method is to prove corresponding results for the functors

$$\text{FÉt}_{X_{\overline{k}}} \leftarrow \text{FÉt}_X \leftarrow \text{FÉt}_{\text{Spec}(k)}.$$

7. Lecture 7 (September 29, 2015)

7.1. **Short exact sequence of fundamental groups.** Today we will consider the fundamental group of a scheme X over k.

$$\begin{array}{ccc} X_{\overline{k}} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\overline{k}) & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

If $X_{\overline{k}}$ is connected, then there is a short exact sequence of fundamental groups. But before involving the geometric side, we will consider what this means for Galois categories.

Consider functors of Galois categories

$$\mathcal{C} \overset{H}{\to} \mathcal{C}' \overset{H'}{\to} \mathcal{C}'',$$

which give rise to the 2-commutative diagram

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{H} & \mathcal{C}' & \xrightarrow{H'} & \mathcal{C}'' \\ F\downarrow\cong & & F'\downarrow\cong & & F''\downarrow\cong \\ \text{Finite-}G\text{-Sets} & \xrightarrow{h} & \text{Finite-}G'\text{-Sets} & \xrightarrow{h'} & \text{Finite-}G''\text{-Sets} \end{array}$$

We want to study when the sequence of profinite groups

$$G \overset{h}{\leftarrow} G' \overset{h'}{\leftarrow} G''$$

is exact.

Properties of these two diagrams are related as follows:

$G \overset{h}{\leftarrow} G' \overset{h'}{\leftarrow} G''$	$\mathcal{C} \overset{H}{\to} \mathcal{C}' \overset{H'}{\to} \mathcal{C}''$
h surjective	(A) H fully faithful $\Leftrightarrow$ $X \in \mathcal{C}$ connected and if there exists $*_{\mathcal{C}'} \to H(X)$, then $X = *_{\mathcal{C}}$.
h' injective	(B) For all $X'' \in \mathcal{C}''$ connected, there exists $H'(X') \overset{\text{mono}}{\leftarrow} Y'' \overset{\text{epi}}{\to} X''$.
$h \circ h'$ trivial	(C) $H'(H(X)) \cong \coprod_{j=1,\cdots,m} *_{\mathcal{C}''}$ for all $X \in \mathcal{C}$.
$\mathrm{Im}(h')$ normal	(D) For all $X \in \mathcal{C}'$ connected, if there exists $*_{\mathcal{C}''} \to H'(X')$ then $H'(X') \cong \coprod_{j=1,\cdots,m} *_{\mathcal{C}''}$.
h surjective + kernel h is smallest normal closed subgroup containing $\mathrm{Im}(h')$.	(E) H fully faithful + essential image of H is exactly those X' such that $H'(X') \cong \coprod_{j=1,\cdots,m} *_{\mathcal{C}''}$.

We can prove that:

Fact. (E) $\Leftrightarrow$ (A) + (C) + ($H'(X') \cong \coprod *_{\mathcal{C}''} \Rightarrow$ there exists $X \in \mathcal{C}$ and an epi $H(X) \twoheadrightarrow X'$).

Now apply this to

$$\mathrm{F\acute{E}t}_{\mathrm{Spec}(k)} \overset{H}{\to} \mathrm{F\acute{E}t}_X \overset{H'}{\to} \mathrm{F\acute{E}t}_{X_{\overline{k}}}$$

with X/k quasicompact, quasiseparated and geometrically connected (for example, a geometrically connected k-variety) and k perfect.

(A) Have to show: for k'/k finite separable field extension we have $\mathrm{Spec}(k') \times_{\mathrm{Spec}(k)} X = X_{k'}$ is connected. This is clear because $X_{\overline{k}} \twoheadrightarrow X_{k'}$ and $X_{\overline{k}}$ is connected by assumption.

(B) First we have the following

Claim. For every $Z \to X_{\overline{k}}$ finite étale there exists a $\overline{k}/k'/k$ finite and $Y \to X_{k'}$ finite étale such that $Z \cong Y \times_{X_{k'}} X_{\overline{k}} = Y \otimes_{k'} \overline{k}$.

If the claim is true, then the composition $Y \to X_{k'} \to X$ is finite étale and

$$Z = \text{union of connected components of } Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$$

The multiplication map $k' \otimes_k \overline{k} \to \overline{k}$ gives a connected component

$$\mathrm{Spec}(\overline{k}) \hookrightarrow \mathrm{Spec}(k') \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}),$$

and Z is the fibre product

$$\begin{array}{ccc} Z & \xhookrightarrow{\text{open+closed}} & Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\overline{k}) & \xhookrightarrow{\text{open+closed}} & \mathrm{Spec}(k') \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}) \end{array}$$

i.e., $Z = Y \times_{\mathrm{Spec}(k')} \mathrm{Spec}(\overline{k})$.

About the claim: $\overline{k} = \mathrm{colim}_{\overline{k}/k'/k}\, k'$, so

$$\mathrm{Spec}(\overline{k}) = \lim_{\overline{k}/k'/k} \mathrm{Spec}(k')$$

and

$$X_{\overline{k}} = \lim X_{k'}$$

in the category of schemes.

In general, if $(I, \geq)$ is a directed partially ordered set, and $(X_i, \varphi_{ii'})$ is an inverse system of schemes over I such that $\varphi_{ii'}$ are affine, then $X = \lim_{i \in I} X_i$ exists in schemes.

Lemma 7.1. *In this situation, if X_i is quasicompact and quasiseparated for all i, then $\mathrm{F\acute{E}t}_X = \mathrm{colim}\, \mathrm{F\acute{E}t}_{X_i}$, and if X_i is connected for all $i \gg 0$, then $\pi_1(X) = \lim \pi_1(X_i)$.*

Affine case. Suppose $A = \mathrm{colim}\, A_i$ filtered. Then

$$\left\{\begin{matrix}\text{category of } A\text{-algebras} \\ \text{of finite presentation}\end{matrix}\right\} = \mathrm{colim} \left\{\begin{matrix}\text{category of } A_i\text{-algebras} \\ \text{of finite presentation}\end{matrix}\right\}.$$

An A-algebra of finite presentation looks like $A[x_1, \cdots, x_n]/(f_1, \cdots, f_m)$, where $f_j = \sum_{\text{finite}} a_{j,I} x^I$. Pick i large enough such that $a_{j,I} \in \mathrm{Im}(A_i \to A)$.

The same holds for categories of étale, smooth, finite+fp, flat+fp, and combinations of these. ∎

The lemma implies the claim by above.

(C) This is clear from

$$\begin{array}{ccc} X_{\overline{k}} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\overline{k}) & \longrightarrow & \operatorname{Spec}(k) \end{array}$$

and the fact that $\operatorname{Spec}(\overline{k})$ has trivial π_1.

(D) Suppose $U \to X$ finite étale, U connected, and there is a section $s : X_{\overline{k}} \to U_{\overline{k}}$.

$$\begin{array}{ccc} U_{\overline{k}} & \longrightarrow & U \\ s \uparrow\downarrow & & \downarrow \\ X_{\overline{k}} & \longrightarrow & X \end{array}$$

Then you can consider

$$\overline{T} = \bigcup_{\sigma \in \operatorname{Gal}(\overline{k}/k)} s^{\sigma}(X_{\overline{k}}) \subset U_{\overline{k}}$$

which is open and closed, and $\operatorname{Gal}(\overline{k}/k)$-invariant. With some work, $\overline{T}$ is the inverse image of an open and closed $T \subset U$. Since U is connected, this implies $T = U$.

(E) Suppose $U \to X$ is finite étale and $U_{\overline{k}} \cong \coprod_{i=1,\cdots,n} X_{\overline{k}}$. Then there exists $\overline{k}/k'/k$ (with k'/k finite) such that

$$U_{k'} \cong \coprod_{i=1,\cdots,n} X_{k'}.$$

Then

$$U \leftarrow U_{k'} \cong \coprod_{i=1,\cdots,n} X_{k'} = X \times_{\operatorname{Spec}(k)} \left(\coprod_{i=1,\cdots,n} \operatorname{Spec} k' \right)$$

where the last coproduct is in FÉt_k.

Remark. We've proved the exact sequence

$$1 \to \pi_1(X_{k^{\text{sep}}}) \to \pi_1(X) \to \operatorname{Gal}(k^{\text{sep}}/k) \to 1$$

for X/k as before and k not necessarily perfect.

Fact. $\pi_1(X_{\overline{k}}) = \pi_1(X_{k^{\text{sep}}})$.

Example 7.2. Let $X = \mathbb{G}_{m,k} = \mathbb{A}^1_k \backslash \{0\}$ with $\operatorname{char}(k) = 0$. Then $X_{\overline{k}} = \mathbb{G}_{m,\overline{k}}$ and consider

$$X_n = \mathbb{G}_{m,\overline{k}} \xrightarrow{(\cdot)^n} \mathbb{G}_{m,\overline{k}} = X_{\overline{k}}$$
$$x \mapsto x^n.$$

Using Riemann–Hurwitz, you can show these are cofinal in $\text{FÉt}_{X_{\overline{k}}}$. This implies

$$\pi_1(X_{\overline{k}}) = \lim_{n} \mu(\overline{k}) \cong \hat{\mathbb{Z}}(1).$$

So we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(X_{\overline{k}}) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(\mathrm{Spec}(k)) & \longrightarrow & 1 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & \hat{\mathbb{Z}} & \longrightarrow & ? & \longrightarrow & \mathrm{Gal}(\overline{k}/k) & \longrightarrow & 1 \end{array}$$

and get

$$\mathrm{Gal}(\overline{k}/k) \to \mathrm{Out}(\mathbb{Z}) = \hat{\mathbb{Z}}^{\times}.$$

This is the *cyclotomic character*.

Example 7.3. If $X \to \mathrm{Spec}(k)$ is an elliptic curve with identity $\mathcal{O} : \mathrm{Spec}(k) \to X$ and $\mathrm{char}(k) = 0$, then

$$\pi_1(X_{\overline{k}}) \cong \hat{\mathbb{Z}}^{\oplus 2}.$$

Again multiplying by n gives a cofinal system in $\mathrm{F\acute{E}t}_{X_{\overline{k}}}$. The exact sequence is

$$0 \longrightarrow \hat{\mathbb{Z}}^{\oplus 2} \longrightarrow \pi_1(X) \underset{\curvearrowleft \mathcal{O}_X}{\longrightarrow} \mathrm{Gal}(\overline{k}/k) \longrightarrow 1.$$

Then

$$\mathrm{Gal}(\overline{k}/k) \to \mathrm{Aut}(\hat{\mathbb{Z}}^{\oplus 2}) = \prod_{\ell} \mathrm{GL}_2(\mathbb{Z}_\ell).$$

8. Lecture 8 (October 1, 2015)

Today we will talk about ramification theory and topological invariance of the étale topology.

8.1. **Ramification theory.** A will be a discrete valuation ring with fraction field $K = \mathrm{f.f.}(A)$. Let L/K be finite separable, and B the integral closure of A in L. By a previous lemma, we know B is finite over A. Ramification theory will take some work to prove, and the proofs will be omitted.

Fact. B is a Dedekind domain with a finite number of maximal ideals $\mathfrak{m}_1, \cdots, \mathfrak{m}_n$.

Throughout today's lecture n will stand for the number of maximal ideals.

$A \subset B_{\mathfrak{m}_i}$ is an extension of DVRs; let e_i be the ramification index and $f_i = [\kappa(\mathfrak{m}_i), \kappa_A]$.

Fact. $[L : K] = \prod_{i=1}^{n} e_i f_i$.

Fact. If A is complete or more generally henselian, then $n = 1$.

Definition 8.1. We say L/K is:

- *unramified with respect to* $A \Leftrightarrow$ all $e_i = 1$ and $\kappa(\mathfrak{m}_i)/\kappa_A$ is separable.
- *tamely ramified with respect to* $A \Leftrightarrow$ all e_i are prime to the characteristic of κ_A and $\kappa(\mathfrak{m}_i)/\kappa_A$ is separable.
- *totally ramified with respect to* $A \Leftrightarrow n = 1$ and $f_1 = 1$.

Assume L/K Galois with $G = \mathrm{Gal}(L/K)$.

Fact. G acts on transitively on $\{\mathfrak{m}_1, \cdots, \mathfrak{m}_n\} \Rightarrow e_1 = \cdots = e_n = e$, $f_1 = \cdots = f_n = f$ and $[L : K] = nef$.

Pick $\mathfrak{m} = \mathfrak{m}_1$. Set

$$\{1\} \subset P \subset I \subset D \subset G$$

where

- $D = \{\sigma \in G \mid \sigma(\mathfrak{m}) = \mathfrak{m}\}$ is the *deccomposition group of* $\mathfrak{m}$.
- $I = \{\sigma \in D \mid \sigma \mod \mathfrak{m} = \mathrm{id}_{\kappa(\mathfrak{m})}\}$ is the *inertia group of* $\mathfrak{m}$.
- $P = \{\sigma \in D \mid \sigma \mod \mathfrak{m}^2 = \mathrm{id}\}$ is the *wild inertia group of* $\mathfrak{m}$.

Fact.

- $[G : D] = n$
- $\kappa(\mathfrak{m})/\kappa_A$ is normal.

  Warning. Not necessarily separable.
- $I \lhd D$ and $D/I \xrightarrow{\sim} \mathrm{Aut}(\kappa(\mathfrak{m})/\kappa_A)$.
- $P \lhd D$ and $P = \{1\}$ if $\mathrm{char}(\kappa_A) = 0$; P is a p-group if $\mathrm{char}(\kappa_A) = p > 0$.
- I/P is cyclic of order the prime-to-p part of e. (There is a canonical isomorphism $I/P \xrightarrow{\sim} \mu_e(\kappa(\mathfrak{m}))$.)

Definition 8.2. $I_t = I/P$ is the *tame inertia group of* $\mathfrak{m}$.

Here is an application. If A is henselian (so $n = 1$), then

$$P \subset I \subset D = G = \mathrm{Gal}(K^{\mathrm{sep}}/K)$$

by passing to the limit. (Write $K^{\mathrm{sep}} = \bigcup L$ where L runs over $K^{\mathrm{sep}}/L/K$ Galois, then for each L we get

$$P_L \subset I_L \subset D_L = \mathrm{Gal}(L/K)$$

and then

$$P = \lim P_L \to I = \lim I_L \to \mathrm{Gal}(K^{\mathrm{sep}}/K) = \lim \mathrm{Gal}(L/K) = \lim D_L.)$$

Both P and I are normal in G and

(1) $G/I \cong \mathrm{Gal}(\kappa_A^{\mathrm{sep}}/\kappa_A)$.
(2) $I_t = I/P \underset{(*)}{\cong} \prod_{\ell \neq \mathrm{char}(\kappa_A)} \mathbb{Z}_\ell$.
(3) The short exact sequence

$$1 \to I_t \to G/P \to \mathrm{Gal}(\kappa_A^{\mathrm{sep}}/\kappa_A) \to 1$$

gives an action of $\mathrm{Gal}(\kappa_A^{\mathrm{sep}}/\kappa_A)$ on I_t which is via the cyclotomic character.
(4) $P = \{1\}$ if $\mathrm{char}(\kappa_A) = 0$.

The isomorphism (*) is noncanonical. Canonically,

$$I_t = \lim_{e} \mu_e(\kappa_A^{\mathrm{sep}}).$$

We will soon explain why $I_L/P_L \cong \mu_{e_L}(\kappa_L)$ at the finite level.

Example 8.3. Let $A = \kappa[[t]]$ with $\kappa = \overline{\kappa}$ of characteristic 0. The extensions are $L_e = K(t^{1/e})$ ($e \geq 1$) with $\mathrm{Gal}(L_e/K) = \mu_e(\kappa)$ via $(t^{1/e} \mapsto \zeta t^{1/e}) \leftrightarrow \zeta$. (Note $A = \kappa[[t]] \subset B_e = \kappa[[t^{1/e}]]$.) Given $\sigma \in \mathrm{Gal}(L_e/K)$, the corresponding root of unity is the ζ_σ such that $\sigma(t^{1/e}) = \zeta_\sigma t^{1/e}$.

Remark. Back in the finite case, the map

$$\theta : I/P \to \mu_e(\text{residue field } \kappa(\mathfrak{m}) \text{ of } \mathfrak{m} \subset B)$$

is given by the rule that

$$\sigma(\pi) = \widetilde{\theta(\sigma)} \cdot \pi \mod \pi^2 B_{\mathfrak{m}}$$

where $\pi \in B_{\mathfrak{m}}$ is a uniformizer and $\widetilde{\theta(\sigma)} \in B_{\mathfrak{m}}$ is any lift of $\theta(\sigma) \in \kappa(\mathfrak{m})$. The reason this works is that $\pi^e = (\text{unit})\pi_A$.

We are interested in studying the fundamental groups of curves using this. We will see how little we get and this is disappointing!

Example 8.4. Let $\overline{X}$ be a smooth curve over $k = \overline{k}$, $\text{char}(k) = 0$. Let $X = \overline{X} \backslash \{x_1, \cdots, x_m\}$. Apply with $A_i = \widehat{\mathcal{O}}_{X,x_i}$ (completion) or $A_i = \mathcal{O}^h_{X,x_i}$ (henselization). We get

$$a_i : \text{Spec}(\text{f.f.}(A_i)) \to X.$$

Hence

$$\hat{\mathbb{Z}} \cong \pi_1(\text{Spec}(\text{f.f.}(A_i))) \stackrel{\pi_1(a_i)}{\to} \pi_1(X),$$

well-defined up to inner conjugation on the right hand side, measures the ramification of finite étale cover above x_i. It is possible to choose base-points to make this well-defined, but there is no canonical choice.

Lemma 8.5. *The kernel of the surjective map*

$$\pi_1(X) \to \pi_1(\overline{X})$$

is the smallest closed normal subgroup of $\pi_1(X)$ *containing the images of* $\pi_1(a_i)$.

Corollary 8.6. *There exists a short exact sequence*

$$\hat{\mathbb{Z}}^{\oplus m} \to \pi_1(X)^{\text{ab}} \to \pi_1(\overline{X})^{\text{ab}} \to 0.$$

Example 8.7. Let $\overline{X} = \mathbb{A}^1_k$ and $X = \mathbb{G}_{m,k}$, so $m = 1$ and $x_1 = 0$. Suppose you knew the lemma and $\pi(\mathbb{A}^1_k) = \{1\}$. Then $\pi_1(X)$ is topologically generated by 1 conjugacy class. This is rather weak.

If we use that $\pi_1(X) \cong \hat{\mathbb{Z}}$ (see earlier), then

$$\text{inertia at } 0 \stackrel{\cong}{\to} \pi_1(X) \stackrel{\cong}{\leftarrow} \text{inertial at } \infty.$$

The isomorphism between the two inertia is via inverse. This is compatible with the topological picture.

In characteristic p, we can do a similar thing but we need to use the tame inertia group instead.

8.2. **Topological invariance of étale topology.**

Theorem 8.8. *Let* $f : X \to Y$ *be a* universal homeomorphism *of schemes. Then the base change functor*

$$\textit{schemes étale over } Y \to \textit{schemes étale over } X$$

is an equivalence. Same for $\text{FÉt}_Y \to \text{FÉt}_X$ *and if* X *and* Y *are connected then* $\pi_1(X) \stackrel{\cong}{\to} \pi_1(Y)$.

