Exercise 2.1.7. Find a way of identifying pairs of faces of $\Delta^3$ to produce a $\Delta$-complex structure on $S^3$ having a single 3-simplex, and compute the simplicial homology group of this $\Delta$-complex.

Proof. We identify faces $[0, 1, 2]$ with $[0, 1, 3]$ and $[0, 2, 3]$ with $[1, 2, 3]$ (see picture). To show that the resulting space is homeomorphic to $S^3$, it is helpful to consider the center $-1$ of the 3-simplex. We can see that two tetrahedra $[-1, 0, 1, 2]$ and $[-1, 0, 1, 3]$ are glued together along one pair of faces, hence creating a homeomorphic copy of $D^3$, and the same goes for gluing $[-1, 0, 2, 3]$ with $[-1, 1, 2, 3]$. Now, the two copies of $D^3$ are glued together via the faces $[-1, 0, 2], [-1, 0, 3], [-1, 1, 2]$, and $[-1, 1, 3]$, which is the same as gluing the boundary of one $D^3$ with the boundary of the other $D^3$ (in a consistent manner). This will create $S^3$.

We now compute the simplicial homology of this space. The chain complex is

$$0 \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} A \oplus \mathbb{Z} B \xrightarrow{\partial_1} \mathbb{Z} a \oplus \mathbb{Z} b \oplus \mathbb{Z} c \xrightarrow{\partial_0} \mathbb{Z} x \oplus \mathbb{Z} y \xrightarrow{\partial_{-1}} 0$$

where $A = [0, 1, 2] = [0, 1, 3]$, $B = [0, 2, 3] = [1, 2, 3]$ $a = [0, 1]$, $b = [2, 3]$, and $c = [1, 2] = [1, 3] = [0, 2] = [0, 3]$, and $x = 0 = 1$, $y = 2 = 3$. The boundary maps are

$$\begin{cases} 
\partial_3 T = A - A + B - B = 0 \\
\partial_2 A = a + c - c = a, \\
\partial_2 B = b + c - c = b \\
\partial_1 a = \partial_1 b = 0, \\
\partial_1 c = x - y
\end{cases}$$
Therefore,
\[
\begin{align*}
H_0(X) &= \mathrm{Ker}(\partial_0)/\mathrm{Im}(\partial_1) = \langle x, y \rangle / \langle x - y \rangle \cong \mathbb{Z} \\
H_1(X) &= \mathrm{Ker}(\partial_1)/\mathrm{Im}(\partial_2) = \langle a, b \rangle / \langle a, b \rangle = 0 \\
H_2(X) &= \mathrm{Ker}(\partial_2)/\mathrm{Im}(\partial_3) = 0/0 = 0 \\
H_3(X) &= \mathrm{Ker}(\partial_3)/\mathrm{Im}(\partial_4) = \langle T \rangle \cong \mathbb{Z}
\end{align*}
\]
and \( H_n(X) = 0 \) for \( n \geq 4 \). \( \square \)

**Exercise 2.1.8.** Compute the simplicial homology of the space \( X \) as constructed in Hatcher p.131.

**Proof.** Note that the \( \Delta \)-complex of \( X \) has two 0-simplices \( x \) (in the center) and \( y \), \( n + 2 \) 1-simplices \( a, b, c_1, \ldots, c_n \), \( 2n \) 2-simplices \( A_1, \ldots, A_n, B_1, \ldots, B_n \) and \( n \) 3-simplices \( T_1, \ldots, T_n \) (see picture).

The chain complex is thus
\[
0 \xrightarrow{\partial_4} \mathbb{Z}^n \xrightarrow{\partial_3} \mathbb{Z}^{2n} \xrightarrow{\partial_2} \mathbb{Z}^{n+2} \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{\partial_0} 0
\]

