Last time: Free products of groups, van Kampen's theorem

Today: Proof of van Kampen, applications

**van Kampen Theorem**: \( X = \bigcup A_x, \quad A_x \text{ path-connected}, \quad x \in A_x \)

- If \( A_x \cap A_y \) path-connected for all \( x, y \),
  - then \( \ast i_x : \pi_1(A_x) \to \pi_1(X) \) is surjective
- If \( A_x \cap A_y \cap A_z \) path-connected for all \( x, y, z \),
  - then \( \ker(\ast i_x) \) is normal subgroup generated by \( \{ i_{xy}(w) i_{yz}(w) i_{xz}(w) \} \),
  - \( w \in \pi_1(A_x \cap A_y) \to \pi_1(A_x) \ast \pi_1(A_y) \to \ast \pi_1(A_x) \)

**Proving Surjectivity**

- \( f: \pi_1(X) \to \mathbb{R}_+, \quad f([0, 1]) \to X \), break up \([0, 1]\) into \( \bigcup_{i=0}^m [s_i, s_{i+1}] \)
- so that \( f([s_i, s_{i+1}]) \subset A_i; \) call \( f|_{[s_i, s_{i+1}]} = f_i \)

A2

A1
Since $A \cap A_1$ is path-connected, take path $g_i$ in $A \cap A_1$ from $x_0$ to $f_i(s_i)$

- then $F$ homotopic (in $X$) to $f, g_1, g_2, g_3, \ldots, g_m, f_m, g_m$ 
  $\sim$ 

- $\pi_1(A_1), \pi_1(A_2), \pi_1(A_m)$

is in image of $*e_x : *\pi_1(A_x) \to \pi_1(X) \vee$

**Injectivity**

- $F \in \pi_1(X) \Rightarrow$ factorization is $[F, f_1, \ldots, f_n]$

  - select of $*\pi_1(A_x)$ homotopic to $F$ in $X$

  - want to show any two factorizations of $F$ are equivalent

  - combine $[F_1, f, f_i] \sim [f, f_i, f_{i+1}]$ if $[F_1, f, f_i] \in \pi_1(A_x)$ same group

  - view $[F_i] \in \pi_1(A_x)$ as in $\pi_1(AB)$, $F_i \in \pi_1(A_x \cap AB)$

**Goal**: any two factorizations of $F$ are equivalent
\[ F : \frac{E_0}{E_0, \frac{E_0}{E_0}} \quad \text{Factorizations} \]

- Break up \( F \) into homotopy via \( F : I \times I \to X \)
- Assume that each vertex \( \frac{5 6}{11 12} \) maps via \( F \) to the basepoint \( x_0 \)
- Let \( \gamma \) be any path in \( I \times I \) with endpoints on \( 0 \times I, 1 \times I \)
- Consisting of horizontal, vertical edges

\[ \text{Ex} \]

- Then \( I \xrightarrow{\gamma} I \times I \xrightarrow{F} X \) is homotopic to \( F \)
- And hence is a factorization of \( F \)
- All vertices map to \( x_0 \)
- By sliding our squares, we get that the factorizations associated to \( p, q \) are equivalent
in particular, the factorizations associated to the bottom horizontal, i.e., \([f_i] \cdot [f_u]\)

is equivalent to top horizontal, i.e., \([f'_i] \cdot [f'_u]\)

Finally, given an arbitrary \(F : I \times I \to X\), we can homotope it (by adding paths) \(g_{ij}\) from \(x_0\) to \(F(v_{ij})\), the vertex \(v_{ij}\)

so that the vertices actually map to \(x_0\). This uses the fact that \(A_i \cap A_j \cap A_k \cap A_l\) is connected, which can be improved to just \(A_i \cap A_j \cap A_k \cap A_l\) connected.
Applications to CW complexes

Recall $X = U \cup X'$, where $X'$ obtained by attaching $n$-cells $D^n$ to $X^{n-1}$ via attaching maps $\phi : \partial D^n \to X^{n-1}$.

Q. If $Y = X \cup \phi D^n = X \cup \{D^n/\sim\phi(x) \mid x \in \partial D^n\}$, how are $\pi_*(X)$, $\pi_*(Y)$ related?

Prop. If $n \geq 3$, $i_* : \pi_*(X) \to \pi_*(Y)$.

Proof.

Cover $Y$ by $Y \cap \{p\}$ and $\text{Int}_+ D^n$ (open subsets).

Then $Y \setminus \{p\} \sim X$.

$\text{Int}_+ D^n \sim p^+$.

$\phi(0,1)$ is path-connected.

$\Rightarrow \pi_*(X) \times \pi_*(p^+) \to \pi_*(Y)$ has kernel from $\pi_*(S^{n-1}) = 0$, if $n \geq 2$, injective.
Cor. if \( i : X^2 \to X \) is 2-skeleton of \( X \)

then \( i : \pi_1(X^2) \to \pi_1(X) \)

\( \Rightarrow \) suffices to study 2-skeleton

- If \( Y = X \cup D^2 \), \( \phi : \partial D^2 = S^1 \to X \)
  pick paths \( \gamma \) in \( X \) from \( x_0 \) to \( \phi(0) \)

  \( \Rightarrow \phi : \gamma \phi \gamma^{-1} \in \pi_1(X) \)

Proof. kernel of \( i : \pi_1(X) \to \pi_1(Y) \)

is normal subgroup generated by \( \phi_{\gamma} \in \pi_1(X) \)

But if pick different \( \gamma' \), then

\( \phi_{\gamma'} = \gamma' \phi(\gamma')^{-1} = \gamma' \gamma^{-1} \phi \gamma^{-1} \gamma(\gamma')^{-1} \)

\( = \alpha \phi_{\gamma} \alpha^{-1} \) where \( \alpha = \gamma' \gamma^{-1} \in \pi_1(X) \)

and normal subgroups generated by \( \phi_{\gamma} \) and \( \alpha \phi_{\gamma} \alpha^{-1} \) are same

If again cover \( Y \) by \( Y \setminus p \cup \text{Int } D^2 \)

now \( Y \setminus p \cap \text{Int } D^2 \cong S^1 \)

\( Z \cong \pi_1(S^1) \to \pi_1(\text{Int } D^2) = 0 \)

\( Z \to \pi_1(Y \setminus p) \cong \pi_1(X) \)

1 \( \to \phi \), attaching map

so kernel of \( \pi_1(X) \to \pi_1(D^2) = \pi_1(1X) \to \pi_1(Y) \)

\( \phi \in \pi_1(X) \)