Definition 8.9. $f : X \to Y$ is a *universal homeomorphism* if for all $Y' \to Y$, $X \times_Y Y' \to Y'$ is a homeomorphism.

Fact. f is a universal homeomorphism if and only if f is integral, surjective and universally injective.

Example 8.10.

(1) Thickenings: $i : X \to X'$ is called a *thickening* if and only if i is a closed immersion and bijective on points.

Example 8.11. $\mathrm{Spec}(A/I) \hookrightarrow \mathrm{Spec}(A)$ if $I \subset A$ is an ideal and:
- $I^2 = 0$ (square zero ideals), or
- $I^n = 0$ for some n (nilpotent ideal), or
- for all $x \in I$, there exists $n = n(x)$ such that $x^n = 0$ (locally nilpotent ideal).

Étale cohomology doesn't see multiplicities.

(2) Purely inseparable field extensions.

(3) Suppose $f : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ where A, B are $\mathbb{F}_p$-algebras, and there exists a $g : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ such that

$$f \circ g = \mathrm{Spec}(B \to B, b \mapsto b^q),$$
$$g \circ f = \mathrm{Spec}(A \to A, a \mapsto a^q),$$

where $q = p^n$. Here you get the theorem because base change by g will be the quasi-inverse functor.

Example 8.12. Let $A = \mathbb{F}_p[x_1, \cdots, x_n]/I$ and $B = \mathbb{F}_p[x_1, \cdots, x_n]/I^{39}$. Clearly there is a surjection $B \to A$. To get a map $A \to B$, we send $x_i \mapsto x_i^{p^m}$ where m is such that $p^m > 39$. We can always use this if $f : X \to Y$ is a finite morphism of varieties over $\mathbb{F}_q$ inducing a purely inseparable function field extension.

9. Lecture 9 (October 6, 2015)

9.1. **Special case of Theorem last time.** If $X_0 \hookrightarrow X$ is a closed immersion of schemes inducing a homeomorphism $|X_0| \to |X|$ on the underlying topological spaces, then the functor

$$\begin{aligned} \mathrm{F\acute{E}t}_X &\to \mathrm{F\acute{E}t}_{X_0} \\ Y &\mapsto Y_0 = X_0 \times_X Y \end{aligned}$$

is an equivalence.

Remark. Even $X_{\text{ét}} \xrightarrow{\cong} X_{0,\text{ét}}$ (small étale sites).

We often call this the topological equivalence of étale topology. For example, we see that the étale site of a scheme only depends on the reduction of the scheme.

Today we will prove the following

Theorem 9.1. *Let $f : X \to \mathrm{Spec}\, A$ be a proper morphism of schemes. If $(A, \mathfrak{m}, \kappa)$ is a* henselian *local ring and $X_0 = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\kappa)$, then the base change functor*

$$\mathrm{F\acute{E}t}_X \to \mathrm{F\acute{E}t}_{X_0}$$

$$Y \mapsto Y_0 = X_0 \times_X Y$$

is an equivalence.

This is an amazing theorem! I will spend some time discussing what this theorem means, how we will use it, and finally sketch the proof. If we want to prove this theorem starting from basic commutative algebra, it will take a year!

Remark.

(1) What does it mean? The fundamental group of the total space X is the same as the fundamental group of the special fibre:

$$\pi_1(X) \cong \pi_1(X_0).$$

This suggests that a small neighborhood of the special fibre X_0 contracts onto X_0. This is true in the $\mathbb{C}$-world.

(2) How will we use it? If $\eta \in \operatorname{Spec}(A)$ is a generic point then we (this "we" is really Grothendieck) will look at

$$\pi_1(X_{\overline{\eta}}) \to \pi_1(X_\eta) \to \pi_1(X) \overset{\text{Thm}}{=} \pi_1(X_0).$$

This composition is called the specialization map. This allows us to compare the fundamental group of the generic fibre with that of the special fibre. In characteristic p, this map has a kernel.

9.2. **Structure of proof.** I will focus on "essential surjectivity". Assume $Y_0 \to X_0$ finite étale is given and we try to construct $Y \to X$ finite étale with $Y_0 = X_0 \times_X X$.

The argument is standard, and the other parts of this proof (such as full faithfulness) will involve similar ingredients:

- deformation (A will be complete local Noetherian),
- algebraization (A will be complete local Noetherian),
- approximation (A will be excellent and henselian),
- limits (A will be arbitrary henselian).

9.2.1. *Deformation.* Assume $(A, \mathfrak{m}, \kappa)$ is complete local Noetherian. Set

$$X_n = \operatorname{Spec}(A/\mathfrak{m}^{n+1}) \times_{\operatorname{Spec}(A)} X.$$

Then we have a sequence of first-order thickenings

$$\cdots \hookleftarrow X_2 \hookleftarrow X_1 \hookleftarrow X_0.$$

By topological invariance, we get for each n a finite étale scheme $Y_n \to X_n$ and isomorphisms $X_n \times_{X_{n+1}} Y_{n+1} \xrightarrow{\sim} Y_n$ over X_n.

Remark. Can put these together to get a formal scheme

$$\mathcal{Y} = \text{“}\operatorname{colim} Y_n\text{”}.$$

9.2.2. *Algebraization.* We will use the

Theorem 9.2 (Grothendieck's existence theorem)**.** *Suppose given $X \to \operatorname{Spec}(A)$ proper, A Noetherian complete local and*

$$\begin{array}{ccccccc} \cdots \hookleftarrow & Z_2 & \hookleftarrow & Z_1 & \hookleftarrow & Z_0 \\ & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_0} \\ \cdots \hookleftarrow & X_2 & \hookleftarrow & X_1 & \hookleftarrow & X_0 \end{array}$$

with cartesian squares and h_n finite, *then there exist $h : Z \to X$ finite and isomorphisms $Z_n = X_n \times_X Z$ compatible with h, h_n and the squares.*

This can be deduced from:

Theorem 9.3 (Sheaf version)**.** *Given $\mathcal{F}_n$* coherent *$\mathcal{O}_{X_n}$-modules and isomorphisms $\mathcal{F}_{n+1} \otimes_{\mathcal{O}_{X_{n+1}}} \mathcal{O}_{X_n} \xrightarrow{\sim} \mathcal{F}_n$, there exist a coherent $\mathcal{O}_X$-module $\mathcal{F}$ and isomorphisms $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n} \xrightarrow{\sim} \mathcal{F}_n$ compatible with transition maps.*

Back to our Theorem in case $(A, \mathfrak{m}, \kappa)$ is complete local Noetherian. Let $Y \to X$ be the finite morphism such that $Y \times_X X_n \cong Y_n$ for all n where Y_n is as in deformation step. Then $Y \to X$ is étale by:

Lemma 9.4. *Let $f : Y \to X$ be lft with X locally Noetherian. Let $X_0 \hookrightarrow X$ be a closed subscheme with n-th infinitesimal neighborhood $X_n \hookrightarrow X$. If $X_n \times_X Y \to X_n$ are smooth, resp. étale for all $n \gg 0$, then f is smooth, resp. étale at all points of $f^{-1}(X_0)$.*

Finally: $X \to \operatorname{Spec}(A)$ proper and A local. Then any open neighborhood of X_0 is all of X.

9.2.3. *Approximation (Artin, ..., Popescu).*

Lemma 9.5. *Let $(A, \mathfrak{m}, \kappa)$ be a local ring. The* henselization *is*

$$A^h = \operatorname*{colim}_{(A \to B, \mathfrak{q})} B = \operatorname*{colim}_{(A \to B, \mathfrak{q})} B_{\mathfrak{q}}$$

where the colimit is over the category of $A \to B$ étale and $\mathfrak{q} \subset B$ prime lying over $\mathfrak{m}$ such that $\kappa = \kappa(\mathfrak{m}) = \kappa(\mathfrak{q})$. The index category is filtered.

Note that a filtered colimit of local rings is a local ring.

Definition 9.6. A local ring $(A, \mathfrak{m}, \kappa)$ is *henselian* if Hensel's lemma holds: for all $P \in A[x]$ monic, and $\alpha_0 \in \kappa$ a simple root of $\overline{P} \in \kappa[x]$, there exists a root $\alpha \in A$ of P such that $\alpha \bmod \mathfrak{m} = \alpha_0$.

Lemma 9.7. *A^h is henselian.*

Lemma 9.8. *If A is Noetherian, then*

$$A \subset A^h \subset A^\wedge = (A^h)^\wedge$$

are Noetherian and both inclusions are flat (hence faithfully flat).

Fact (Artin–Popescu)*.* If A is an excellent Noetherian local ring (for example the local ring of a scheme of finite type over a field or $\mathbb{Z}$, or over any Dedekind domain of characteristic 0), then *approximation* holds for $A^h \subset A^\wedge = (A^h)^\wedge$.

Deformation theory combined with algebraization produces objects over the completion $A^\wedge$, and now we want to descend to A^h.

Definition 9.9. Let R be a Noetherian ring, $I \subset R$ an ideal and $R^\wedge = \lim R/I^n$ be the completion. We say *approximation holds* for $R \subset R^\wedge$ if and only if for all $N \geq 1$, all n, m, all $f_1, \cdots, f_m \in R[x_1, \cdots, x_n]$, and all $b_1, \cdots, b_n \in R^\wedge$ such that $f_j(b_1, \cdots, b_n) = 0$ for $j = 1, \cdots, m$, there exist $a_1, \cdots, a_n \in R$ such that

$$f_j(a_1, \cdots, a_n) = 0$$

for $j = 1, \cdots, m$, and $a_i \equiv b_i \mod I^N R^\wedge$.

With approximation, we see that bad properties of $A^\wedge$ are inherited from A^h, for examples zero divisors and nilpotence.

9.2.4. *Apply.* Theorem holds if $A = C^h$ where C is a Noetherian local ring essentially of finite type over $\mathbb{Z}$ (more generally if approximation holds for $A \to A^\wedge$).

Proof. By deformation and algebraization, we have $Y' \to \operatorname{Spec}(A^\wedge) \times_{\operatorname{Spec}(A)} X$ recovering Y_0 finite étale. Say $A^\wedge = \operatorname{colim} A_i$ with $A \to A_i$ of finite type. Then

$$\operatorname{Spec}(A^\wedge) \times_{\operatorname{Spec}(A)} X = \lim \operatorname{Spec}(A_i) \times_{\operatorname{Spec}(A)} X.$$

By a previous result, there exist i and $Y_i \to \operatorname{Spec}(A_i) \times_{\operatorname{Spec}(A)} X$ finite étale whose base change is Y'. Write (for the map coming from $A^\wedge = \operatorname{colim} A$)

$$\begin{aligned} A_i = A[x_1, \cdots, x_n]/(f_1, \cdots, f_m) &\to A^\wedge \\ x_i &\mapsto b_i \in A^\wedge \end{aligned}$$

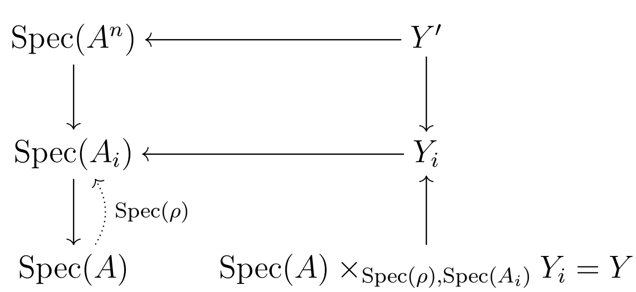
and $(b_1, \cdots, b_n)$ is a solution!

By approximation, we get $(a_1, \cdots, a_n) \in A$ with $f_j(a_1, \cdots, a_n) = 0$ for $j = 1, \cdots, m$ and $a_i \equiv b_i \mod \mathfrak{m}A^\wedge$, so an A-algebra homomorphism

$$\begin{aligned} \rho : A_i = A[x_1, \cdots, x_n]/(f_j) &\to A \\ x_i &\mapsto a_i. \end{aligned}$$

Take

$$Y = \operatorname{Spec}(A) \times_{\operatorname{Spec}(\rho), \operatorname{Spec}(A_i)} Y_i.$$



Since $a_i \equiv b_i \mod \mathfrak{m}A^\wedge$, we get that the special fibre of Y is the same as the special fibre of Y', so equal to our given Y_0. □

9.2.5. *Limits.* Any A henselian is a filtered colimit of A's as above.

10. Lecture 10 (October 8, 2015)

10.1. **Purity of branch locus.** Suppose X is a curve over $\mathbb{Z}_p$, with special fibre X_0 (which is a curve in characteristic p), generic fibre X_η over $\mathbb{Q}_p$, and geometric fibre $X_{\overline{\eta}}$ over $\overline{\mathbb{Q}_p}$. We want to analyze how close $\pi_1(X_{\overline{\eta}})$ is to $\pi_1(X_0)$. This will have to do with the purity of branch locus, which is one of Grothendieck's big ideas.

Lemma 10.1. *Let $f : X \to Y$ be lft. Let $x \in X$ with image $y = f(x) \in Y$. TFAE:*

(1) *f is* quasifinite *at x.*
(2) *x is isolated in X_y ($\overset{def}{\Leftrightarrow}$ $\{x\}$ is open in X_y).*
(3) *x is closed in X_y and no $x' \rightsquigarrow x$ (specialization) in X_y except $x' = x$.*
(4) *For some (any) affine opens*

$$\begin{array}{ccc} \mathrm{Spec}(A) & \hookrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(B) & \hookrightarrow & Y \end{array}$$

and $x \in \mathrm{Spec}(A)$ corresponding to $\mathfrak{p} \subset A$, the ring map $B \to A$ is quasifinite at $\mathfrak{p}$.

Definition 10.2.

- $f : X \to Y$ is *locally quasifinite* (lqf) if f is quasifinite at all x.
- $f : X \to Y$ is *quasifinite* if f is locally quasifinite and quasicompact.

Let me state the first version of purity of branch locus, which is really about the ramification locus. Roughly speaking, "purity" means "if it happens, then it happens in codimension 1".

Lemma 10.3 (Easy case)**.** *Let $f : X \to Y$ and $x \in X$. Assume*

(1) *X and Y are locally Noetherian,*
(2) *f is lft and quasifinite at x,*
(3) *f is* flat*,*
(4) *x is not a generic point,*
(5) *for all $x' \rightsquigarrow x$ with $\dim(\mathcal{O}_{X,x'}) = 1$ we have f unramified at x'.*

Then f is étale at x.

The flatness assumption is what makes the proof easy.

Now we will translate this into algebra, which requires lots of technical garbage:

- Étale locally a quasifinite morphism becomes finite.
- Zariski's main theorem.

Lemma 10.4 (Algebra problem)**.** *Let $A \to B$ be a finite flat ring map, with A a Noetherian local ring. If*

- $\dim(A) \geq 2$*,*
- $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is étale over the punctured spectrum* $\mathrm{Spec}(A) \backslash \{\mathfrak{m}_A\}$*,*

then $A \to B$ is étale.

We can see how this is related to the above. We can look at $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. It is not finite, but maybe after some completion we get a finite map.

Proof. Flatness implies $B \cong A^{\oplus n}$ for some n, so we can define the trace pairing $Q_{B/A} : B \times B \to A$ by $(b_1, b_2) \mapsto \mathrm{Tr}_{B/A}(b_1 b_2)$ (trace of the matrix of multiplication by $b_1 b_2$ on B). Let

$$\mathrm{disc}_{B/A} = \det(\text{the matrix of } Q_{B/A} \text{ with respect to some basis}).$$

We know that the

$$\text{branch locus} := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{Spec}(B) \to \mathrm{Spec}(A) \text{ not étale at all points over } \mathfrak{p}\}$$

is equal to

$$\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{disc}_{B/A} \in \mathfrak{p}\} = V(\mathrm{disc}_{B/A})$$

(i.e., a prime of A is in the branch locus precisely when the trace pairing is degenerate over this prime).

If $d = \mathrm{disc}_{B/A} \in \mathfrak{m}$, then $\dim V(d) \geq 1$, so there exists $\mathfrak{p} \subset A$, $\mathfrak{p} \neq \mathfrak{m}$, $\mathfrak{p} \in V(d)$. This contradicts the assumption. Therefore $d \in A^\times$, which proves that f is étale. □

Lemma 10.5 (Difficult case)**.** *Let $f : X \to Y$, $x \in X$, $y = f(x)$. Assume*

(1) *X and Y are locally Noetherian,*
(2) *$\mathcal{O}_{X,x}$ is normal,*
(3) *$\mathcal{O}_{Y,y}$ is regular,*
(4) *f is qf at x,*
(5) $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y}) \geq 1$,
(6) *for all $x' \rightsquigarrow x$ with* $\dim(\mathcal{O}_{X,x}) = 1$*, f is unramified at x'.*

Then f is étale at x.

Again it takes lots of technical garbage to translate this into the following

Lemma 10.6 (Algebra problem)**.** *Let $A \subset B$ be a finite extension, with A regular local ring of* $\dim \geq 2$ *and B normal. If* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is étale over the punctured spectrum, then $A \to B$ is étale.*

Sketch of proof.

- Case I: $\dim(A) = 2$. In this case $A \to B$ will be flat. This is because of the

  Fact.
  (1) Let $f : X \to Y$ be a qf and dominant morphism of integral schemes, with Y regular, $\dim Y = 2$ and X normal. Then f is flat.
  (2) Let $f : X \to Y$ be a qf and dominant morphism of integral schemes, with Y regular and X Cohen–Macauley. Then f is flat.

  The idea is that if x_1, x_2 are a regular system of parameters of A, then $\varphi(x_1)$ and $\varphi(x_2)$ form a regular sequence in B. Then use $\mathfrak{m}_B = \sqrt{\mathfrak{m}_A B}$.
- Case II: $\dim(A) \geq 3$. To prove this Grothendieck uses a local Lefschetz theorem. Pick $f \in \mathfrak{m}_A \backslash \mathfrak{m}_A^2$. Contemplate

$$\begin{array}{ccc} \mathrm{Spec}(A) = X & \hookleftarrow & X_0 = \mathrm{Spec}(A/fA) \\ \uparrow & & \uparrow \\ \mathrm{Spec}(A)\backslash\{\mathfrak{m}\} = U & \hookleftarrow & U_0 = \mathrm{Spec}(A/fA)\backslash\{\mathfrak{m}\} \end{array}$$

Note A/fA is regular local of dimension 1 less, so by induction we have the result for A/fA. Reformulated, then result is the statement that $\text{FÉt}_{X_0} \to \text{FÉt}_{U_0}$ is an equivalence.

If A is henselian, then we have

$$\text{FÉt}_X \cong \text{FÉt}_{X_0} \cong \text{FÉt}_{\text{Spec}(\kappa_A)}.$$

So we're done if we can show

$$\text{FÉt}_X \to \text{FÉt}_{U_0}$$

is an equivalence when A is henselian or something. □

Theorem 10.7. [1] *Let $(A, \mathfrak{m})$ be a local ring and $f \in \mathfrak{m}$ a nonzerodivisor. Let $X = \text{Spec}(A)$ and U the punctured spectrum of A. Let $X_0 = \text{Spec}(A/fA)$ and U_0 the punctured spectrum of A/fA. Let V be finite étale over U. Assume*

(1) *f is a nonzerodivisor,*
(2) *$H^1_{\mathfrak{m}}(A)$ is a finite A-module,*
(3) *a power of f annihilates $H^2_{\mathfrak{m}}(A)$,*
(4) *$V_0 = V \times_U U_0$ is equal to $Y_0 \times_{X_0} U_0$ for some $Y_0 \to X_0$ finite étale.*

Then $V = Y \times_X U$ for some $Y \to X$ finite étale.

