Last Time:

- deck transformations \( G(X) \): isomorphisms \( \tilde{X} \overset{\text{r}}{\rightarrow} X \)
- normal covering space: for any \( \tilde{x}_0, \tilde{x}_0' \in p^{-1}(x_0) \)
- \( H \subset \pi_1(X) \) normal \( \iff \) \( \tilde{X} \rightarrow X \) normal covering space

Prop: if \( \tilde{X} \rightarrow X \) is normal, \( G(\tilde{X}) \cong \pi_1(X)/\pi_1(X)_{\text{r}} \)

Today: what spaces can \( Y \) cover? group actions

Def.: a group \( G \) action on a set \( S \) (write \( G \acts \! \! \! \! \! \! \! \! \! S \)) is a map \( G \times S \rightarrow S \)

\[ (g, s) \mapsto g(s) \]

and \( g, (g_2(s)) = (g, g_2)(s), \) \( \text{Id}(s) = s \)

so get group homomorphism \( G \rightarrow \text{Aut}(S) \)

if \( Y \) is a space, \( g : Y \rightarrow Y \) homeomorphism and so \( G \rightarrow \text{Homeo}(Y) \)

Ex.: group action of \( G(X) \) on \( \tilde{X} \), i.e. group homomorphism \( G(\tilde{X}) \rightarrow \text{Homeo}(\tilde{X}) \)
If $g \in G(\tilde{x})$, then $g : p^\sim(\tilde{x}) \to p^\sim(g(x))$, action on $p^\sim(\tilde{x})$, so $G(\tilde{x}) \to \text{Aut}(p^\sim(\tilde{x})) = S_n$ if $|p^\sim(\tilde{x})| = n$.

**HW:** Relate this description with previous description.

**Ex.**

$Z = \pi_1(S') \to S_3 = \text{Aut}(\{1, 2, 3\})$

**Now study converse:**

given $G \subseteq Y$, can form $Y/G : = \{y - g(y) : g \in G\}$ orbit space

**Ex.** if $S$ is a group, $G \subseteq S$ subgroup,
then $G \subseteq S$ by multiplication and $S/G$
are cosets (quotient group if $G \subseteq S$ normal)

**Ex.** $G(\tilde{x}) \supseteq \tilde{x}$ and $\tilde{x}/G(\tilde{x}) \cong X$ quotient map $S \to S/G$ $s \to [s]$ 

**Q.** when is this map a covering space?

**Ex.** $\mathbb{Z}/2 \subseteq \mathbb{R}$ by $\phi(x) = -x$ ($\phi^2 = \text{Id}$)

$\xymatrix{ \mathbb{Z}/2 \ar[r]^\phi \ar[d]^{p^\sim(\mu)} & \mathbb{R} \ar[d]^\mu \\
\not\text{not homeomorphic} & \text{not homeomorphic} }$
Ex. \( Z \cong \mathbb{R}, \ \phi(x) = x + 1 \)

then \( \mathbb{R} \xrightarrow{\subseteq} \mathbb{S} \)

covering space

Def. \( G \curvearrowright Y \) is a covering space action if every \( y \in Y \) has a neighborhood \( U \ni y \) so that \( g(U) \cap U = \emptyset \) for all \( g \neq \text{Id} \in G \) \( \Rightarrow \) no fixed points (if \( G \) finite, \( \Leftarrow \))

Prop. if \( G \curvearrowright Y \) covering space action, then

1) \( Y \xrightarrow{\pi_1} Y/G \) is a normal covering space

2) deck transformation \( G(\tilde{y}) \cong G \) if \( Y \) is path-connected

\[ \Rightarrow \pi_1(Y/G) / \pi_1(Y) \cong G \]

\( \tilde{y} \rightarrow Y \xrightarrow{\pi_1} Y/G \) maps \( \bigsqcup_{g \in G} g(U) \xrightarrow{\sim} U \)

and \( g(U) \cong U \)

- normal: \( \pi^{-1}(x_0) = \bigsqcup_{g \in G} g(\tilde{x}_0) \) so all related by \( G \)-action

- \( G \subseteq G(\tilde{y}) \) and \( G(\tilde{y}) \subseteq G \) since deck transformation \( \phi \) deformed by \( \phi(\tilde{x}_0) \in \pi^{-1}(\tilde{x}_0) \) and exists \( g \) with \( g(\tilde{x}_0) = \phi(\tilde{x}_0) \)
normal covering spaces \( Y \to X \iff \) covering space action \( G \subseteq Y \) for some \( G \)

with \( \pi_1(X) / \rho_\pi_1(Y) \cong G \)

Ex. \( \mathbb{Z}/2 \triangleleft S^n, \ x \to -x \)

\( S^n \to S^n/(\mathbb{Z}/2) = \mathbb{R}P^n \) covering space

\( \implies \mathbb{Z}/2 \cong G(S^n) \cong \pi_1(\mathbb{R}P^n) / \rho_\pi_1(S^n) \cong \pi_1(\mathbb{R}P^n) \)

so way to compute \( \pi_1(\mathbb{R}P^n) \)

Q: for which \( G \) is there covering space action \( G \triangleleft S^n \)

( i.e., if \( X_{\text{new}} \cong S^n \), what can \( \pi_1(X) \) be ?)

Thm. if \( n \) even, \( G \cong \mathbb{Z}/2 \) is only possibility

(Milnor): every abelian subgroup of \( G \) is cyclic

and \( G \) has at most one element of order 2

(Madsen, Thomas, Wall) given a group \( G \) satisfying *:

exists \( m \) so that \( G \triangleleft S^n \) is covering action

Ex. \( \mathbb{Z}/m \triangleleft S^{2m-1} \to \mathbb{R}^{2k} = S^k \)

by \( \phi(v) = e^{2\pi i/m} v, \phi^m = \text{Id} \)

no fixed points \( \implies \) finite \( \implies \) covering action

\( S^{2m-1}/\mathbb{Z}/m \) called lens space, generalise \( \mathbb{R}P^n \)
can get non-cyclic groups

Ex. \( n=3 \), \( S^3 \) is actually a group!
(called Lie group)

Quaternion algebra \( H = \mathbb{R}^4 = \mathbb{R}< 1, i, j, k > \)

\[ i^2 = j^2 = k^2 = -1, \quad ij = k, \quad ji = -k \]

- \( S^3 = \{ |a| = 1 \} \subset H \), \( \| \cdot \| \) Euclidean norm and \( |ab| = |a| \cdot |b| \), so \( S^3 \subset H \) is subgroup

so any finite subgroup \( G \subset S^3 \)

gives \( G \subset S^3 \) covering space action

Quaternion group \( Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \} \subset S^3 \)

\( Q_{um} = \langle e^{2\pi i m}, j \rangle = \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle \)