Last time: \( \pi_1(S') \cong \mathbb{Z} \)
\[ W_n \leftarrow \cdot \cdot \cdot \circ \cdot \cdot \cdot \]
\( W_n \) wraps \( n \) times.

Then every polynomial \( p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0, n \geq 1 \)
has a zero in \( \mathbb{C} \).

Proof sketch:
- If \( p(z) \neq 0 \) all \( z \), then \( F_r(s) = \frac{p(re^{2\pi is})}{p(re^{2\pi is})} \)
  - Loop in \( S' \) based at 1
- \( F_0(s) = 1 \) constant loop \( \omega_0 \)
- For \( |z| >> 1 \), \( p(z) \approx z^n \) and so
  - For \( r >> 1 \), \( F_r(s) \approx (e^{2\pi is})^n = \omega_n \)
so \( \omega_0 \) homotopic to \( \omega_n \) by \( F_r \), contradiction \( \square \)

Today compute \( \pi_1 \) for any CW complex
using just \( \pi_1(S') \cong \mathbb{Z} \)!

Goal: if \( j_a: A_x \rightarrow X \) open subsets covering \( X \), want
  to compute \( \pi_1(X) \) from \( j_a: \pi_1(A_x) \rightarrow \pi_1(X) \)
  (assume \( x_0 \in X \) basepoint and \( x_0 \in A_i \))
  
\[ a_1, a_2, a_3 \]
\[ e_1, e_2, e_3, e_4, \ldots \]
Group Theory

- Let $G_\alpha$ be a (possibly infinite) set of groups.

**Def.** The free product $\ast G_\alpha$ of $G_\alpha$ is set of

- finite length, reduced words $g_1 \cdot \ldots \cdot g_m$, $g_i \in G_\alpha$.
  - reduced: $g_i, g_{i+1}$ belong to different $G_\alpha$, $\alpha_\neq \alpha_{i+1}$
  - and $g_i \neq \pm 1$

- product is concatenation

$$(g_1 \cdot \ldots \cdot g_m) \ast (h_1 \cdot \ldots \cdot h_n) = g_1 \cdot \ldots \cdot g_m \cdot h_1 \cdot \ldots \cdot h_n$$

- empty word is identity

**Prop.** Product associative

$\Rightarrow$ all reductions of a word are the same.

**Note.** That in $G_\alpha \ast G_\alpha$ is a subgroup.

**Universal property.** For any homomorphisms $G_1 \to H$, $G_2 \to H$

exists a unique hom $\phi_1 \ast \phi_2 : G_1 \ast G_2 \to H$

Can be factored as $G_i \to \phi_i$ and $G_i \ast G_i \to \phi_i$.

Return to topology

X covered by $A_\alpha$

- $\alpha : \pi_1(A_\alpha) \to \pi_1(X)$ induces $\ast \pi_1(A_\alpha) \to \pi_1(X)$

- when surjective, injective.

$A_\alpha \cap A_\beta \to A_\alpha \to X$ and so $\alpha \circ \beta \circ \alpha = \beta \circ \beta \circ \alpha$

as maps $\pi_1(A_\alpha \cap A_\beta) \to \pi_1(X)$.
so for $w \in \pi_1(A_x \cap A_y)$, $(i_{A}^{-1}(w)) i_{A}^{-1}(w)^{-1}$ is in kernel of $\pi_1(A_x) \ast \pi_1(A_y) \rightarrow \pi_1(X)$.

van Kampen Theorem

$X = \bigcup_{x} A_x$, $A_x$ path-connected

and $x_0 \in A_x$ for all $x$.

- if $A_x \cap A_y$ path-connected for all $x$, then $i_x \ast i_y : \pi_1(A_x) \ast \pi_1(A_y) \rightarrow \pi_1(X)$ is surjective.

- if $A_x \cap A_y \cap A_z$ path-connected, kernel $N$ is (normal subgroup generated by) $i_{A}^{-1}(w) i_{A}^{-1}(w)^{-1}$.

so $\pi_1(X) / N \cong \pi_1(X)$.

Ex.1 wedge sums $\vee_{x} X_x = \bigcup_{x} X_x / x_0 \sim x_0$ identify basepoints $x_1$, $x_2$, $x_3$.

- cover by $A_x = X_x \cup U(x_0)$. Neighborhood of $x_0$ $A_x \cong X_x$ and $A_x \cap A_y \cap A_z = U(x_0) \cong X_0$, $\pi_1 \cong 0$ and $A_x \cap A_y \cap A_z$ path-connected.

$\Rightarrow \pi_1(X) \cong \bigast \pi_1(A_x)$.

- $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} \ast \mathbb{Z}$ all loops homotopic to unique (reduced) word $w_1 \ast w_2 \ast \cdots \ast w_n$.

- not abelian!
Ex. 2 \[ \bigcirc A_2 \quad \pi_1(A_1) = \pi_1(\varpi A_2) = 0 \]
but \[ \pi_1(A, UA_2) = \pi_1(S') \cong \mathbb{Z} \]
so \[ \pi_1(A_1) \times \pi_2(A_2) \to \pi_1(S') \text{ not surjective} \]
if \( A \cap \varnothing A_2 \) not path-connected

Ex. 3 \[ \bigtriangleup \bigcirc \bigtriangleup \bigcirc \bigtriangleup \quad \text{comm} \quad \bigtriangleup \square \cong S' \cup S' \]

\( A \cap \varnothing A_1 \cong \triangle, \quad A \cap \varnothing A_2 \cap \varnothing A_3 = \bigtriangleup \quad \text{not connected} \)
\[ \Rightarrow \pi_1(A_1) \times \pi_1(\varnothing A_2) \times \pi_1(A_3) \to \pi_1(S' \cup S') \]
\[ \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \]
not injective (abelianization \( \mathbb{Z}^3 \to \mathbb{Z}^2 \))

Universal property \( \Phi_1 : K \to G_1, \Phi_2 : K \to G_2, \) exists

\( G_2 \times G_1 \cong G_1 \times G_2 \) satisfying universal property:
any commutative diagram can be factored as

\[ K \to G_2 \quad \downarrow \quad \downarrow \quad \sim \]
\[ G_2 \to H \]

\[ \Phi_1(K) \Phi_2(K) \]

\[ G_1 \times G_2 \]

\[ \ker \Phi_1 \cdot \ker \Phi_2 \]

is the pushout of \( K \to G_1 \)

if \( \times \) covered by \( A_1, A_2 \) then \( \times \) is
pushout of \( A_1 \cap \varnothing A_2 \to A_2 \) (i.e., satisfies universal property for this diagram)
Van Kampen:

```
1. continuous maps
   spaces \rightarrow groups
   homomorphisms

pushouts \rightarrow pushouts
```

school to school