Remark.

$$\text{depth}(A) \geq t \Leftrightarrow H^i_{\mathfrak{m}}(A) = 0 \text{ for } i = 0, \cdots, t-1.$$

So if $\dim A \geq s$ and Cohen–Macauley (e.g. regular), then the assumptions are satisfied.

Corollary 10.8. *Local purity of branch locus holds for A local,* $\dim(A) \geq 3$ *and A complete intersection.*

Roughly speaking, complete intersection means that A is a regular local ring modulo a regular sequence. This result is sharp because it doesn't work when $\dim A = 2$.

Example 10.9. $A = \mathbb{C}[x^2, xy, y^2] \subset B = \mathbb{C}[x, y]$. This looks like the plane mapping to the cone via quotienting out by $\{\pm 1\}$. Then B is not finite étale over A.

11. Lecture 11 (October 13, 2015)

11.1. Local Lefschetz.

Notation. $(A, \mathfrak{m}, \kappa)$ will be a Noetherian local ring, and $f \in \mathfrak{m}$ will be a nonzerodivisor.

Consider

$$\begin{array}{ccc} \text{Spec}(A)\backslash\{\mathfrak{m}\} = U & \hookleftarrow & U_0 = \text{Spec}(A/fA)\backslash\{\mathfrak{m}\} \\ \downarrow & & \downarrow \\ \text{Spec}(A) = X & \hookleftarrow & X_0 = \text{Spec}(A/fA) \end{array}$$

[1] PH: This is a corrected version of the theorem stated in lecture (see here), as pointed out to me by Johan. To quote him: "*It should not be an equivalence between the categories of finite étale schemes from U to U_0. This theorem is enough to prove purity in the regular local case (the implication direction being that if the thing extends after restriction to the hypersurface, then it extends). But, it is not enough to get the application to Grothendieck's thing about complete intersection rings. The result you need there is this. But this is a bit harder to state... I may come back to this in the next lecture, but probably not.*"

If U and U_0 are connected, we get a diagram of fundamental groups

$$\begin{array}{ccc} \pi_1(U) & \longleftarrow & \pi_1(U_0) \\ \downarrow & & \downarrow \\ \pi_1(X) & \longleftarrow & \pi_1(X_0) \end{array}$$

Observation: If A is *strictly henselian* (i.e., A is henselian and κ is separably closed, for example A is complete with algebraically closed residue field), then $\pi_1(X) = \{1\}$ and $\pi_1(X_0) = \{1\}$.

Purity says that the map on the left is an isomorphism, hence in the strictly henselian case $\pi_1(U) = \{1\}$ and any finite étale cover of U is trivial. Lefschetz is about the top map $\pi_1(U) \leftarrow \pi_1(U_0)$.

Local Lefschetz type questions: Find conditions on A such that

- $\pi_1(U_0) \twoheadrightarrow \pi_1(U)$, or
- $\pi_1(U_0) \xrightarrow{\cong} \pi_1(U)$, or
- $\mathrm{Pic}(U) \hookrightarrow \mathrm{Pic}(U_0)$, or
- $\mathrm{Pic}(U) \xrightarrow{\cong} \mathrm{Pic}(U_0)$, ...

The misstated theorem from last time has been corrected in the notes.

Theorem 11.1. *Assume A is henselian, $H^1_{\mathfrak{m}}(A)$ is finite, and f^N kills $H^2_{\mathfrak{m}}(A)$ for some N. Then*

$$\mathrm{F\acute{E}t}_U \to \mathrm{F\acute{E}t}_{U_0}$$

is fully faithful, i.e., $\pi_1(U_0) \twoheadrightarrow \pi_1(U)$ if U and U_0 are connected.

In the situation of Theorem 11.1, if A is strictly henselian and if local purity holds for A/fA, i.e., $\pi_1(U_0) = \{1\}$, then local purity holds for A, i.e., $\pi_1(U) = \{1\}$. (This is an example of going from knowing local purity for lower dimension to knowing local purity for higher dimension.)

We have to show

$$\begin{aligned} \mathrm{F\acute{E}t}_U &\to \mathrm{F\acute{E}t}_{U_0} \\ V &\mapsto V_0 = V \times_U U_0 \end{aligned}$$

is fully faithful.

Set $V_n = V \times_U U_n$ where $U_n = V(f^{n+1}) \subset U$.

$$\begin{array}{ccccccccc} V & \hookleftarrow & \cdots & \hookleftarrow & V_2 & \hookleftarrow & V_1 & \hookleftarrow & V_0 \\ \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ U & \hookleftarrow & \cdots & \hookleftarrow & U_2 & \hookleftarrow & U_1 & \hookleftarrow & U_0 \\ \big\downarrow & & & & \big\downarrow & & \big\downarrow & & \big\downarrow {\scriptstyle \text{open}} \\ \mathrm{Spec}(A) & \hookleftarrow & \cdots & \hookleftarrow & \mathrm{Spec}(A/f^3A) & \hookleftarrow & \mathrm{Spec}(A/f^2A) & \hookleftarrow & \mathrm{Spec}(A/fA) \end{array}$$

Lemma 11.2. *The functor is fully faithful if and only if for all $V \in \mathrm{F\acute{E}t}_U$, we have $\pi_0(V) \xrightarrow{bij.} \pi_0(V_0)$.*

Looking at the diagram above, we get

$$\pi_0(V) \to \cdots \to \pi_0(V_2) \xrightarrow{\cong} \pi_0(V_1) \xrightarrow{\cong} \pi_0(V_0).$$

Note that

$$\text{subsets of } \pi_0(V_i) \xrightarrow{\text{bij.}} \text{set of idempotents of } \Gamma(V_i, \mathcal{O}_{V_i})$$

for each i, so we have

$$\begin{array}{ccccccc}
\text{subsets of } \pi_0(V) & \longrightarrow \cdots \longrightarrow & \text{subsets of } \pi_0(V_2) & \xrightarrow{\cong} & \text{subsets of } \pi_0(V_1) & \xrightarrow{\cong} & \text{subsets of } \pi_0(V_0) \\
\downarrow{\scriptstyle \text{bij.}} & & \downarrow{\scriptstyle \text{bij.}} & & \downarrow{\scriptstyle \text{bij.}} & & \downarrow{\scriptstyle \text{bij.}} \\
\text{set of idempotents of } \Gamma(V, \mathcal{O}_V) & \longrightarrow \cdots \longrightarrow & \text{set of idempotents of } \Gamma(V_2, \mathcal{O}_{V_2}) & \xrightarrow{\cong} & \text{set of idempotents of } \Gamma(V_1, \mathcal{O}_{V_1}) & \xrightarrow{\cong} & \text{set of idempotents of } \Gamma(V_0, \mathcal{O}_{V_0})
\end{array}$$

Therefore it is *enough* to show

$$\Gamma(V, \mathcal{O}_V) \xrightarrow{\cong} \lim \Gamma(V_n, \mathcal{O}_{V_n}).$$

Here completeness is crucial.

Lemma 11.3 (Tag 0BLD)**.** *In the setting above there exist mysterious modules H^p and short exact sequences*

$$0 \to R^1 \lim H^{p-1}(\mathcal{O}_{V_n}) \to H^p \to \lim H^p(\mathcal{O}_{V_n}) \to 0$$

and

$$0 \to H^0(H^p(\mathcal{O}_V)^\wedge) \to H^p \to T_f H^{p+1}(\mathcal{O}_V) \to 0,$$

where

- *$H^P(\mathcal{O}_V)^\wedge$ is the derived completion of $H^p(\mathcal{O}_V)$ with respect to F, and for any A-module M one has a short exact sequence,*

$$0 \to R^1 \lim M[f^n] \to H^0(M^\wedge) \twoheadrightarrow \widehat{M} \to 0$$

  where $\widehat{M} = \lim M/f^n M$ is the usual completion, and $M^\wedge = R\lim(M \xrightarrow{f^n} M)$ (at degrees -1 and 0 respectively) is the derived completion.
- *$T_f(M) = \varprojlim_n M[f^n]$ is the f-Tate module of M:*

$$\cdots \to M[f^2] \xrightarrow{\times f} M[f] = \{x \in M \mid fx = 0\}.$$

We will take $p = 0$:

- $H^0 = \lim H^0(\mathcal{O}_{V_n})$ by the first exact sequence.
- If $H^0(\mathcal{O}_V)$ has bounded f-power torsion (e.g. when V is Noetherian), then

$$H^0(H^0(\mathcal{O}_V)^\wedge) = \widehat{H^0(\mathcal{O}_V)}.$$

We get

$$0 \to \widehat{H^0(V, \mathcal{O}_V)} \to \lim H^0(\mathcal{O}_{V_n}) \to T_f H^1(\mathcal{O}_V) \to 0.$$

Corollary 11.4. *To show $\mathrm{FÉt}_U \to \mathrm{FÉt}_{U_0}$ is fully faithful it suffices to show for all $V \to U$ finite étale that:*

1. *$H^0(\mathcal{O}_V)$ is f-adically complete.*
2. *f-power torsion on $H^1(\mathcal{O}_V)$ is bounded.*

Proof of (1) when A is complete. $H^1_{\mathfrak{m}}(A)$ finite and $V \to U$ finite étale imply that $H^0(V, \mathcal{O}_V)$ is finite over A. This uses finiteness theorems in coherent cohomology. □

Example 11.5.

- If $\dim A = 1$, then $H^0(U, \mathcal{O}_U)$ is never finite: pick $x \in \mathfrak{m}$ nonzerodivisor, and then $\frac{1}{x^n} \in H^0(U, \mathcal{O}_U)$ for all $n \geq 1$.
- If A is a normal domain of dim ≥ 2, then $H^0(U, \mathcal{O}_U) = A$ is finite. We have $H^1_{\mathfrak{m}}(A) = 0$.

In general, there is an exact sequence

$$0 \to H^0_{\mathfrak{m}}(A) \to A \to H^0(U, \mathcal{O}_U) \to H^1_{\mathfrak{m}}(A) \to 0$$

and

$$H^i(U, \mathcal{O}_U) \xrightarrow{\cong} H^{i+1}_{\mathfrak{m}}(A)$$

if $i \geq 1$. (Look in Hartshorne for

$$0 \to H^0_Z(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \to H^1_Z(X, \mathcal{F}) \to \cdots.)$$

Proof of (2). We have to show $\pi : V \to U$ étale finite implies

$$H^1(V, \mathcal{O}_V) = H^1(U, \pi_* \mathcal{O}_V) = H^1(U, \text{some vector bundle on } U)$$

has bounded f-torsion. This follows from the next lemma. □

Lemma 11.6. *If $H^1(U, \mathcal{O}_U) \cong H^2_{\mathfrak{m}}(A)$ is annihilated by f^N for some N, then for any finite locally free sheaf $\mathcal{E}$ on U there is an $M = M(\mathcal{E})$ such that f^M annihilates $H^1(U, \mathcal{E})$.*

Proof. For all $u \in U$ there exists $u \in U' \underset{\text{open}}{\subset} U$ such that

$$\mathcal{E}|_{U'} \underset{\varphi}{\overset{\cong}{\leftarrow}} \mathcal{O}_U^{\oplus r}|_{U'}.$$

We can pick $U' = D(g)$ for some $g \in A$.

There exists a map $\alpha : \mathcal{O}_U^{\oplus r} \to \mathcal{E}$ such that $\alpha|_{U'} = g^{N_1}\varphi$. Dually, there exists a map $\beta : \mathcal{E} \to \mathcal{O}_U^{\oplus r}$ such that $\beta|_{U'} = g^{N_2}\varphi^{-1}$. Thus the composition

$$\mathcal{E} \xrightarrow{\beta} \mathcal{O}_U^{\oplus r} \xrightarrow{\alpha} \mathcal{E}$$

is multiplication by $g^{N_1+N_2}$ (maybe after increasing N_1 and N_2).

Now for all $u \in U$, there exists $g \in A$ such that $u \in D(g)$ and $g^{N_1+N_2} f^N \in \operatorname{Ann}(H^1(U, \mathcal{E}))$. The ideal generated by all these $g^{N_1+N_2}$ is $\mathfrak{m}$-primary, so

$$\mathfrak{m}^{N_{101}} \cdot f^N \subset \operatorname{Ann} H^1(U, \mathcal{E})$$

and hence

$$f^{N_{101}+N} \in \operatorname{Ann} H^1(U, \mathcal{E}).$$ □

Upshot: We are done with Theorem 11.1 in the complete case. Then you still have to reduce to the complete case.

Next time we will talk about specialization for fundamental groups.

12. Lecture 12 (October 15, 2015)

12.1. **Specialization of the fundamental group.** We will talk about the specialization of the fundamental group in the smooth proper case.

Theorem 12.1. *Let $k \subset k'$ be an extension of algebraically closed fields. Let X/k be a proper connected scheme. Then $X_{k'}$ is connected and $\pi_1(X) = \pi_1(X_{k'})$.*

Sketch of proof. Recall "cohomology and base change": if X/R is quasicompact and quasiseparated and $R \to R'$ is flat, then

$$H^i(X, \mathcal{O}_X) \otimes_R R' = H^i(X_{R'}, \mathcal{O}_{X_{R'}}).$$

To prove connectedness, we use

$$H^0(X, \mathcal{O}_X) \otimes_k k' = H^0(X_{k'}, \mathcal{O}_{X_{k'}}).$$

Note $H^0(X, \mathcal{O}_X)$ is a finite-dimensional k-algebra, hence artinian. It is a product of local rings. Since k is algebraically closed, the number of local components remains the same after base change to k'.

The proof of $\pi_1(X) = \pi_1(X_{k'})$ uses our previous theorem about $\pi_1(X)$ where X is proper over a henselian local ring. □

Remark. It would be easy to prove the theorem if you knew that $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$ when X and Y are varieties over $k = \overline{k}$. But this is not true:

$$\pi_1(\mathbb{A}^2_{\overline{\mathbb{F}_p}}) \neq \pi_1(\mathbb{A}^1_{\overline{\mathbb{F}_p}}) \times \pi_1(\mathbb{A}^1_{\overline{\mathbb{F}_p}}).$$

In characteristic 0, this is true but I don't know a truly simple proof.

Theorem 12.2 (Specialization of fundamental group)**.** *Let $X \to S$ be a proper smooth morphism with geometrically connected fibres. Let $s \rightsquigarrow s'$ be a specialization of points of S. Then there is a map*

$$\mathrm{sp} : \pi_1(X_{\overline{s}}) \to \pi_1(X_{\overline{s}'})$$

which is surjective and

- *an isomorphism if* $\mathrm{char}(\kappa(s')) = 0$.
- *an isomorphism on prime-to-p quotients if* $\mathrm{char}(\kappa(s')) = p$.

Remark. If X is connected, then

$$\begin{array}{ccc} \pi_1(X_{\overline{s}}) & \xrightarrow{\mathrm{sp}} & \pi_1(X_{\overline{s}'}) \\ & \searrow \quad \swarrow & \\ & \pi_1(X) & \end{array}$$

commutes.

Remark (Missing material). Let $f : X \to S$ be a flat proper morphism with geometrically connected and reduced fibres. If S is Noetherian and connected and $\overline{s} \in S$, then there is an exact sequence

$$\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S) \to 1.$$

This is the "first homotopy sequence". See Murre.

Remark. sp exists only assuming $X \to S$ proper.

Step 0: Reduce to S Noetherian (technical).

Step 1: Reduce to $S = \mathrm{Spec}(A)$ where A is a dvr.

Proof. Use that given a specialization $s \rightsquigarrow s'$ in S Noetherian, we can find

$$\begin{aligned} \mathrm{Spec}(A) &\to S \\ \eta &\mapsto s, \\ 0 &\mapsto s', \end{aligned}$$

where A is a dvr, $\eta \in \mathrm{Spec}(A)$ is the generic point, and $0 \in \mathrm{Spec}(A)$ is the closed point. (Roughly, this follows by blowing-up to get codimension 1, taking the generic point of the exceptional fibre, and normalizing to get a dvr.) $\square$

By Theorem 12.1, the residue field extensions don't matter.

Step 2: We may assume A is a complete dvr with algebraically closed residue field (same arguments).

Step 3+4: Now $s = \eta$, $s' = 0$, $S = \mathrm{Spec}(A)$, A as in Step 2, and we get

$$\pi_1(X_{\overline{\eta}}) \to \pi_1(X_\eta) \to \pi_1(X) \cong \pi_1(X_0)$$

whose composition is the specialization map sp. The isomorphism is given by a previous theorem.

Step 5: sp is surjective.

Proof. Let $K = \mathrm{f.f.}(A)$. Let $Y_0 \to X_0$ be connected finite étale. Let $Y \to X$ be the corresponding finite étale cover. Then Y is connected (which follows from $\pi_1(X) = \pi_1(X_0)$). We have to show $Y_{\overline{\eta}}$ is connected.

If not, then there exists L/K finite such that Y_L is disconnected (because $\kappa(\overline{\eta}) = \mathrm{colim}_{\substack{\kappa(\overline{\eta})/L/K \\ \text{finite}}} L$ and ...). Let $B \subset L$ be the integral closure of A in L. Note $A/\mathfrak{m}_A = B/\mathfrak{m}_B$ and B is local.

$$\begin{array}{ccccccc} Y_L & \hookrightarrow & Y_B & \longrightarrow & Y & \longrightarrow & X \\ \downarrow & & \downarrow & & & \searrow & \downarrow \\ \mathrm{Spec}(L) & \hookrightarrow & \mathrm{Spec}(B) & & \longrightarrow & & \mathrm{Spec}(A) \end{array}$$

Now $Y_B \to \mathrm{Spec}(B)$ is smooth proper with disconnected generic fibre. Two components have to meet in Y_B, but this cannot happen because Y_B is normal (even regular). The next lemma implies $Y_0 = Y_{B/\mathfrak{m}_B}$ is disconnected, a contradiction. $\square$

Lemma 12.3. *If $Z \to T$ is smooth and proper. Then the function $t \mapsto \#\pi_0(Z_{\overline{t}})$ is locally constant on T.*

Step 6: The kernel of sp?

Claim. Let $\gamma : \pi_1(X_{\overline{\eta}}) \to G$ be a continuous finite quotient with $\#G$ prime to $\mathrm{char}(A/\mathfrak{m}_A)$. Then γ factors through sp.

$$\begin{array}{ccc} \pi_1(X_{\overline{\eta}}) & \xrightarrow{\mathrm{sp}} & \pi_1(X_0) \\ & \searrow \quad \swarrow & \\ & G & \end{array}$$

Proof. Let G acting on $Y' \to X_{\overline{\eta}}$ be the corresponding finite étale cover. By a limit argument (see Step 5), there exists $\kappa(\overline{\eta})/L/K$ finite such that Y' is defined over L. After replacing A by $B \subset L$ (see Step 5), we may assume we have G acting on $Y' \to X_\eta$.