The boundary maps are
\[
\begin{align*}
\partial_2 T_i &= A_i - A_{i-1} + B_{i+1} - B_i \\
\partial_2 A_i &= b - c_{i+1} + c_i, \quad \partial_2 B_{i+1} = c_{i+1} - c_i + a \\
\partial_1 a &= \partial_1 b = 0, \quad \partial_1 c_i = x - y
\end{align*}
\]
Therefore,

\[
\begin{align*}
H_0(X) &= \text{Ker}(\partial_0)/\text{Im}(\partial_1) = \langle x, y \rangle / \langle x - y \rangle \simeq \mathbb{Z} \\
H_1(X) &= \text{Ker}(\partial_1)/\text{Im}(\partial_2) = \langle a, b, c_2 - c_1, \ldots, c_n - c_{n-1} \rangle / \langle a + c_{i+1} - c_i, b + c_i - c_{i+1} \rangle = \mathbb{Z}/n\mathbb{Z} \\
H_2(X) &= \text{Ker}(\partial_2)/\text{Im}(\partial_3) = \langle A_i - A_{i-1} + B_{i+1} - B_i \rangle / \langle A_i - A_{i-1} + B_{i+1} - B_i \rangle = 0 \\
H_3(X) &= \text{Ker}(\partial_3)/\text{Im}(\partial_4) = \langle T_1 + T_2 + \cdots + T_n \rangle \simeq \mathbb{Z}
\end{align*}
\]

and \( H_n(X) = 0 \) for \( n \geq 4 \). To explain the computation for \( H_1(X) \), note that:

\[
\langle a, b, c_2 - c_1, \ldots, c_n - c_{n-1} \rangle = \langle a, a + b, a + c_2 - c_1, a + c_3 - c_2, \ldots, a + c_n - c_{n-1} \rangle
\]

since we can always subtract \( a \) from the other generators, and

\[
\langle a + c_{i+1} - c_i, b + c_i - c_{i+1} \rangle = \langle a + c_{i+1} - c_i, a + b \rangle = \langle a + c_2 - c_1, a + c_3 - c_2, \ldots, a + c_n - c_{n-1}, na, a + b \rangle
\]

and hence \( \text{Ker}(\partial_1) \) quotient by \( \text{Im}(\partial_2) \) this group is \( \langle a \rangle / \langle na \rangle \simeq \mathbb{Z}/n\mathbb{Z} \).

\[\square\]

**Exercise 6.** Find a \( \Delta \)-complex that is homotopy equivalent to the CW complex

\[ S^1 \cup D^2 / \{ x \sim \varphi(x), x \in \partial D^2 \} \]

where \( \varphi: \partial D^2 = S^1 \to S^1 \) is the map that winds \( n \) times around \( S^1 \). Compute the simplicial homology of this space.

**Proof.** The CW complex structure is of a \( n \)-gon with all vertices identified, and all edge going counterclockwise identified. We give it a \( \Delta \)-complex structure as follows:
The chain complex is thus:

\[ 0 \xrightarrow{\partial_3} \mathbb{Z}A_1 \oplus \ldots \oplus \mathbb{Z}A_n \xrightarrow{\partial_2} \mathbb{Z}a \oplus \mathbb{Z}b_1 \oplus \ldots \oplus \mathbb{Z}b_n \xrightarrow{\partial_1} \mathbb{Z}x \oplus \mathbb{Z}y \xrightarrow{\partial_0} 0 \]

The boundary maps are

\[ \begin{align*}
\partial_2 A_i &= a + b_i - b_{i+1} \\
\partial_1 a &= 0, \quad \partial_1 b_i = x - y
\end{align*} \]

Therefore,

\[ \begin{align*}
H_0(X) &= \text{Ker}(\partial_0)/\text{Im}(\partial_1) = \langle x, y \rangle / \langle x - y \rangle \simeq \mathbb{Z} \\
H_1(X) &= \text{Ker}(\partial_1)/\text{Im}(\partial_2) = \langle a, b_2 - b_1, \ldots, b_n - b_{n-1} \rangle / \langle a + b_i - b_{i+1} \rangle = \mathbb{Z}/n\mathbb{Z} \\
H_2(X) &= \text{Ker}(\partial_2)/\text{Im}(\partial_3) = 0/0 = 0
\end{align*} \]

and \( H_n(X) = 0 \) for \( n \geq 3 \). Here, the computation for \( H_1(X) \) is similar to the last problem; in particular, we have:

\[ \langle a, b_2 - b_1, \ldots, b_n - b_{n-1} \rangle = \langle a + b_2 - b_1, \ldots, a + b_n - b_{n-1} \rangle \]

and

\[ \langle a + b_i - b_{i+1} \rangle = \langle na, a + b_2 - b_1, \ldots, a + b_n - b_{n-1} \rangle. \]