Let $Y \to X$ be the normalization of X in f.f.(Y'). Then we get

$$\begin{array}{ccc} Y' & \hookrightarrow & Y \\ {\scriptstyle \pi_\eta}\downarrow & & \downarrow{\scriptstyle \pi} \\ X_\eta & \hookrightarrow & X \end{array}$$

where π_η is finite étale, π is finite and Y is normal. As X is a regular scheme (smooth over dvr), by purity of branch locus, if π is not unramified then π is not unramified at a codimension 1 point. Such a point always lies over $\xi \in X_0$, the generic point of X_0.

Let $y_1, \cdots, y_n \in Y_0$ be the generic points of Y_0. Then $\pi^{-1}(\{\xi\}) = \{y_1, \cdots, y_n\}$. Now $\#G$ prime to char$(\kappa(\xi))$ implies that the ramification of $\mathcal{O}_{Y,y_i}/\mathcal{O}_{X,\xi}$ is tame.

(We will continue the proof next time.) ☐

To elucidate, here we have exactly the situation as in the section on ramification theory:

$$\begin{array}{ccc} \text{f.f.}(X) & \hookrightarrow & \text{f.f.}(Y) \\ \uparrow & & \uparrow \\ \mathcal{O}_{X,\xi} & \hookrightarrow & C \end{array}$$

where f.f.$(X) \subset$ f.f.(Y) is a Galois extension with group G, and C is the integral closure of $\mathcal{O}_{X,\xi}$ in f.f.(Y). We have maximal ideals $\mathfrak{m}_1, \cdots \mathfrak{m}_n$ with $C_{\mathfrak{m}_i} = \mathcal{O}_{Y,y_i}$.

Next time: Abhyankar's lemma guarantees that we can lower the e of this after base change with a suitable B/A finite as before.

13. Lecture 13 (October 20, 2015)

13.1. **Abhyankar's lemma.** The Stacks Project (Tag 0BRM) has a general version of Abhyankar's lemma, which is about lowering ramification index by base change. Today we will lower this down to 1 so that there is no ramification. In fact the general version allows us to lower the ramification index to any specified divisor.

The situation is as follows. Let $A \subset B$ be an extension of dvr's; we only require that $A \hookrightarrow B$ be a local ring map, i.e., $\mathfrak{m}_A \subset \mathfrak{m}_B$. Let $L = $ f.f.(B) and $K = $ f.f.(A). Then L/K is an extension (not necessarily finite). We have the invariants:

- e = ramification index e: $\pi_A = (\text{unit})\pi_B^e$,
- κ_B/κ_A extension of residue fields (not necessarily finite).

Reminder: An extension of fields κ'/κ is called *separable* if and only if every finitely generated subextension $\kappa'/\ell/\kappa$ is *separably generated*: there exists $x_1, \cdots, x_r \in \ell$ such that

$$\ell \supset \kappa(x_1, \cdots, x_r) \supset \kappa$$

where the first extension is finite separable and the second extension is purely transcendental. Under this definition, it is true (but not obvious) that a separably generated extension is separable!

Proposition 13.1. *TFAE:*

- *κ'/κ is separable.*
- *κ' is geometrically reduced over κ.*
- *$\kappa' \otimes_\kappa \Omega_{\kappa/\mathbb{Z}} \hookrightarrow \Omega_{\kappa'/\mathbb{Z}}$.*
- *Any derivation of κ extends to a derivation of κ'.*
- *κ' is formally smooth over κ.*
- *$H_1(L_{\kappa'/\kappa}) = 0$.*

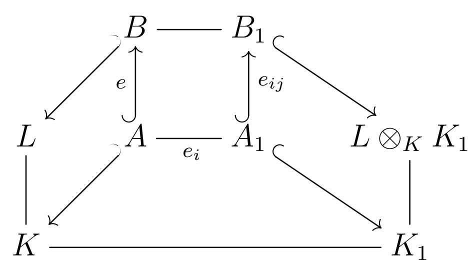
If κ'/κ is a finitely generated field extension, then these are also equivalent to:

- *κ'/κ is separably generated.*
- *$\kappa' = \kappa(X)$ where X/κ is a smooth variety.*
- *$\dim_{\kappa'}(\Omega_{\kappa'/\kappa}) = \mathrm{Tr}(\kappa'/\kappa)$.*

Example 13.2. $\mathbb{F}_p \subset \bigcup_n \mathbb{F}_p(t^{1/p^n})$ is separable: the only derivation of $\mathbb{F}_p$ is trivial!

Example 13.3. Let $\kappa = \mathbb{F}_p(t, s, r)$. Then the variety $X : tx^p + sy^p + rz^p = 0$ is nowhere smooth over κ, so $\kappa(X)/\kappa$ is *not* separable. Such a variety does arise in nature: it is the generic fibre of a smooth morphism $\mathfrak{X} \to \mathbb{A}^3_{\mathbb{F}_p}$ over $\mathrm{Spec}(\mathbb{F}_p(t, s, r))$.

Going back to $A \subset B$, let K_1/K be a finite separable extension.



where:

- $L \otimes_K K_1$ is a finite product of finite separable extensions of L,
- A_1 is the integral closure of A in K_1,
- B_1 is the integral closure of B in $L \otimes_K K_1$,
- $\mathfrak{m}_1, \cdots, \mathfrak{m}_n$ are the maximal ideals of A_1, and
- $\mathfrak{m}_{i1}, \cdots, \mathfrak{m}_{im_i}$ are the maximal ideals of B_1 lying over $\mathfrak{m}_i$.

We have new invariants:

- e_i = ramification of $A \to (A_1)_{\mathfrak{m}_i}$,
- e_{ij} = ramification of $(A_1)_{\mathfrak{m}_i} \to (B_1)_{\mathfrak{m}_{ij}}$,
- $\kappa(\mathfrak{m}_{ij})/\kappa(\mathfrak{m}_i)$.

Theorem 13.4 (Abhyankar's lemma)**.** *Assume e is prime to the characteristic of κ_A and that κ_B/κ_A is separable. If $e \mid e_i$, then $e_{ij} = 1$ for all $j = 1, \cdots, m_i$, and $\kappa(\mathfrak{m}_{ij})/\kappa(\mathfrak{m}_i)$ is separable.*

Remark. In fact under some hypothesis you get

$$e_{ij} = \frac{e}{\gcd(e, e_i)}.$$

This is not yet in the Stacks Project.

Example 13.5.

$$x \longmapsto (z, \cdots, z)$$

$$\begin{array}{ccccc}
A[x]/(x^5-t) & \longrightarrow & \prod_{i=1}^5 k[[z]] & & (z, \zeta_5 z, \zeta_5^2 z, \zeta_5^3 z, \zeta_5^4 z) \\
\uparrow & & \uparrow & & \uparrow \\
A = k[[t]] & \longrightarrow & A[y]/(y^5-t) & & y
\end{array}$$

where $\zeta_5 \in k$ is a primitive 5-th root of unity. Here $\prod_{i=1}^5 k[[z]]$ is the normalization of $A[x,y]/(x^5-t, y^5-t)$.

13.2. **End of proof of Theorem last time.** Recall that A is a complete dvr with $\kappa_A = \overline{\kappa_A}$ and $\mathrm{char}(\kappa_A) = p$. $X \to \mathrm{Spec}(A)$ is a smooth proper geometrically connected fibre. $Y \to X$ is the normalization of X in a Galois extension, with $\mathrm{f.f.}(X) \subset \mathrm{f.f.}(Y)$ with group G of order prime to p and $Y_\eta \to X_\eta$ étale.

$$Y \to X \to \mathrm{Spec}(A) \ni \eta$$

$$Y_\eta \xrightarrow{G} X_\eta$$

Let $\xi \in X_0$ be the generic point of the special fibre, and $y_1, \cdots, y_n \in Y_0$ are points lying over ξ.

Claim. The hypotheses of Abhyankar's lemma apply to $A \subset \mathcal{O}_{Y,y_i}$ for $i = 1, \cdots, n$.

Proof. Look at $A \subset \mathcal{O}_{X,\xi} \subset \mathcal{O}_{Y,y_i}$. Since $\mathrm{f.f.}(Y)/\mathrm{f.f.}(X)$ is tamely ramified with respect to $\mathcal{O}_{X,\xi}$, we see that $e(\mathcal{O}_{Y,y_i}/\mathcal{O}_{X,\xi})$ is prime to p. Note $e(\mathcal{O}_{X,\xi}/A) = 1$ as $X \to \mathrm{Spec}(A)$ is smooth. Hence

$$e(\mathcal{O}_{Y,y_i}/A) = e(\mathcal{O}_{Y,y_i}/\mathcal{O}_{X,\xi}) \cdot e(\mathcal{O}_{X,\xi}/A)$$

is prime to p. Since $\kappa_A = \overline{\kappa_A}$, we get that the residue field extension is separable. □

Pick $A_1 = A[\pi_A^{1/e}]$ where $e \in \mathbb{N}$ is sufficiently divisible (e.g. $\#G \mid e$). (This was called B in the previous lecture.) Then

$$\begin{array}{ccc}
Y & \longleftarrow & Y_1 \\
\downarrow & & \downarrow \\
X & \longleftarrow & X_{A_1} \\
\downarrow & & \downarrow \\
\mathrm{Spec}(A) & \longleftarrow & \mathrm{Spec}(A_1)
\end{array}$$

where Y_1 is the normalization of Y_{A_1}. By Abhyankar's lemma, $Y_1 \to X_{A_1}$ has ramification index 1 at all points of closed fibre of Y_1.

By tameness of $\mathrm{f.f.}(Y)/\mathrm{f.f.}(X_{A_1})$ with respect to $\mathcal{O}_{X_{A_1},\xi}$, we see that $Y_1 \to X_{B_1}$ is unramified in codimension 1. By purity, $Y_1 \to X_{A_1}$ is étale. This means G is a quotient of $\pi_1(X_0)$ as desired.

13.3. **Applications.** Let us now apply this famous theorem of Grothendieck. An example of a variety in characteristic p that lifts to characterisitc 0 is complete intersection.

13.3.1. *Complete intersections.* Let $X \subset \mathbb{P}^n_k$ be a smooth complete intersection and $k = \overline{k}$. If $\dim(X) \geq 2$, then $\pi_1(X) = \{1\}$.

Proof. Over $k = \mathbb{C}$, we use Lefschetz. Then we get it over any $k = \overline{k}$ of characteristic 0, so we can lift (using the Witt ring) and apply the theorem. $\square$

13.3.2. *Curves.* Let X be a smooth projective curve of genus g over $k = \overline{k}$. Then there is a surjection

$$\left(\frac{\langle \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g\rangle}{[\alpha_1, \beta_1]\cdots[\alpha_g, \beta_g]}\right)^{\wedge} \twoheadrightarrow \pi_1(X)$$

which induces an isomorphism on maximal prime-to-p quotients.

This is one of the victories of Grothendieck's school.

Theorem 13.6 (Tamagawa, Raynaud, ...)**.** *Fix p and $g > 1$. Then the association*

$$\begin{matrix}\textit{Isomorphism classes of genus } g \\ \textit{smooth projective curves over } \overline{\mathbb{F}}_p\end{matrix} \to \begin{matrix}\textit{Isomorphism classes of} \\ \textit{profinite groups}\end{matrix}$$

given by

$$X \mapsto \pi_1(X)$$

is finite to 1.

Example 13.7. If (E, O) is an elliptic curve over $k = \overline{k}$, then

$$\pi_1(E) = \begin{cases} \widehat{\mathbb{Z}}^{\oplus 2} & \text{if } \mathrm{char}(k) = 0, \\ \prod_{\ell \neq p} \mathbb{Z}_\ell^{\oplus 2} \times \mathbb{Z}_p & \text{if } \mathrm{char}(k) = 0, E \text{ ordinary}, \\ \prod_{\ell \neq p} \mathbb{Z}_\ell^{\oplus 2} \times \{0\} & \text{if } \mathrm{char}(k) = 0, E \text{ supersingular}. \end{cases}$$

Example 13.8. If A is an abelian variety, then

$$\pi_1(A) = \prod_{\ell \neq p} \mathbb{Z}_\ell^{\oplus 2g} \times \mathbb{Z}_p^f$$

where $f \in \{0, \cdots, g\}$ is the p-rank of A.

14. Lecture 14 (October 22, 2015)

First we will discuss quasi-unipotent monodromy over $\mathbb{C}$, but this will be so much easier with algebraic geometry. This will also fit well with the Remynar.

14.1. **Quasi-unipotent monodromy over $\mathbb{C}$.** Let $f : X \to S$ be a smooth proper morphism of schemes l.f.t. over $\mathbb{C}$.

Fact. $f^{\mathrm{an}} : X^{\mathrm{an}} \to S^{\mathrm{an}}$ is a fibre bundle: for all $s \in S^{\mathrm{an}} = S(\mathbb{C})$, there exists $s \in U \subset S^{\mathrm{an}}$ open such that $(f^{\mathrm{an}})^{-1}(U) \cong X_s \times U$ as homeomorphism of spaces over U.

Corollary 14.1. *The sheaves $R^i f^{\mathrm{an}}_*(\underline{\mathbb{Z}})$ are locally constant on S^{an} (such a thing is often called a local system) with stalk $H^i(X_s, \underline{\mathbb{Z}})$ at s. In particular, there is a monodromy representation*

$$\rho^i : \pi_1(S^{\mathrm{an}}, s) \to \mathrm{Aut}_{\mathbb{Z}}(H^i(X_s, \underline{\mathbb{Z}})).$$

Theorem 14.2. *The representation ρ^i sends "loops around ∞" to quasi-unipotent operators.*

Proof. Hodge theory; but see later. □

Definition 14.3. An element $g \in \mathrm{GL}_n(\text{Field})$ is *quasi-unipotent* if some power of g is unipotent ($\Leftrightarrow$ the eigenvalues of g are roots of unity).

What do we mean by "loops around ∞"?

(1) If S is a smooth curve, choose a smooth projective compactification $S \subset \overline{S}$. Then for each $x \in \overline{S} \backslash S$ there is a well-defined conjugacy class in $\pi_1(S^{\text{an}}, s)$ consisting of loops around x (counterclockwise).
(2) If $\dim(S)$ is arbitrary, we consider finite morphisms $C \to S$ for C smooth, and look at the images of loops around ∞ in $\pi_1(C)$ in $\pi_1(S)$.

14.2. **Étale cohomology version.** Let $f : X \to S$ be a smooth proper morphism of Noetherian schemes. Let ℓ be a prime number invertible on S. Then we get $R^i f_{\text{ét},*}(\mathbb{Z}_\ell)$ locally constant (technically we haven't discusses étale sites yet, but $f_{\text{ét}}$ is the étale analogue of the analytification for the complex topology), with

$$R^i f_{\text{ét},*}(\mathbb{Z}_\ell)_{\overline{s}} = H^i_{\text{ét}}(X_{\overline{s}}, \mathbb{Z}_\ell)$$

is a finitely generated $\mathbb{Z}_\ell$-module. There is a monodromy representation

$$\rho^i_\ell : \pi_1(S, \overline{s}) \to \mathrm{Aut}_{\mathbb{Z}_\ell}(H^i_{\text{ét}}(X_{\overline{s}}, \mathbb{Z}_\ell)).$$

Theorem 14.4 (Grothendieck)**.** *ρ^i_ℓ have quasi-unipotent local monodromy, i.e., for every morphism* $\mathrm{Spec}(K) \to S$ *where* $K = \text{f.f.}(A)$ *and A is a dvr, the action*

$$\rho^i_{X_K/K,\ell} : \mathrm{Gal}(K^{\text{sep}}/K) \to \mathrm{Aut}_{\mathbb{Z}_\ell}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell))$$

when restricted to an inertia subgroup has the following properties, provided ℓ is invertible in κ_A:

(1) *the image of wild inertia is finite: $\#\rho^i_{X_K/K,\ell}(P) > \infty$.*
(2) *if $\tau \in I$ is an element mapping a topological generator of I_t, then $\rho^i_{X_K/K,\ell}(\tau)$ is quasi-unipotent.*

Recall $\{1\} \subset P \subset I \subset D \subset \mathrm{Gal}(K^{\text{sep}}/K)$, where P is the wild inertia, I is the inertia, D is the decomposition, $P_I = I/P \cong \prod_{\ell' \neq \text{char}(\kappa_A)} \mathbb{Z}_{\ell'}$ is the tame inertia.

The argument of Grothendieck shows that this is a consequence of the structure of inertia groups (which can be thought of as fundamental groups), without knowing about étale cohomology and the local systems $R^i f_{\text{ét},*}(\mathbb{Z}_\ell)$! This theorem really gets used in geometric situations when we are given a smooth proper morphism, even though the proof goes through the inertia groups.

Remark. You can deduce the theorem on quasi-unipotent monodromy over $\mathbb{C}$ from Grothendieck's version by using comparison theorems.

Remark. Immediate reduction to $S = \mathrm{Spec}(K)$, where $K = \text{f.f.}(A)$.

I am going to prove part (1).

Lemma 14.5. *Let ℓ, p be two distinct primes. Let H, G be profinite groups with H pro-ℓ and G pro-p. Then there is no nontrivial continuous group homomorphism $G \to H$.*

Proof. The same is true for finite groups. ☐

Lemma 14.6. *Let M be a finitely generated $\mathbb{Z}_\ell$-module. Endow M with the ℓ-adic topology. Then*

$$\mathrm{Aut}_{\mathbb{Z}_\ell}(M) = \mathrm{Aut}_{\mathrm{cont}}(M) = \lim \mathrm{Aut}(M/\ell^n M)$$

is a profinite group and the kernel of the continuous

$$\mathrm{Aut}_{\mathbb{Z}_\ell}(M) \to \mathrm{Aut}(M/\ell M)$$

is a pro-ℓ group.

Proof. Omitted. (This is clear from the description $\lim \mathrm{Aut}(M/\ell^n M)$.) ☐

Lemma 14.7. *If X/K is a variety, then $\rho^i_{X/K,\ell} : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}_{\mathbb{Z}_\ell}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell))$ is continuous.*

Proof. By definition,

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell) = \lim_n H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/\ell^n\mathbb{Z})$$

and the action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on these is continuous. ☐

Remark. Even though $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/\ell^n\mathbb{Z})$ is finite, continuity is not automatic because $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ is a profinite group!

Proof of part (1) of Theorem 14.4. Done by combining the lemmas and using that the wild inertia is a pro-p group where $p = \mathrm{char}(\kappa_A)$. This is why $\ell \neq \mathrm{char}(\kappa_A)$ is *needed.* ☐

15. Lecture 15 (October 27, 2015)

Let A be a dvr with fraction field K and residue field κ, $X/\operatorname{Spec}(K)$ be a smooth and proper variety, and $\ell \neq \mathrm{char}(\kappa)$ be a prime number. We have the picture

$$1 \subset P \subset I \subset D \subset \mathrm{Gal}(K^{\mathrm{sep}}/K)$$

where P is the wild inertia, I is the inertia and D is the decomposition (if K is henselian local, then $D = \mathrm{Gal}(K^{\mathrm{sep}}/K)$), together with representations

$$\rho^i_{X/K,\ell} : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)).$$

Last time we showed that $\rho^i_{X/K,\ell}(P)$ is finite, using $\ell \neq \mathrm{char}(\kappa)$ and the continuity of $\rho^i_{X/K,\ell}$.

Theorem 15.1. *If $\tau \in I$ maps to a topological generator of $I_t = I/P$, then $\rho^i_{X/K,\ell}(\tau)$ is quasi-unipotent.*

We will do this by mapping $\mathrm{Aut}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell))$ into

$$\mathrm{Aut}_{\mathbb{Q}_\ell}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \cong \mathrm{GL}_{B^i_X}(\mathbb{Q}_p),$$

where B^i_X is the i-th Betti number of X.

Remark. This just means every $\tau \in I$ maps to a quasi-unipotent element of GL.

Today we will consider the case when κ has a finite number of ℓ-power roots of 1, and say a few words about why the general case does not follow by simply taking limits.

Lemma 15.2. *Assume κ has finitely many ℓ-power roots of* 1. *Then there exist $\sigma \in D$ and $\tau \in I$ such that*

- *τ maps to a topological generator of I_t,*
- *$\sigma\tau\sigma^{-1}$ and τ^α map to the same element of I_t where $\alpha \in \widehat{\mathbb{Z}}^\times$ is $\alpha = (\alpha_2, \alpha_3, \alpha_5, \alpha_7, \cdots)$ with $\alpha_\ell \equiv 1 \pmod{\ell}$ but $\alpha_\ell \neq 1$.*

Aside: Suppose G is a profinite group, $\tau \in G$, $\alpha \in \widehat{\mathbb{Z}}$. Then τ^α is defined.

Proof. We may assume G is finite. Then $\tau^n = 1$ for some n. Pick $a \in \mathbb{Z}$ such that $a \equiv \alpha \pmod{n}$. Then set $\tau^\alpha = \tau^a$ in G.

More conceptually, there is a map $\mathbb{Z} \to G$ given by $a \mapsto \tau^a$. By the universal property of profinite completion, we get $\widehat{\mathbb{Z}} \to G$ by $\alpha \mapsto \tau^\alpha$. □

Proof of lemma. There are canonical maps $D \to \mathrm{Gal}(\kappa^{\mathrm{sep}}/\kappa)$ and

$$\theta_{\mathrm{can}} : I \to \lim_{n \text{ prime to } p} \mu_n(\kappa^{\mathrm{sep}})$$

where $p = \mathrm{char}(\kappa)$ (or 1 when $\mathrm{char}(\kappa) = 0$) (and the limit on the right is noncanonically isomorphic to $\prod_{\ell' \neq p} \mathbb{Z}_{\ell'}$), such that

$$\theta_{\mathrm{can}}(\sigma\tau\sigma^{-1}) = \sigma(\theta_{\mathrm{can}}(\tau))$$

for $\tau \in I$ and $\sigma \in D$. Note that θ_{can} factors through I_t, and any $\sigma \in D$ acts on $I_t \cong \lim_{n \text{ prime to } p} \mu_n(\kappa^{\mathrm{sep}}) \simeq \prod_{\ell'} \mathbb{Z}_{\ell'}$ by multiplication by some $\alpha_\sigma \in \prod_{\ell' \neq p} \mathbb{Z}_{\ell'}^\times$. So we just pick a $\sigma \in D$ such that

(1) $\sigma(\zeta_\ell) = \zeta_\ell$,
(2) $\sigma(\zeta_{\ell^n}) \neq \zeta_{\ell^n}$ for some $n > 1$,

where ζ_ℓ and ζ_{ℓ^n} are primitive ℓ- and ℓ^n-th roots of 1 in κ^{sep}. This is possible by our assumptions on κ. □

Corollary 15.3. *Theorem 15.1 holds if κ has only a finite number of ℓ-power roots of 1, and in fact it holds for any continuous $\rho : \mathrm{Gal}(\kappa^{\mathrm{sep}}/\kappa) \to \mathrm{Aut}_{\mathbb{Z}_\ell}(M)$ and $\to \mathrm{Aut}_{\mathbb{Q}_\ell}(V)$ where M is a finite generated $\mathbb{Z}_\ell$-module and V is a finite-dimensional $\mathbb{Q}_\ell$-vector space.*

Proof. Pick σ, τ as in the previous lemma. Then $\rho(\tau)$ is conjugate to $\rho(\sigma\tau\sigma^{-1})$ (here we need the fact that ρ is defined on D, not just I). Since $\#\rho(P) < \infty$ and $\sigma\tau\sigma^{-1}$ and τ^α have the same image mod P, this implies by thinking plus group theory that

$$\rho((\sigma\tau\sigma^{-1})^N) = \rho((\tau^\alpha)^N)$$

for some $N > 0$. Then $\rho(\tau)^N$ is conjugate to $(\rho(\tau)^\alpha)^N = (\rho(\tau)^N)^\alpha$. Finish by the following lemma. □

Lemma 15.4. *If $g : M \to M$ is an automorphism of a finite $\mathbb{Z}_\ell$-module such that g is conjugate to g^α with α as in the previous lemma, then g is quasi-unipotent.*

To prove this for $\mathbb{Q}_\ell$-vector spaces, pick a stable lattice, but we will not do this in detail.

Proof. Look at eigenvalues $\lambda_1, \cdots, \lambda_t$; these will be in $\overline{\mathbb{Z}_\ell}^\times$. Then we get: for all i there exists j such that $\lambda_i^\alpha = \lambda_j$. Then for all i, there exists $0 \leq r < s$ such that $\lambda_i^{\alpha^r} = (\lambda_i^{\alpha^r})^{\alpha^s}$, so

$$(\lambda_i^{\alpha^r})^{\alpha^s - 1} = 1.$$

Note $\alpha^r \neq 0$ and $\alpha^s - 1 \neq 0$, by looking at the ℓ-component of α: there are no roots of 1 in $1 + \ell\mathbb{Z}_\ell$ if $\ell > 2$, or in $1 + 4\mathbb{Z}_2$ if $\ell = 2$.

We still have to prove that if $\lambda \in \overline{\mathbb{Z}_\ell}^\times$ and $\lambda^\beta = 1$ with $\beta \in \widehat{\mathbb{Z}}$ and $\beta \neq 0$, then λ is a root of 1. This will be omitted. $\square$

Example 15.5. Let K be a local field, $A = \mathcal{O}_K$ be the ring of integers and $X \to \operatorname{Spec}(K)$ be an abelian variety of dimension g. Consider the Tate module

$$T_\ell X = \lim X[\ell^n](\overline{K}) \cong \mathbb{Z}_\ell^{\oplus 2g}$$

with a continuous action ρ of $\operatorname{Gal}(K^{\text{sep}}/K)$. Then we have shown, after replacing K by a finite separable extension, we have $\rho(P) = \{1\}$ and ρ of the tame inertia is given by a unipotent operator.

Fact (I don't know an easy proof). Each Jordan block in our unipotent thing is 1×1 or 2×2.

Remark. If X has good reduction (i.e., X extends to an abelian scheme over $\operatorname{Spec}(A)$), then $\rho(I) = \{1\}$. The converse is true, but a lot harder to prove.

The picture is

$$\rho(\tau) = \begin{pmatrix} \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} & 0 & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} \\ 0 & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} & 0 \\ 0 & 0 & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} \end{pmatrix}$$

of size $2g \times 2g$, with the middle block of size $2g' \times 2g'$. Namely, it will turn out that X extends to a semi-abelian scheme G over $\operatorname{Spec}(\mathcal{O}_K)$ with special fibre

$$0 \to T \to G_0 \to Y \to 0$$

(extension of commutative group schemes) where Y is an abelian variety of dimension g', and T is a torus over κ:

$$T \otimes_\kappa \overline{\kappa} \cong \mathbb{G}_{m,\overline{\kappa}}^{g-g'}.$$

Now we consider the general case, when κ has lots of roots of 1, e.g. $\kappa = \overline{\kappa}$ or $A = \mathbb{C}[[t]]$. Here is a naive idea: there is a $K_0 \subset K$ which is finitely generated over $\mathbb{Q}$ or $\mathbb{F}_p$ and X_0/K_0 is smooth proper such that $X \cong X_0 \times_{\operatorname{Spec}(K_0)} \operatorname{Spec}(K)$.

Proof. See SGA. $\square$

We have a commutative diagram

$$\begin{array}{ccc} \operatorname{Gal}(K^{\text{sep}}/K) & \longrightarrow & \operatorname{Gal}(K_0^{\text{sep}}/K_0) \\ {\scriptstyle \rho^i_{X/K,\ell}} \big\downarrow & & \big\downarrow {\scriptstyle \rho^i_{X_0/K_0,\ell}} \\ \operatorname{Aut}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)) & \xrightarrow{\sim} & \operatorname{Aut}(H^i_{\text{ét}}(X_{0,\overline{K_0}}, \mathbb{Q}_\ell)) \end{array}$$

Now the discrete valuation on K induces a discrete valuation v on K_0.

Warning. This idea FAILS because the residue field $\mathcal{O}_{K_0,v}$ may have too many roots of 1 for the previous argument to work.

Example 15.6. Consider $K_0 = \mathbb{Q}(x,y) \to K = \mathbb{C}[[t]]$ given by $x \mapsto t$ and $y \mapsto \sum \zeta_n t^n$, where $\zeta_n = e^{\frac{2\pi i}{n}}$. Then $\kappa = \bigcup_n \mathbb{Q}(\zeta_n)$.

To prove the general case, we will write $A = \operatorname{colim} A_0$ as a colimit of regular rings, and apply Abhyankar's lemma.

16. Lecture 16 (October 29, 2015)

Recall that we are trying to prove the quasi-unipotent monodromy theorem.

Lemma 16.1. *To prove the quasi-unipotent monodromy theorem, it suffices to prove it when A is a complete dvr with $\kappa = \overline{\kappa}$.*

This is counterintuitive: in the previous lecture we needed κ to be small!

Proof. Omitted. Reason: any A can be completed and we can always find an extension of dvr's $A \subset A'$ with residue field of A' algebraically closed. The tame inertia of the Galois group of the fraction field of A' will map onto that of A. □

Néron desingularization is the same process needed for constructing Néron models. The references are Tag 0BJ7 of the Stacks Project and Artin's paper on approximation.

Theorem 16.2 (Néron desingularization)**.** *Let $R \subset \Lambda$ be an extension of dvr's with $e = 1$,* f.f.$(\Lambda)/$f.f.(R) *separable, and κ_Λ/κ_R separable. Then*

$$\Lambda = \operatorname{colim} A_i$$

is a filtered colimit of smooth R-algebras.

This theorem is amazing, but Néron's procedure of proving it is even more amazing. This is proved using Néron blowups, which I claim can be proved on a napkin!

Suppose $X \to \operatorname{Spec}(R)$ is a morphism of finite type with a section σ. Let $\eta \in \operatorname{Spec}(R)$ be the generic point and $0 \in \operatorname{Spec}(R)$ be a closed point. $\sigma(0)$ may not be smooth, so we consider the affine blowup at $\sigma(0)$ to get $X_1 \to \operatorname{Spec}(R)$ with a new section σ_1. The point $\sigma_1(0)$ may still not be smooth, so we do it again.

Fact. If $\sigma(\eta)$ is in the smooth locus of f, then after a finite number (n) of these Néron blowups, $\sigma_n(0)$ is in the smooth locus of $X_n \to \operatorname{Spec}(R)$.

Application (in the equal characteristic case): Suppose A is a complete dvr, $\kappa = \overline{\kappa}$ and A contains a field. Then (Cohen structure theorem) $A \cong \kappa[[t]]$ with $\kappa = \overline{\kappa}$. Say $\operatorname{char}(\kappa) = p > 0$. Then we may apply Néron desingularization to $\mathbb{F}_p[[t]] \subset \kappa[[t]]$.

We need to check the hypotheses are satisfied: indeed $\Omega^1_{\mathbb{F}_p[[t]]/\mathbb{F}_p} = \mathbb{F}_p[[t]]\, dt$ shows the extension of fraction fields is separable, and the other parts are clear.

Corollary 16.3. *$A = \operatorname{colim} A_i$ is a filtered colimit with A_i's smooth over $\mathbb{F}_p[[t]]$ or $\mathbb{Q}[[t]]$ and t maps to a uniformizer π of A.*

Remark. In particular, $K = \text{f.f.}(A) = A[\frac{1}{\pi}] = \operatorname{colim}_i A_i[\frac{1}{t}]$.

Remark. We can replace A_i by the henselization of $(A_i)_{A_i\cap\mathfrak{m}_A}$ and then we see

$$(A,\pi) = \operatorname{colim}(A_i, t)$$

is filtered, where

- A_i is a henselian regular local ring,
- $A_i/\mathfrak{m}_{A_i}$ is a finitely generated field extension of $\mathbb{F}_p$ or $\mathbb{Q}$,
- $t \in \mathfrak{m}_{A_i}$, $t \notin \mathfrak{m}_{A_i}^2$ (because $\mathbb{F}_p[[t]] \to A_i$ or $\mathbb{Q}[[t]] \to A_i$ is smooth).

Important question: What is the structure of

$$\pi_1(\operatorname{Spec}(A_i)\backslash V(t)) = \pi_1\left(\operatorname{Spec}\left(A_i\left[\frac{1}{t}\right]\right)\right)?$$

Theorem 16.4 (Generalized Abhyankar's lemma)**.** *Let $(A,\mathfrak{m},\kappa)$ be a regular henselian local ring. Let $t_1,\cdots,t_d$ be a regular system of parameters. Let $0 \le r \le d$. Then there exists a quotient*

$$\pi_1\left(\operatorname{Spec}\left(A_i\left[\frac{1}{t_1\cdots t_r}\right]\right)\right) \twoheadrightarrow \pi_1^t$$

with the following properties:

1. *The kernel of this map is topologically generated by pro-p-groups, where $p = \operatorname{char}(\kappa)$ (a pro-0-group is $\{1\}$).*
2. *There is a short exact sequence*

$$0 \to \prod_{i=1}^{r}\left(\lim_{n \text{ prime to } p} \mu_n(\kappa^{\text{sep}})\right) \to \pi_1^t \to \operatorname{Gal}(\kappa^{\text{sep}}/\kappa) = \pi_1(\operatorname{Spec}(A)) \to 1$$

where $\lim_{n \text{ prime to } p}\mu_n(\kappa^{\text{sep}})$ is non-canonically isomorphic to $\prod_{\ell\neq p}\mathbb{Z}_\ell$, such that the action of $\operatorname{Gal}(\kappa^{\text{sep}}/\kappa)$ on the group on the left is as indicated.

Remark. π_1^t is the *tame fundamental group* of $(\operatorname{Spec}(A), D = V(t_1\cdots t_r))$. There is a general definition of tame fundamental groups for pairs (S,D) where $D \subset S$ is a normal crossings divisor.

Idea of proof. The map comes from considering the Galois closure of

$$\bigcup_{n \text{ prime to } p} \text{f.f.}(A[\sqrt[n]{t_1},\cdots,\sqrt[n]{t_r}])$$

and combining with ramification theory (at generic points of $t_i = 0$), purity and Abhyankar's lemma. □

Corollary 16.5. *In the situation of generalized Abhyankar's lemma, assume $\ell \neq \operatorname{char}(\kappa)$ and κ has a finite number of ℓ-power roots of 1. Then any continuous*

$$\rho : \pi_1\left(\operatorname{Spec}\left(A_i\left[\frac{1}{t_1\cdots t_r}\right]\right)\right) \to \operatorname{GL}_n(\mathbb{Q}_\ell)$$

whose image is pro-ℓ, factors through π_1^t and maps elements of $\ker(\pi_1^t \to \operatorname{Gal}(\kappa^{\text{sep}}/\kappa))$ to quasi-unipotent elements.

Proof. Exactly the same as last lecture except we need to assume our image is pro-ℓ so that pro-p-groups map to 1. □

Lemma 16.6. *To prove the quasi-unipotent monodromy theorem, it suffices to prove it for* $(A, X/K, \ell, i)$ *when*

- *A is a complete dvr,*
- *the action $\rho^i_{X/K,\ell}$ is trivial mod ℓ, i.e., the action of* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ *on* $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/\ell\mathbb{Z})$ *is trivial.*

16.1. **Outline of proof of quasi-unipotent monodromy theorem.** We combine the above to prove the general case of the quasi-unipotent monodromy theorem.

Step 1: Reduce to A complete dvr, $\kappa = \overline{\kappa}$ (Lemma 16.1).

Step 2: Reduce to action on $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/\ell\mathbb{Z})$ trivial (Lemma 16.6).

Step 3: Write $(A, \pi) = \mathrm{colim}(A_i, t)$ with A_i henselian regular local, κ_{A_i} finitely generated over the prime field, and regular parameters (Remark).

Step 4 (limit): Get $X_i \to \mathrm{Spec}(A_i[\frac{1}{t}])$ proper smooth whose base change is X.

$$\rho^i_{X_i/A_i[\frac{1}{t}],\ell} : \pi_1\left(\mathrm{Spec}\left(A_i\left[\frac{1}{t}\right]\right)\right) \to \mathrm{Aut}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)).$$

Step 5 (limit): After possibly increasing i we may assume $\rho^i_{X_i/A_i,\ell}$ mod ℓ is trivial. By the previous lemma, $\mathrm{Im}(\rho^i_{X_i/A_i,\ell})$ is pro-ℓ.

Step 6: Conclude by Corollary 16.5.

Left over: mixed characteristic complete dvr A with $\kappa = \overline{\kappa}$. Consider $\mathbb{Z}_p \subset A$, with absolute ramification index e. Then the Cohen structure theorem gives $A \supset W(\kappa)$ with ramification index e. A uniformizer $\pi \in A$ satisfies an Eisenstein equation

$$\pi^e + \lambda_1 \pi^{e-1} + \cdots + \lambda_e = 0.$$

To finish, consider the diagram

$$\begin{array}{ccc} R & \longrightarrow & A \\ \uparrow & & \uparrow \\ W(\kappa_0) & \longrightarrow & W(\kappa) \end{array}$$

where

- $\kappa_0 \subset \kappa$ is the perfection of a finitely field extension of $\mathbb{F}_p$,
- $R \cong W(\kappa_0)[X]/(X^e + \lambda_1' X^{e-1} + \cdots + \lambda_e')$ with λ_i' close to λ_i.

This uses Krasner's lemma.

17. Lecture 17 (November 5, 2015)

Next week: no lectures.

Last lecture we have seen that Grothendieck's quasi-unipotent monodromy theorem follows from a statement of the form

Fact. Let A be a complete dvr with algebraically closed residue field κ and uniformizer π. Then we can write

$$(A, \pi) = \mathrm{colim}(A_i, t)$$

as a filtered colimit, where each A_i is regular local henselian, $t \in \mathfrak{m}_{A_i}/\mathfrak{m}_{A_i}^2$, with residue field $\kappa_i \in A_i/\mathfrak{m}_{A_i}$ a purely inseparable extension of a finitely generated extension of its prime field.

Last time we saw that the Fact is true in the equicharacteristic case.

17.1. **Proof in the mixed characteristic case.** We will prove the Fact in the mixed characteristic case.

Step 1: For every perfect field κ of characteristic $p > 0$ there exists a canonical complete dvr $W(\kappa)$ with uniformizer p such that (universal property): for any complete local ring $(B, \mathfrak{m}_B)$ and $\kappa \to B/\mathfrak{m}_B$ there exists a *unique* lift $W(\kappa) \to B$.

(References for the Witt ring include Lang, Serre and Zink.)

Step 2: Apply to our complete dvr A with residue field $\kappa = \overline{\kappa}$ to get

$$W(\kappa) \to A.$$

Then A will be finite flat over $W(\kappa)$ with some ramification index e. Pick a minimal equation

$$\pi^e + \lambda_1 \pi^{e-1} + \cdots \lambda_e = 0$$

with $\lambda_i \in W(\kappa)$.

Lemma 17.1 (Krasner's lemma)**.** *There exists $N > 0$ such that if $\lambda'_1, \cdots, \lambda'_e \in W(\kappa)$ satisfy $\lambda_i - \lambda'_i \in p^N W(\kappa)$, then*

$$P(x) = x^e + \lambda'_1 x^{e-1} + \cdots + \lambda'_e$$

has a root $\pi' \in A$ with $\pi' \equiv \pi \pmod{\mathfrak{m}_A^2}$.

Last lecture we considered Néron desingularization, which corresponds to the case $e = 1$ and gives A as a colimit of nice $\mathbb{Z}_p$-algebras. We want to work with general e.

Step 3: Given $\lambda \in W(\kappa)$ and $N > 0$ we can find $\kappa_0 \subset \kappa$, where κ_0 is the perfection of a finitely generated extension of $\mathbb{F}_p$, and $\lambda' \in W(\kappa_0)$ such that $\lambda - \lambda' \in p^N W(\kappa)$. Here we think of $W(\kappa_0) \subset W(\kappa)$ via the map corresponding to $\kappa_0 \to \kappa$.

Step 4: We get

$$\begin{array}{ccccc} W(\kappa)[x]/(P(x)) & = & A_0 & \xrightarrow{\text{Krasner}} & A \\ & & \uparrow & & \uparrow \\ & & W(\kappa_0) & \longrightarrow & W(\kappa) \end{array}$$

- Pick N as in Krasner's lemma.
- Pick $\kappa_0, \lambda'_1, \cdots, \lambda'_e$ using Step 3 (repeatedly).
- Set $A_0 = W(\kappa_0)[x]/(x^e + \lambda'_1 x^{e-1} + \cdots + \lambda'_e)$.
- Because we started with an Eisenstein polynomial we see that A_0 is a complete dvr with uniformizer the class $\overline{x}$ of x.
- We get $A_0 \to A$ by mapping $\overline{x}$ to π'.

Step 5: Néron desingularization applies to $A_0 \to A$. Thus

$$(A, \pi') = \operatorname{colim}(A_i, t)$$

with A_i the henselization of smooth algebras over A_0 at a prime. Then $\kappa_i = A_i/\mathfrak{m}_{A_i}$ has the desired property.

Remark. This argument does not reduce the difficult case of Grothendieck's quasi-unipotent monodromy theorem back to the easy case, but it uses the generalized Abhyankar lemma instead.

17.2. **Birational invariance of π_1.**

Lemma 17.2. *Let $f : X \to Y$ be a birational proper morphism of varieties with X normal and Y* nonsingular*. Then*

$$\pi_1(X) \cong \pi_1(Y).$$

Proof. Let $U \subset X$ be the largest open such that $f|_U : U \to f(U)$ is an isomorphism. Then $\operatorname{codim}(Y \backslash f(U), Y) \geq 2$ by the valuative criterion of properness for f and the fact that $f(U) \subset Y$ is the largest open over which f^{-1} lives. By purity, the top map is an isomorphism in the diagram

$$\begin{array}{ccc} \pi_1(Y) & \xleftarrow{\cong} & \pi_1(f(U)) \\ \uparrow & & \| \\ \pi_1(X) & \twoheadleftarrow & \pi_1(U) \end{array}$$

The bottom map is surjective because X is normal. This proves that $\pi_1(X) \cong \pi_1(Y)$. $\square$

Corollary 17.3. *"π_1 is a birational invariant."*

Idea: Given $K/\mathbb{C}$ finitely generated, Pick X smooth projective over $\mathbb{C}$ with $\mathbb{C}(X) \cong K$. Then $\pi_1(X)$ is independent of the choice of X (only depends on K).

Corollary 17.4. *If X is a rational smooth projective variety over $k = \overline{k}$, then $\pi_1(X) = \{1\}$.*

Proof. Choose a birational map $\varphi : \mathbb{P}^n_k \dashrightarrow X$, and let X' be the normalization of $\Gamma_\varphi \subset \mathbb{P}^n_k \times X$. Apply birational invariance twice. $\square$

Recall (not discussed): Let $f : X \to Y$ be a proper flat morphism of varieties all of whose geometric fibres are connected and reduced. Then there exists an exact sequence

$$\pi_1(X_{\overline{t}}) \to \pi_1(X) \to \pi_1(Y) \to 1$$

where $X_{\overline{t}}$ is a geometric fibre of f.

As a special case, let $Y = \mathbb{P}^1_k$ and X be a smooth projective surface over $k = \overline{k}$. Then $\pi_1(X)$ is a quotient of $\pi_1(X_{\overline{t}})$ for all $t \in \mathbb{P}^1$.

Example 17.5. If one fibre is a tree of $\mathbb{P}^1$'s then $\pi_1(X) = \{1\}$. For example, this is the case if $g(X_{\overline{\eta}}) = 0$.

Example 17.6. Suppose $g(X_{\overline{\eta}}) = 1$ and all fibres are at worst nodal. Then $X_{\overline{\eta}}$ is an elliptic curve E, and (with a grain of salt) either

- $X = E \times \mathbb{P}^1$ and $\pi_1(X) \cong \pi_1(E)$, or
- $\pi_1(X)$ is finite cyclic.

Proof. No singular fibres $\Leftrightarrow X = E \times \mathbb{P}^1 \Leftrightarrow j$-invariant of fibres is constant. The rest of the cases occur when there exists a bad fibre $X_{\overline{t}}$. Then $\pi_1(X_{\overline{t}}) \cong \widehat{\mathbb{Z}}$ and $\pi_1(X)$ is procyclic. We have the correspondence

$$\left\{ \begin{matrix} \text{connected cyclic finite étale covers} \\ \text{with group } \mathbb{Z}/n\mathbb{Z} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \mathcal{L} \text{ invertible on } X \text{ with } \mathcal{L}^{\otimes n} \cong \mathcal{O}_X \\ \text{but } \mathcal{L}^{\otimes m} \ncong \mathcal{O}_X \text{ for } 0 < m < n \end{matrix} \right\}$$

$$(g : X' \to X) \mapsto \mathcal{L}_\chi$$

by choosing a primitive n-th root of 1 in k and the corresponding $\chi : \mathbb{Z}/n\mathbb{Z} \to k^\times$. More precisely, the action of $\mathbb{Z}/n\mathbb{Z}$ gives the decomposition $(g_* \mathcal{O}_{X'}) = \bigoplus_{\chi : \mathbb{Z}/n\mathbb{Z} \to k^\times} \mathcal{L}_\chi$.

Let $\eta \in Y = \mathbb{P}^1$ be the generic point, and $K = \kappa(\eta) = k(\mathbb{P}^1) \cong k(t)$. Then $\mathcal{L}|_{X_\eta}$ gives a K-point of $E = \underline{\mathrm{Pic}}^0_{X_\eta/\eta}$ (an elliptic curve over K) whch has order n.

By Lang–Néron, $E(K)$ is a finitely generated abelian group, because $j(E/K) \notin k$. This implies $E(K)$ has a finite amount of torsion. □

Warning. Prof. de Jong is worried that something went wrong!!

18. Lecture 18 (November 17, 2015)

Last time we finished our discussion on the monodromy theorem. Today we will talk about the semi-stable reduction theorem for curves.

18.1. **Semi-stable reduction theorem.** This is what the theorem says in the complex-analytic category. Suppose we have a family of curves over the punctured disk D^*, and we want to fill this in with a smooth proper curve at the center. In general this is not possible, but after base change by $z \mapsto z^n$, we can fill in a nodal curve.

This complex version is not that hard to prove. I will formulate this in the correct generality in algebraic geometry, with dvr's.

Definition 18.1. Let X be a locally Noetherian scheme. A *strict normal crossings divisor on* X is an effective Cartier divisor $D \subset X$ such that for all $p \in D$, the local ring $\mathcal{O}_{X,p}$ is regular and there exists a system of parameters $x_1, \cdots, x_d \in \mathfrak{m}_p$ and $1 \le r \le d$ such that

$$D \times_X \mathrm{Spec}(\mathcal{O}_{X,p}) = V(x_1 \cdots x_r).$$

Lemma 18.2. *This is equivalent to:*

- *$D \subset X$ effective Cartier divisor;*
- *each irreducible component D_i of D is regular;*
- *for $J = \{i_1, \cdots, i_t\}$ where $t = \#J$, $D_J = D_{i_1} \cap \cdots \cap D_{i_t}$ is regular of codimension t in X;*
- *$D = \sum_{i \in I} D_i$ (i.e., D is reduced).*

Example 18.3.

- Two lines intersecting at a point: Yes.
- Three lines intersecting at point: No.
- Two curves meeting tangentially at a point: No.
- A self-intersecting curve: No.
- Cone: No.
- Two plane curves intersecting transversally at two points: Yes.

Definition 18.4. For X locally Noetherian, an effective Cartier divisor $D \subset X$ is a *normal crossings divisor* if there exists an étale covering $\{U_k \to X\}_{k \in K}$ such that $D \times_X U_k \subset U_k$ is a strict normal crossings divisor for all $k \in K$.

Remark. Normal crossings divisors "can have self-intersections".

Example 18.5. The curve $Y^2 + X^3 + X^2 = 0$ in $\mathbb{A}^2_{\mathbb{R}}$ is a normal crossings divisor.

Definition 18.6. Let R be a dvr, and $S = \mathrm{Spec}(R)$ be the trait[2] with generic point η and closed point s.

[2] This is French!

(a) An *S-variety* is an integral scheme, separated and of finite type over S with $X_\eta \neq \emptyset$.
(b) $X_s = f^{-1}(s) = X \otimes \kappa(s) = V(\pi) \subset X$ is the *special fibre*.
(c) We say X is *strictly semi-stable* over S if (c1) to (c4) hold.
  (c1) X_η is smooth over $\kappa(\eta)$.
  (c2) X_s is reduced.
  (c3) Each irreducible component X_i of X_s is an effective Cartier divisor on X. (By (c2) this implies $X_s = \sum_{i \in I} X_i$.)
  (c4) For all nonempty $J \subset I$ we have that the scheme-theoretic intersection $X_J = \bigcap_{i \in J} X_i$ is *smooth* over $\kappa(s)$ and has codimension $\#J$ in X.
  (In particular X_s is a strict normal crossings divisor on X.)

Fact. If $x \in X_s$, then

$$\widehat{\mathcal{O}}_{X,x} \cong B[[t_1, \cdots, t_r]]/(t_1 \cdots t_r - \pi)$$

where B is a complete local R-algebra which is formally smooth over R.

Fact. If $\kappa(s)$ is perfect, then

$$\text{(c1), (c2), (c3)} \Leftrightarrow X_s \text{ is an s.n.c.d. in } X.$$

Example 18.7. Let $R = \kappa[[t]] \subset A = \kappa'[[t]]$ with κ'/κ not separable. Then for $X = \mathrm{Spec}(A)$, X_s is an s.n.c.d. but (c4) is violated.

Definition 18.8. We say X is *semi-stable over S* if étale locally on X we have X is strictly semi-stable (s.s.-s.) over S.

We have the following diagram of implications:

$$\begin{array}{ccccc}
(X_s)_{\mathrm{red}} \text{ n.c.d.} & \Longleftarrow & X_s \text{ n.c.d.} & \underset{\Longleftarrow}{\overset{\kappa(s) \text{ is perfect}}{\dashrightarrow}} & X \text{ s.s.}/S \\
\Uparrow & & \Uparrow & & \Uparrow \\
(X_s)_{\mathrm{red}} \text{ s.n.c.d.} & \Longleftarrow & X_s \text{ s.n.c.d.} & \underset{\Longleftarrow}{\overset{\kappa(s) \text{ is perfect}}{\dashrightarrow}} & X \text{ s.s.s.}/S.
\end{array}$$

Definition 18.9. Let S be a scheme. A *semi-stable curve over S* is a morphism $X \to S$ which is flat, proper, of finite presentation, such that all geometric fibres are connected, and of dimension 1 with singularities at worst nodes.

Recall that for X finite type over $k = \overline{k}$ and $\dim(X) = 1$, a closed point $x \in X$ is a node if and only if $\widehat{\mathcal{O}}_{X,x} \cong k[[u, v]]/(uv)$.

Definition 18.10. A *split semi-stable curve X over a field k* is a semi-stable curve over k whose irreducible components are all geometrically irreducible, and whose nodes are all k-rational. A *split semi-stable curve over S* is a semi-stable curve over S such that all fibres are split.

Lemma 18.11. *Let R be a dvr, $S = \mathrm{Spec}(R)$, and $X \to S$ a proper S-variety of relative dimension 1 with geometrically connected fibres. If X/S is semi-stable as an S-variety, then X is a semi-stable curve over S.*

Warning. The converse is not true.

Remark. For the converse it is true that a blowup of an X on the RHS will be in the LHS. (Grain of salt!)

In terms of moduli theory, we want to produce a semi-stable model for every curve over K by lifting $\mathrm{Spec}(K) \to \mathcal{M}_g$ to $\mathrm{Spec}(R) \to \overline{\mathcal{M}_g}$

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(R) & & \\ \downarrow & & \vdots & \searrow & \\ \mathcal{M}_g & \hookrightarrow & \overline{\mathcal{M}_g} & \longrightarrow & \mathrm{Spec}(\mathbb{Z}). \end{array}$$

As stated this is wrong, and the issue is that $\overline{\mathcal{M}_g}$ is an algebraic stack. The valuative criterion for stacks should allow for finite extensions. With this modification, such a lifting becomes possible.

Theorem 18.12 (Semi-stable reduction of curves)**.** *Given R dvr with fraction field K, and C/K smooth, proper, geometrically connected curve there exist:*

- $R \subset R'$ *extension of dvr's,*
- $X' \to S' = \mathrm{Spec}(R')$ *an s.s.s. S-variety,*
- $C \otimes_K K' \cong X' \otimes_{R'} K'$.

Additional properties we would like:

- K'/K finite separable,
- $Y \to \mathrm{Spec}(B)$ where B is the integral closure of R in K' such that $Y \otimes_B B_{\mathfrak{m}'}$ is an s.s.s. variety over $B_{\mathfrak{m}'}$ for all maximal ideals $\mathfrak{m}' \subset B$. (A standard argument allows one to work with one $\mathfrak{m}'$ at a time; we will not discuss further.)

Here is the strategy.

Step 1: Pick some $K^{\mathrm{sep}}/K'/K$ finite separable such that some property holds for the $\mathrm{Gal}_{K'}$-action on $\mathrm{Pic}^0(C_{K^{\mathrm{sep}}})_{\mathrm{tors}}$.

Step 2: Let $R' = B_{\mathfrak{m}'}$ with $B \supset \mathfrak{m}'$ as before. We will show there exists $X' \to \mathrm{Spec}(R')$ (relatively minimal model) which is proper flat, X' regular, $(X_s)_{\mathrm{red}}$ s.n.c.d. and without (-1)-curves, with $X' \otimes K' \cong C \otimes K'$.

Step 3: Show that property in Step 1 implies the model of Step 2 is s.s.s.

Artin–Winters picks $\ell \gg g$ and trivializes the action on ℓ-torsion. Saito makes the action on $T_\ell(\mathrm{Pic}(C_{\overline{K}}))$ unipotent.

19. Lecture 19 (November 19, 2015)

The proof of the semistable reduction of curves involves many ingredients, one of which is the resolution of singularities of surfaces.

19.1. **Resolution of singularities.**

Definition 19.1. Let Y be a Noetherian integral scheme. A *resolution of singularities* is a modification $X \to Y$ such that X is regular.

Definition 19.2. A *modification* is a proper birational morphism of integral schemes.

In general, Noetherian schemes are horrible and might not admit resolutions of singularities. For example, take the spectrum of a Noetherian domain of dimension 1 whose completion is not reduced. This motivated Grothendieck to introduce excellent rings and excellent schemes, which characterize in terms of commutative algebra when resolutions of singularities exist. There is the celebrated

Theorem 19.3 (Lipman)**.** *Let Y be an integral Noetherian 2-dimensional scheme such that:*

(1) *the normalization morphism $Y^\nu \to Y$ is finite;*
(2) *Y^ν has finitely many singular points $y_1, \cdots, y_n$ and $\mathcal{O}^\wedge_{Y^\nu, y_i}$ is normal.*

Then there exists a resolution of singularities of Y. The converse is also true.

Remark. If Y is of finite type over a field or $\mathbb{Z}$ or a characteristic 0 Dedekind domain or a complete Noetherian local ring, then (1) and (2) hold (in fact, Y is a quasi-excellent scheme).

A nice exposition is Artin's article in *Arithmetic Geometry*, edited by Cornell and Silverman.

Notation. Fix R a dvr with $K = \text{f.f.}(R)$, and C/K a proper smooth geometrically connected curve.

Proposition 19.4. *There exists*

$$\begin{array}{ccc} C & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec}(K) & \longrightarrow & \operatorname{Spec}(R) \end{array}$$

with f flat proper and X regular.

Proof. Choose $C \hookrightarrow \mathbb{P}^n_K$ and let $Y \subset \mathbb{P}^n_R$ be the Zariski closure.

Fact (Technical mumbo-jumbo)*.* Y satisfies (1) and (2) of Lipman's theorem.

Then we get $X \to Y$ a resolution of singularities. Note, since $C \subset Y$ is open and C is regular of dimension 1, we see that $X \to Y$ is an isomorphism over C. Therefore, $X_K \cong C$.

(Hint (alteration): If $f : X \to Y$ is a proper, generically finite, dominant morphism of Noetherian integral schemes and if $y \in Y$ has codimension ≤ 1, then there exists $y \in V \underset{\text{open}}{\subset} Y$ such that $f^{-1}(V) \to V$ is finite.) ☐

Example 19.5. Consider $\mathbb{Z}_p \to \mathbb{Z}_p[x, y]/(xy(x+y)^{10} - p)$. This is regular at the maximal ideal (x, y, p).

Theorem 19.6 (Embedded resolutions)**.** *Let Y be a regular 2-dimensional scheme. Let $Z \subset Y$ be a closed subscheme all of whose irreducible components Z_i are either points or 1-dimensional schemes whose normalization is finite. Then there exists a sequence of blowups $f : X \to Y$ such that $f^{-1}(Z)_{\text{red}}$ is a strict normal crossings divisor (s.n.c.d.).*

This is a lot easier to show than Lipman's theorem.

Proposition 19.7. *There exists*

$$\begin{array}{ccc} C & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec}(K) & \longrightarrow & \operatorname{Spec}(R) \end{array}$$

such that f is projective and flat, X is regular, $(X_s)_{\text{red}}$ is a s.n.c.d.

Notation. Let $X \to \operatorname{Spec}(R) = S$ be as in the proposition, $C_1, \cdots, C_n$ the irreducible components of X_s. Write $X_s = \sum r_i C_i$ as Cartier divisors, where r_i is the multiplicity of C_i in X_s. Then X_s is reduced if and only if $r_i = 1$ for all $1 \leq i \leq r$.

Claim. If $\operatorname{char}(\kappa(s)) = 0$, then set $r = \operatorname{lcm}(r_i)$, $R' = R[\pi']$ with $\pi' = \pi^{1/r}$, and X' the normalization of $X \times_{\operatorname{Spec}(R)} \operatorname{Spec}(R')$. Then

$$X' \to \operatorname{Spec}(R') = S'$$

is a semi-stable curve over R'.

This is not the same thing as saying X' is a semi-stable S'-variety! However, we have the

Lemma 19.8 (Addendum to last time)**.** *If X is a semi-stable curve over a dvr R with smooth generic fibre, then*

- *the singularities of X as a surface are (A_k), $k \geq 1$;*
- *there exists a repeated blowup $X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$ such that X_n is a semi-stable S-variety.*

Applying the Claim followed by the addendum, we get a semi-stable S'-variety.

Remark. Claim is also true if:

- $\operatorname{char}(\kappa(s)) = p > 0$,
- $p \nmid r_i$ for all $i = 1, \cdots, n$,
- $C_i \to \operatorname{Spec}(\kappa(s))$ is smooth,
- if $x \in C_i \cap C_j$ $(i \neq j)$ then $\kappa(x)/\kappa(s)$ is separable.

Example 19.9. Let $\kappa = \mathbb{F}_p(t)$ where $p \neq 2$. Then $C = \operatorname{Spec} \kappa[x,y]/(y^2 - x^p - t)$ is a regular non-smooth curve over κ, as follows. One can check that

$$\operatorname{Sing}(C \to \operatorname{Spec}(\kappa)) = V(y, y^2 - x^p - t) = \{(y, x^p + t)\}.$$

Denote $Q = (y, x^p + t) \in C$. Then $\kappa(Q) = \kappa[x]/(x^p + t)$, and $\mathcal{O}_{C,Q}$ is a regular local ring as $\mathfrak{m}_Q = (y)$. But over the algebraic closure,

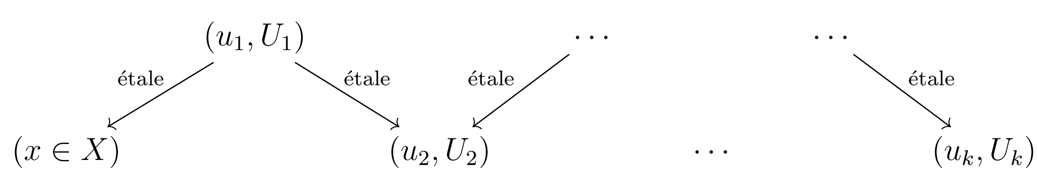
$$C \times_{\operatorname{Spec}(\kappa)} \operatorname{Spec}(\overline{\kappa}) = \operatorname{Spec}(\overline{\kappa}[x,y]/(y^2 - (x + t^{1/p})^p)) \cong \operatorname{Spec}(\overline{\kappa}[x,y]/(y^2 - x^p))$$

is not smooth.

Example 19.10 (Worse)**.** Let $\kappa' = \kappa[x]/(x^p - t)$. Take $C = \operatorname{Spec}(\kappa'[y]) \to \operatorname{Spec}(\kappa)$.

Sketch of proof of Claim. The idea is to describe étale local structure of X and use that normalization commutes with étale localization.

We will not define "étale local structure", but the picture is



where (u_k, U_k) should be in a standard form.

Example 19.11. A smooth morphism is étale locally like $\mathbb{A}^n_S \to S$.

Let $x \in C_i \cap C_j$ ($i \neq j$) with

(1) $\kappa(x)/\kappa(s)$ separable,

(2) r_i, r_j prime to $\mathrm{char}(\kappa(s))$.

We have $R \to \mathcal{O}_{X,x} \supset \mathfrak{m}_x$. Let $a, b \in \mathfrak{m}_x$ be local equations for C_i, C_j, and $n = r_i$ and $m = r_j$. Then

$$\pi = u \cdot a^n b^m$$

where $u \in \mathcal{O}_{X,x}^\times$. Because $\mathrm{char}(\kappa(s)) = \mathrm{char}(\kappa(x))$ does not divide n, after replacing X by an étale cover we may assume $\pi = a^n b^m$ (adjoin n-th root of u to $\mathcal{O}_{X,x}$).

Now $\kappa(x)/\kappa(s)$ is separable, so

$$\begin{aligned} \mathcal{O}_{X,x} &\leftarrow R[u,v]/(u^n v^m - \pi) \\ a &\leftarrow\!\shortmid u \\ b &\leftarrow\!\shortmid v \end{aligned}$$

defines an étale morphism $(X, x) \to \mathrm{Spec}(R[u,v]/(u^n v^m - \pi))$.

In characteristic 0, consider $A_n = R[u,v]/(u^n - \pi)$ and $A_{n,m} = R[u,v]/(u^n v^m - \pi)$.

Final step: Compute the étale local structure of

$$A'_{n,m,d} = (R'[u,v]/(u^n v^m - (\pi')^d))^{\mathrm{norm}}$$

where $R' = R[\pi]$ and $(\pi')^d = \pi$, $n \mid d$, $m \mid d$.

Step 1: If $e = \gcd(n,m) > 1$ and $\zeta_e \in R'$, then

$$A'_{n,m,d} \cong \prod_{i=1}^{e} A'_{n/e, m/e, d/e}.$$

Why? $(u^{n/e} v^{m/e})^e = ({\pi'}^{d/e})^e$ in $A'_{n,m,d}$.

Step 2: If $\gcd(n,m) = 1$, then

$$u^n v^m = (\pi')^d$$

in $A'_{n,m,d}$, i.e.,

$$u^n = \frac{(\pi')^d}{v^m} = \left(\frac{{\pi'}^{d/m}}{v}\right)^m,$$

so there exists $u' \in A'_{n,m,d}$ such that

$$(u')^m = u \text{ and } (u')^n = \frac{(\pi')^{d/m}}{v}.$$

Similarly, there exists $v' \in A'_{n,m,d}$ such that

$$(v')^n = v \text{ and } (v')^m = \frac{(\pi')^{d/n}}{u}.$$

Check that

$$A'_{n,m,d} \equiv R'[u',v']/(u'v' - (\pi')^{d/nm}).$$

This ring has singularities that are at worst nodes. $\square$

20. Lecture 20 (November 24, 2015)

Last time we showed that, conditional on resolution of singularities of surfaces, in characteristic 0 we have semi-stable reduction of curves (in either of the two senses). Today I will say what problems we run into if we try to work in arbitrary characteristic.

Let C be a curve and $J = \mathrm{Jac}(C)$. There is an action ρ of I on

$$T_\ell J = \varprojlim J[\ell^n](K^{\mathrm{sep}}).$$

If C has semi-stable reduction, then the action ρ is unipotent.

20.1. Method of Artin–Winters.

Suppose we have a Cartesian diagram

$$\begin{array}{ccc} C & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(K) & \hookrightarrow & \mathrm{Spec}(R) \end{array}$$

where

- R is a dvr with fraction field K,
- C is smooth, proper, and geometrically connected over K,
- f is proper and flat,
- X is regular.

Let $C_1, \cdots, C_m$ be the irreducible components of X_s. Then

$$X_s = \sum r_i C_i$$

as Cartier divisors. Let $r = \gcd(r_1, \cdots, r_n)$.

Warning. r need not be 1.

Lemma 20.1. *X_s is geometrically connected over $\kappa(s)$.*

Proof. By the conditions on C, we have $H^0(C, \mathcal{O}_C) = K$ and so $f_*\mathcal{O}_X = \mathcal{O}_S$ where $S = \mathrm{Spec}(R)$. By the Zariski main theorem, all fibres are geometrically connected. □

For $\mathcal{L} \in \mathrm{Pic}(X)$ and $i \in \{1, \cdots, n\}$, we define

$$\mathcal{L} \cdot C_i = \deg(\mathcal{L}|_{C_i})$$

where the degree is computed over $\kappa(s)$. (For example, $\mathcal{O}(1)$ on $\mathbb{P}^1_{\mathbb{Q}(i)}$, as a variety over $\mathbb{Q}$, has degree 2.) If $D \subset X$ is an (effective) Cartier divisor, we define

$$D \cdot C_i = \mathcal{O}_X(D) \cdot C_i.$$

Fact.

$$C_i \cdot C_j = C_j \cdot C_i = \begin{cases} \text{degree over } \kappa(s) \text{ of the scheme } C_i \cap C_j & \text{if } i \neq j, \\ \deg(\mathcal{N}_{C_i/X}) & \text{if } i = j. \end{cases}$$

Remark. Because X is regular, $C_i \subset X$ is Cartier and $\mathcal{I} = \mathcal{I}_{C_i/X}$ is invertible and $\mathcal{O}_X(C_i) \cong \mathcal{I}^{-1}$ so $\mathcal{O}_X(C_i)|_{C_i} \cong \mathcal{N}_{C_i/X}$. (It is always true that the normal sheaf

$$\mathcal{N}_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}_{Z/X}/\mathcal{I}_{Z/X}^2, \mathcal{O}_Z)$$

where $\mathcal{I}_{Z/X}/\mathcal{I}_{Z/X}^2$ is the conormal sheaf.)

Lemma 20.2. $\left(\sum r_i C_i\right) \cdot C_j = 0$ *for all* j.

Proof. $\mathcal{O}_X(\sum r_i C_i) = \mathcal{O}_X(X_s) \cong \mathcal{O}_X$, where the isomorphism $\mathcal{O}_X \xrightarrow{\cong} \mathcal{O}_X(X_s)$ is given by multiplication by a uniformizer $\pi \in R$. □

Set $\Lambda = \bigoplus_{i=1}^n \mathbb{Z}C_i \ni F = \sum r_i C_i$. Define a pairing

$$\Lambda \times \Lambda \to \mathbb{Z}$$
$$\left(D = \sum a_i C_i, D' = \sum b_i C_i\right) \mapsto D \cdot D' = \sum a_i b_j C_i C_j$$

which is a symmetric bilinear form.

Lemma 20.3. *The pairing* $\Lambda \times \Lambda \to \mathbb{Z}$ *is semi-negative definite and if* $Z \in \Lambda$ *with* $Z^2 = 0$, *then* $Z \in \mathbb{Z}(\frac{1}{r}F)$.

Proof. Say $Z = \sum s_i C_i$. Then

$$\begin{aligned} Z^2 = \left(\sum_i s_i C_i\right)^2 &= \sum_i s_i C_i \left(\sum_j s_j C_j\right) \\ &= \sum_i s_i C_i \left(\sum_j s_j C_j - \frac{s_i}{r_i} \sum_j r_j C_j\right) \\ &= \sum_i s_i C_i \left(\sum_{j \neq i} \left(s_j - \frac{s_i r_j}{r_i}\right) C_j\right) \\ &= \sum_{i \neq j} \frac{s_i}{r_i}(r_i s_j - r_j s_i) C_i C_j \\ &= \sum_{i<j} -\frac{(r_i s_j - r_j s_i)^2}{r_i r_j} C_i C_j. \end{aligned}$$

Now use the fact that X_s is connected. □

There is an exact sequence

$$0 \to \mathbb{Z} \to \Lambda \to \operatorname{Pic}(X) \to \operatorname{Pic}(C) \to 0$$
$$1 \mapsto F.$$

There is also the restriction map

$$\operatorname{Pic}(X) \to \operatorname{Pic}(X_s)$$
$$\mathcal{L} \mapsto \mathcal{L}_{X_s}.$$

We want to relate $\operatorname{Pic}(C)$ (for the generic fibre) with $\operatorname{Pic}(X_s)$ (for the special fibre).

Lemma 20.4. *If* n *is prime to* $\operatorname{char}(\kappa(s))$, *then*

$$\operatorname{Pic}(X)[n] \to \operatorname{Pic}(X_s)[n]$$

is injective.

Proof. Skipped. □

Set $\Lambda^* = \operatorname{Hom}(\Lambda, \mathbb{Z})$. Then we get an exact sequence

$$0 \to \mathbb{Z} \to \Lambda \xrightarrow{\alpha} \Lambda^* \to G \to 0$$
$$1 \mapsto \frac{1}{r}F$$

where

$$\alpha(D) = \text{linear form } D' \mapsto D \cdot D'.$$

This implies

$$G \cong G_{\text{tors}} \oplus \mathbb{Z}$$

with $\# G_{\text{tors}} < \infty$.

Lemma 20.5. *There is an exact sequence*

$$0 \to \mathbb{Z}/\gcd(n,r)\mathbb{Z} \to \operatorname{Pic}(X)[n] \to \operatorname{Pic}(C)[n] \to \frac{\alpha^{-1}(n\Lambda^*)}{n\Lambda + \mathbb{Z}F}.$$

Proof. $\ker(\operatorname{Pic}(X) \to \operatorname{Pic}(C)) = \Lambda/\mathbb{Z}F$. The torsion in this is $\mathbb{Z}(\frac{1}{r}F)/\mathbb{Z}F \cong \mathbb{Z}/r\mathbb{Z}$. Hence

$$(\mathbb{Z}/r\mathbb{Z})[n] \cong \mathbb{Z}/\gcd(r,n)\mathbb{Z}.$$

Suppose $\mathcal{L}_C \in \operatorname{Pic}(C)[n]$. Pick $\mathcal{L} \in \operatorname{Pic}(X)$ with $\mathcal{L}_C \cong \mathcal{L}|_C$. Then

$$\mathcal{L}^{\otimes n} \cong \mathcal{O}_C\left(\sum a_i C_i\right)$$

for some $\sum a_i C_i \in \Lambda$ well-defined modulo $\mathbb{Z}F$. Moreover

$$\left(\sum a_i C_i\right) \cdot C_j = \mathcal{L}^{\otimes n} \cdot C_j = n(\mathcal{L} \cdot C_j) \in n\mathbb{Z}.$$

Hence $\sum a_i C_i \in \alpha^{-1}(n\Lambda^*)$ is well-defined modulo $\mathbb{Z}F$. On the other hand we can replace $\mathcal{L}$ by $\mathcal{L}(\sum b_i C_i)$ for some $\sum b_i C_i \in \Lambda$. Then $\sum a_i C_i$ changes to $\sum a_i C_i + n(\sum b_i C_i)$. We see that we get a well-defined class in $\frac{\alpha^{-1}(n\Lambda^*)}{n\Lambda + \mathbb{Z}F}$ which if zero means we can choose $\mathcal{L}$ to be n-torsion. ☐

Lemma 20.6. *There is an exact sequence*

$$0 \to \frac{\alpha^{-1}(n\Lambda^*)}{n\Lambda} \to \Lambda/n\Lambda \to \Lambda^*/n\Lambda^* \to G/nG \to 0$$

and hence

$$\# \operatorname{Pic}(X)[n] \geq \frac{\# \operatorname{Pic}(C)[n]}{\#(G_{\text{tor}}/nG_{\text{tor}})}.$$

Proof. Chase diagrams. ☐

20.2. **Example applications.**

Example 20.7. If R is strictly henselian, $\ell \in \kappa(s)^\times$ is a prime and $\operatorname{Gal}(K^{\text{sep}}/K) = I$ acts trivially on $J[\ell^m](K^{\text{sep}})$ for all $m \geq \ell$, then we can conclude

$$\# \operatorname{Pic}(X_s)[\ell^m] \geq (\ell^m)^{2g}/\text{fixed constant}$$

where J is the Jacobian of C/K and g is the genus of C. This implies

$$\dim_{\mathbb{F}_\ell} \operatorname{Pic}(X_s)[\ell] \geq 2g.$$

We will see later this implies X_s is a tree of smooth curves with $\sum g_i = g$.

Example 20.8. Again say ℓ prime to $\mathrm{char}(\kappa(s))$ and $\mathrm{Pic}(C)[\ell] \cong \mathbb{F}_\ell^{2g}$. Then we get

$$\dim_{\mathbb{F}_\ell} \mathrm{Pic}(X_s)[\ell] \geq 2g - \dim_{\mathbb{F}_\ell}(G_{\mathrm{tor}}/\ell G_{\mathrm{tor}}).$$

Warning. We cannot pick ℓ after choosing model X.

20.3. **Idea of Artin–Winters.** The idea is to work with X such that $(X_s)_{\mathrm{red}}$ n.c.d.

(1) Find an *a priori* bound on the "types" of graphs we can get for fixed genus g.
(2) Show that $\dim_{\mathbb{F}_\ell}(\mathrm{Pic}(X_s)[\ell]) \geq 2g - \beta$ where $\beta = \beta(\text{graph})$ with equality if and only if semi-stable.

21. Lecture 21 (December 1, 2015)

PH: I missed the lecture.

22. Lecture 22 (December 3, 2015)

22.1. **Abstract types of genus g.** Last time we showed that: for $g \geq 2$, the abstract types of genus g with no (-1)-curves is finite up to equivalence.

Definition 22.1. Let G be a finitely generated abelian group. Let $c \geq 1$. Then

$$\rho_c(G) = \min \left\{ r \;\middle|\; \begin{array}{l} \text{there exists a subgroup } H \subset G \text{ of index dividing } c \\ \text{such that } H \text{ can be generated by } r \text{ elements} \end{array} \right\}.$$

Example 22.2. $\rho_1(G)$ is the minimal number of generators of G.

Lemma 22.3. *If $0 \to G_1 \to G_2 \to G_3 \to 0$ is exact, then*

$$\rho_{cc'}(G_2) \leq \rho_c(G_1) + \rho_{c'}(G_3)$$

and

$$\rho_c(G_2) \geq \rho_c(G_3).$$

Example 22.4. If $\ell \nmid c$ is prime, then

$$\dim_{\mathbb{F}_\ell}(G/\ell G) \leq \rho_c(G)$$

and

$$\dim_{\mathbb{F}_\ell}(G[\ell]) \leq \rho_c(G).$$

Theorem 22.5 (Artin–Winters)**.** *For any $g \geq 0$ there exists a $c = c(g)$ such that if T is an abstract type of genus g, then*

$$\rho_c(G) \leq 1 + \beta$$

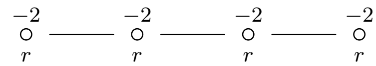
where

- *β is the 1st Betti number of the graph T,*
- *$G = \mathrm{coker}(\Lambda \xrightarrow{(m_{ij})} \Lambda^*)$ with $\Lambda = \bigoplus_{i=1}^n \mathbb{Z}C_i$.*

Proof. The steps are:

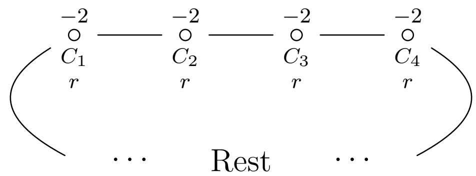
Step 1: Argue that (-1)-curves can be "contracted".
Step 2: Do $g = 0, 1$ separately.
Step 3: Use boundedness of last lecture to do $g \geq 2$ by induction on g.

I will explain Step 3. First, there are finitely many abstract types of genus g with no subgraphs that are chains of (-2) with the same multiplicity r of length 4 (from last time):

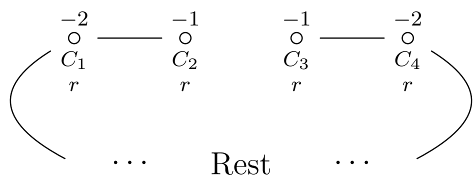


OK for these with in fact $\rho_c(G) \leq 1$ for a suitable c.

If the abstract type T does contain such a chain:



then we remove the middle edge to form T':



Recall that

$$g = 1 + \frac{1}{2}\sum k_i r_i.$$

Since $k_1 = k_2 = k_3 = k_4 = 0$ in T but $k'_2 = k'_3 = -1$ in T', we have $g' < g$. If T' is connected, then it is an abstract type of genus g'. Otherwise it breaks into two connected components, but we will omit this case.

Letting $x_1, \cdots, x_n$ be the basis of Λ^* dual to $C_1, \cdots, C_n$ in Λ, we see

$$M = \operatorname{Im}(\Lambda \overset{m_{ij}}{\to} \Lambda^*) = \operatorname{Span}\left(\sum_j m_{ij}x_j, i \neq 2,3; x_1 - 2x_2 + x_3; x_2 - 2x_3 + x_4\right)$$

and

$$M' = \operatorname{Im}(\Lambda \overset{m'_{ij}}{\to} \Lambda^*) = \operatorname{Span}\left(\sum_j m_{ij}x_j, i \neq 2,3; x_1 - x_2; -x_3 + x_4\right).$$

We see that

$$M' \subset M + \mathbb{Z}(x_2 - x_3),$$

so there is a surjection

$$G = \Lambda^*/M \twoheadrightarrow \Lambda^*/M + \mathbb{Z}(x_2 - x_3) = G/\langle x_2 - x_3\rangle$$

with cyclic kernel, and similarly a surjection

$$G' = \Lambda^*/M' \twoheadrightarrow G/\langle x_2 - x_3\rangle.$$

Hence

$$\rho_c(G) \leq \rho_c(G/\langle x_2 - x_3\rangle) + 1 \leq \rho_c(G') + 1 \leq 1 + (\beta - 1) + 1 = 1 + \beta,$$

where the last inequality follows by induction hypothesis if T' is connected and c works for all lower genera.

The disconnected case is similar (but trickier). □

22.2. **Part 2 of Artin–Winters' argument.** Given C/K, R and $\kappa = \overline{\kappa}$. Pick ℓ prime not dividing $c(g)$ and prime to $\mathrm{char}(\kappa)$ where $g = g(C)$. Let K' be a finite separable extension of K such that

$$\mathrm{Pic}(C_{K'})[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^{2g}$$

and

$$C(K') \neq \emptyset.$$

Replace R, K, C by $R', K', C_{K'}$. We will show C has semi-stable reduction.

Pick

$$\begin{array}{ccc} C & \hookrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

a regular flat proper model without (-1)-curves (omitted: can contract (-1)-curves). We will show *this* model is semi-stable.

The existence of rational point implies:

- $\gcd(r_i) = 1$ (there is an i with $r_i = 1$).
- $\dim_\kappa H^1(X_s, \mathcal{O}) = g$.
- $\dim_\kappa H^1(X_s, \mathcal{O}) \geq \dim H^1((X_s)_{\mathrm{red}}, \mathcal{O})$ with equality if and only if $X_s = (X_s)_{\mathrm{red}}$.

By our choice of ℓ we have

$$2g - \beta \leq \dim_{\mathbb{F}_\ell} \mathrm{Pic}(X_s)[\ell].$$

This uses $\rho_{c(g)}(G) \leq 1 + \beta$ (where β is the 1st Betti number of our graph), $\ell \nmid c(g)$, material of two lectures ago, plus the example of today. It is easy to see that

$$\dim_{\mathbb{F}_\ell} \mathrm{Pic}(X_s)[\ell] = \dim_{\mathbb{F}_\ell} \mathrm{Pic}((X_s)_{\mathrm{red}})[\ell].$$

Set $Y = (X_s)_{\mathrm{red}}$. Then we get a long exact sequence

$$0 \to \Gamma(Y, \mathcal{O}_Y^*) \to \prod_{i=1}^n \Gamma(C_i, \mathcal{O}_{C_i}^*) \to \prod_{\substack{x \in C_i \cap C_j \\ i \neq j}} \mathcal{O}^*_{C_i \cap C_j, x} \to \mathrm{Pic}^{0,0}(Y) \to \prod_{i=1}^n \mathrm{Pic}^0(C_i) \to 0.$$

Fact (Oort). $\dim_{\mathbb{F}_\ell} \mathrm{Pic}^0(C_i)[\ell] \leq g(C_i) + p_a(C_i)$, with equality if and only if C_i is nodal.

Then

$$\dim_{\mathbb{F}_\ell} (\mathcal{O}^*_{C_i \cap C_j, x})[\ell] = 1$$

and equal to $\dim_\kappa(\mathcal{O}_{C_i \cap C_j, x})$ if and only if $\mathcal{O}_{C_i \cap C_j, x} \cong \kappa$.

Putting everything together, we get

$$2g - \beta \leq \dim_{\mathbb{F}_\ell} \mathrm{Pic}(Y)[\ell] \leq \sum (g(C_i) + p_a(C_i)) + \beta$$

which implies

$$2g \leq \sum (g(C_i) + p_a(C_i)) + 2\beta.$$

Now we consider the same long exact sequence as before without $*$'s:

$$0 \to H^0(Y, \mathcal{O}_Y) \to \prod_{i=1}^n \Gamma(C_i, \mathcal{O}_{C_i}) \to \prod_{\substack{x \in C_i \cap C_j \\ i \neq j}} \mathcal{O}_{C_i \cap C_j, x} \to H^1(Y, \mathcal{O}_Y) \to \bigoplus_{i=1}^n H^1(C_i, \mathcal{O}) \to 0.$$

This gives

$$g = \dim_\kappa H^1(X_s, \mathcal{O}) \geq \dim_\kappa H^1(Y, \mathcal{O}_Y) = \sum p_a(C_i) + \beta.$$

Therefore,

$$2\sum p_a(C_i) + 2\beta \leq \sum (g(C_i) + p_a(C_i)) + 2\beta$$

so

$$\sum p_a(C_i) \leq \sum g(C_i).$$

This shows there are no δ-invariants, i.e., all C_i's are smooth and we have equality everywhere. This finishes the proof.

22.3. **Saito.** We have been following Artin–Winters, but there is a paper by Saito which proves the theorem (same as in Deligne–Mumford): if the inertia acts unipotently, then we have semi-stable reduction of curves. Saito's proof uses étale cohomology and vanishing cycle sheaves.

23. Lecture 23 (December 8, 2015)

23.1. **Néron models.** We will discuss the Néron–Ogg–Shafarevich criterion. First we need to talk about Néron models. A reference is Artin's article in Chapter VIII in *Arithmetic Geometry* (edited by Cornell and Silverman).

Let R be a Dedekind domain (dvr) with fraction field K, and A an abelian variety over K.

Definition 23.1. A *Néron model* for A is a smooth group scheme $G \to \operatorname{Spec}(R)$ with $G_K \cong A$ and such that the following universal property holds:

If X is smooth over $\operatorname{Spec}(R)$ then any rational map $X \dashrightarrow G$ extends to a morphism $X \to G$. (*)

An alternative weaker condition is:

If X is smooth over $\operatorname{Spec}(R)$ then any morphism $X_K \to A$ extends to a morphism $X \to G$. (†)

It is clear that (*) implies (†).

Remark. In both cases we have, if R is a dvr, then

$$A(K^{\mathrm{sep}})^I = A(\mathrm{f.f.}(R^{\mathrm{sh}})) = G(R^{\mathrm{sh}}) \xrightarrow{\mathrm{sp}} G_s(\kappa^{\mathrm{sep}}),$$

where $I \subset \operatorname{Gal}(K^{\mathrm{sep}}/K)$ is the inertia with $(K^{\mathrm{sep}})^I = \mathrm{f.f.}(R^{\mathrm{sh}})$, the second equality follows from (*) or (†), and sp is the specialization map.

Theorem 23.2 (Weil)**.** *Finite type Néron models for abelian varieties exist over Dedekind domains.*

Theorem 23.3 (Raynaud)**.** *Locally of finite type smooth models with (†) exist for* semi-abelian varieties*. These models are also called Néron models.*

Example 23.4. Consider $\mathbb{G}_{m,K}$ with R dvr. Pick $\pi \in R$ uniformizer. Set

$$\text{“}G = \bigcup_{n\in\mathbb{Z}} \pi^n \mathbb{G}_{m,R}\text{”}$$

(construct by glueing). Then we have

$$K^\times = \mathbb{G}_{m,K}(K) = G(R) = \bigcup_{n\in\mathbb{Z}} \pi^n R^\times.$$

Example 23.5. Consider elliptic curves with multiplicative reduction and $v(\Delta) = 1$. Assume we have a Weierstrass equation over R with discriminant $\Delta = \pi \cdot \text{unit}$. Then

$$G = \begin{pmatrix}\text{closed subscheme of } \mathbb{P}^2_R \text{ defined} \\ \text{by the Weierstrass equation}\end{pmatrix} \setminus \begin{pmatrix}\text{singular point of} \\ \text{the special fibre}\end{pmatrix}$$

(over $\operatorname{Spec}(R)$) is the Néron model of its generic fibre.

23.2. Group schemes over fields.

Theorem 23.6 (Chevalley)**.** *If G/K is smooth and connected, then there exists a short exact sequence*

$$1 \to L \to G \to A \to 1$$

of group schemes over K, where L is a connected linear algebraic group, and A is an abelian variety.

Remark. If K is perfect then L is smooth.

Theorem 23.7. *If L/K is a commutative, smooth, connected linear algebraic group, then*

$$0 \to U \to L \to T \to 0$$

with U unipotent *and T a* torus*.*

Remark. If K is perfect then $L = U \times T$ canonically.

Unipotent means it fits in

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \subset \mathrm{GL}_n .$$

Torus T means

$$T \otimes_K \overline{K} \cong \mathbb{G}_{m,\overline{K}}^{\oplus r}$$

for some r.

Definition 23.8. A group scheme G over a field K is *semi-abelian* if G is abelian and an extension of an abelian variety by a torus. A group scheme G over a base S is *(semi-)abelian* if $G \to S$ is smooth and all fibres are (semi-)abelian.

23.3. Néron–Ogg–Shafarevich criterion.

Definition 23.9. Let R be a dvr with fraction field K, and A/K an abelian variety. We say A has *good reduction* if its Néron model is an abelian scheme, or equivalently if there exists an abelian scheme $G \to \operatorname{Spec}(R)$ with $A \cong G_K$.

Theorem 23.10 (Néron–Ogg–Shafarevich)**.** *Let R be a dvr with fraction field K, and A/K an abelian variety. Then A has good reduction if and only if there exists ℓ prime to* $\operatorname{char}(\kappa)$ *such that the action of I on $A(K^{\mathrm{sep}})[\ell^\infty]$ is trivial.*

"Proof" of $\Longleftarrow$ *.* First we show that although

$$A(K^{\mathrm{sep}})^I = A(\mathrm{f.f.}(R^{\mathrm{sh}})) = G(R^{\mathrm{sh}}) \stackrel{\mathrm{sp}}{\to} G_s(\kappa^{\mathrm{sep}})$$

is not injective, writing $\mathrm{f.f.}(R^{\mathrm{sh}}) = K^{\mathrm{sh}}$ we have

$$A(K^{\mathrm{sh}})[\ell^n] \hookrightarrow G_s(\kappa^{\mathrm{sep}})[\ell^n].$$

Note that I acts trivially by assumption, so

$$A(K^{\mathrm{sh}})[\ell^n] \cong (\mathbb{Z}/\ell\mathbb{Z})^{2g}$$

where $g = \dim(A)$.

Since $G \to \mathrm{Spec}(R)$ is of finite type, $[G_s : G_s^0] < \infty$ and hence

$$\#G_s^0(\kappa^{\mathrm{sep}})[\ell^n] \geq \ell^{2gn-\mathrm{constant}}.$$

The structure of group schemes over $\overline{\kappa}$ gives

$$0 \to L \to G_s \otimes \overline{\kappa} \to B \twoheadrightarrow 0$$

where B is an abelian variety, and

$$0 \to U \to L \to T \to 0.$$

Write $g = \dim U + \dim T + \dim B = u + t + b$. Then

$$\begin{aligned} \#B(\overline{\kappa})[\ell^n] &= \ell^{2bn}, \\ \#T(\overline{\kappa})[\ell^n] &= \ell^{tn}, \\ \#U(\overline{\kappa})[\ell^n] &= 1. \end{aligned}$$

Combining everything we get

$$\begin{cases} 2g \leq t + 2b, \\ g = u + t + b \end{cases} \implies t = u = 0. \qquad \square$$

Much harder is the following

Theorem 23.11. *I acts unipotently on $T_\ell(A)$ if and only if A has semi-abelian reduction (SGA7 calls this stable reduction).*

24. Lecture 24 (December 10, 2015)

24.1. **Semi-abelian reduction.** Recall the situation: A/K is an abelian variety, $K \supset R$ a dvr with residue field κ.

Theorem 24.1 (SGA7)**.** *A has semi-abelian reduction if and only if there exists ℓ different from* $\mathrm{char}(\kappa)$ *such that the action of I on $T_\ell(A)$ is unipotent.*

The original proof was due to Grothendieck, but I will explain a simpler proof by Deligne.

Theorem 24.2 (Deligne–Mumford)**.** *If $A = \mathrm{Jac}(C)$, then A has semi-abelian reduction if and only if C has semi-stable reduction.*

This is not very hard to show. We have to relate the Picard scheme of the model to the Néron model of the Jacobian, and then using techniques similar to those in Artin–Winters we can prove this theorem. We would in particular get the fact that if the inertia acts unipotently on the Tate module of $A = \mathrm{Jac}(C)$, then C has semi-stable reduction.

Combining these Deligne–Mumford proved (for the first time) the semi-stable reduction theorem for curves.

24.2. **Proof of Theorem 24.1.** We will proof the implication $\Longleftarrow$ following Deligne in SGA7-I, Exp. I App.

For simplicity assume R complete with finite κ. The Tate module

$$V_\ell(A) = T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = \left(\lim_n A(K^{\mathrm{sep}})[\ell^n]\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

is a $\mathbb{Q}_\ell$-vector space of dimension $2g$ endowed with a continuous action of

$$0 \to I \to \mathrm{Gal}(K^{\mathrm{sep}}/K) \twoheadrightarrow \mathrm{Gal}_\kappa = \widehat{\mathbb{Z}} \to 0.$$

By assumption, the action of I is unipotent.

Let $\sigma \in I$ be a topological generator, and recall the tame inertia $I_t = I/P \cong \prod_{\ell' \neq \mathrm{char}(\kappa)} \mathbb{Z}_{\ell'}$. Suppose $\tau \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ maps to the arithmetic Frobenius in $\mathrm{Gal}_\kappa = \widehat{\mathbb{Z}} \ni 1$, i.e.,

$$\tau\sigma\tau^{-1} = \sigma^q \mod P.$$

Let r be the greatest integer such that $(\sigma - 1)^r \neq 0$ on $V_\ell(A)$.

Consider the coinvariants and invariants

$$\begin{array}{ccc} (V_\ell)_I & & (V_\ell)^I \\ \downarrow\!\!\!\downarrow & & \uparrow \\ Q = V_\ell/\ker((\sigma-1)^r) & \xrightarrow[(\sigma-1)^r]{\cong} & \mathrm{Im}((\sigma-1)^r) =: S \end{array}$$

This map $Q \to S$ is not Gal_κ-invariant but it is if we twist

$$Q \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(r) \xrightarrow{\sim} S$$

as Gal_κ-representations. Note τ acts as q on $\mathbb{Q}_\ell(1)$ and as q^r on $\mathbb{Q}_\ell(r)$.

Proof.

$$\begin{aligned} \tau(\sigma-1)^r &= (\tau\sigma\tau^{-1} - 1)^r\tau \\ &= (\sigma^q - 1)^r\tau \\ &= q(\sigma-1)^r \end{aligned}$$

by looking at

$$\sigma = \begin{pmatrix} 1 & 1 & & * \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix}, \sigma^q = \begin{pmatrix} 1 & q & & * \\ & 1 & \ddots & \\ & & \ddots & q \\ & & & 1 \end{pmatrix}, (\sigma^q - 1)^r = \begin{pmatrix} 0 & \cdots & 0 & q^r \\ & 0 & \ddots & 0 \\ & & \ddots & 0 \\ & & & 0 \end{pmatrix}. \qquad \square$$

Definition 24.3. A *q-Weil number of weight w* is $\alpha \in \overline{\mathbb{Z}}$ such that $|\alpha'| = q^{w/2}$ for all conjugates $\alpha' \in \mathbb{C}$ of α.

Lemma 24.4. *The eigenvalues of τ^{-1} on $(V_\ell)^I$ are either:*

- *q-Weil numbers of weight -2 or*
- *q-Weil numbers of weight -1.*

Proof. $(V_\ell)^I = V_\ell(G_s^0)$, where G is the Néron model and G_s is the special fibre. G_s^0 has two parts. On the the torus part, τ acts on $V_\ell(\mathbb{G}_{m,\kappa})$ by the cyclotomic character, so the eigenvalue of τ^{-1} is q^{-1}. On the abelian variety part, we get weight -1 by the Weil conjectures for abelian varieties over κ. □

Lemma 24.5. *The eigenvalues of τ^{-1} on $(V_\ell)_I$ are either:*

- *q-Weil numbers of weight 0, or*
- *q-Weil numbers of weight -1.*

Proof. Let $A^t = \underline{\mathrm{Pic}}^0_A$ be the dual abelian variety. Then there exists a nondegenerate Gal_κ-equivariant bilinear pairing

$$V_\ell(A) \times V_\ell(A^t) \to \mathbb{Q}_\ell(1).$$

Note

$$(V_\ell(A))_I \cong ((V_\ell(A^t))^I)^*(1).$$

By the previous lemma for A^t (with weights $-2, -1$), dualizing gives $2, 1$ and twisting gives $0, -1$. □

Conclusion: $r \leq 1$.

$$\begin{array}{cccc} \begin{smallmatrix} 0-2r \\ -1-2r \end{smallmatrix} & (V_\ell)_I(r) & (V_\ell)^I & \begin{smallmatrix} -2 \\ -1 \end{smallmatrix} \\ & \Big\downarrow\!\!\!\!\downarrow & \Big\uparrow & \\ & Q(r) \xrightarrow{\ \cong\ } & S & \end{array}$$

If $r = 0$, Néron–Ogg–Shafarevich says A has good reduction.

If $r = 1$, the Jordan blocks of σ are $\begin{pmatrix} 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and

$$2g = \#(1 \times 1 \text{ blocks}) + 2 \cdot \# \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Looking at weights we see

$$\dim(\text{torus part of } G_s^0) \geq \# \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

using $V_\ell^I = V_\ell(G_s^0)$ and the fact that weight -2 eigenvalues come from the torus part. The relation $V_\ell^I = V_\ell(G_s^0)$ also shows

$$\begin{aligned} 2\dim(\text{abelian part of } G_s^0) + \dim(\text{torus part of } G_s^0) &= 2g - \# \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &\geq 2g - \dim(\text{torus part}), \end{aligned}$$

which must be an equality because

$$g = \dim(\text{unipotent part}) + \dim(\text{torus part}) + \dim(\text{abelian part}).$$

So we are done!

24.3. **Weight monodromy conjecture.** Let X/K be a variety over a local field, and

$$V = H^i(X_{\overline{K}}, \mathbb{Q}_\ell).$$

There are two filtrations on V:

- filtration coming from the nilpotent operator $N = \sigma - 1$,
- weight filtration.

Conjecture 24.6. *The two filtrations agree.*

We just proved this for abelian varieties on H^1. Scholze proved this for complete intersections using perfectoid spaces.