Homework 2 Solution

Exercise 1.2.8. Compute the fundamental group of the space $X$ obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

Proof. Let $T_1, T_2$ be the two tori, and let $U_1, U_2$ be two open neighborhoods of $S^1 \times \{x_0\}$ in $T_1, T_2$ respectively that deformation retract onto $S^1 \times \{x_0\}$. This implies that $T_1 \cup U_1$ deformation retracts onto $T_1, T_2 \cup U_2$ deformation retracts onto $T_2$, and $U_1 \cup U_2$ deformation retracts onto $S^1 \times \{x_0\}$. Applying van Kampen’s Theorem to $T_1 \cup U_1$ and $T_2 \cup U_2$, we get:

$$\pi_1(X) \simeq \pi_1(T_1) \ast \pi_1(T_2) / \pi_1(S^1)$$

where the identifications are $\pi_1(T_1) \simeq \langle a, b \mid [a, b]\rangle$, $\pi_1(T_2) \simeq \langle c, d \mid [c, d]\rangle$, and $\pi_1(S^1) = \langle a, c \mid ac^{-1}\rangle$. Thus: $\pi_1(X) \simeq \langle a, b, c, d \mid [a, b] = [c, d] = ac^{-1} = 1\rangle \simeq (\mathbb{Z} \ast \mathbb{Z}) \times \mathbb{Z}$. □

Exercise 1.2.9. In the surface $M_g$ of genus $g$, let $C$ be a circle that separates $M_g$ into two compact sub-surfaces $M'_h$ and $M'_k$ obtained from the closed surfaces $M_h$ and $M_k$ by deleting an open disk from each. Show that $M'_h$ does not retract onto its boundary circle $C$, and hence $M'_h$ does not retract onto $C$. But show that $M'_g$ does retract onto the non-separating circle $C'$ in the figure.

Proof. Assume there is a retract $r : M'_h \to C$. We then have the induced homomorphisms on fundamental groups $r_* : \pi_1(M'_h) \to \pi_1(C)$ and $i_* : \pi_1(C) \to \pi_1(M'_h)$ with $r_* \circ i_* = 1$. Taking the abelianization gives us $r^*_{ab} \circ i^*_{ab} = 1$, hence $i^*_{ab} : \pi_1(C)_{ab} \to \pi_1(M'_h)_{ab}$ is injective. Since $\pi_1(C) = \mathbb{Z}$, it is the same as its abelianization, and $\pi_1(M'_h)$ is abelianized by modding the commutator $[a_1, b_1][a_2, b_2] \ldots [a_h, b_h]$ with $a_1, b_1, \ldots, a_h, b_h$ being the sides of $M_h$ in its CW complex structure as a $2h$-gon. Note that the generator of $\pi_1(C)$ is sent to the commutator above, hence $i^*_{ab}$ is actually trivial, a contradiction.

On the other hand, $M'_g$ retracts onto the torus $M_1$, which in turn retracts onto $C'$ which is one of its 1-cells (see picture). □

Exercise 1.2.11. The mapping torus $T_f$ of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$. In the case $X = S^1 \vee S^1$ with $f$ basepoint-preserving, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_* : \pi_1(X) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$.

Proof. First, consider $X = S^1 \vee S^1$. Denote the two circles by $a$ and $b$. Note that $T_f$ has the CW structure of one 0-cell $x_0$, three 1-cells $a, b, c$ attached to $x_0$, and two 2-cells attached
according to $f$. The first attachment goes in the order of $a, c, f_*(a)^{-1}$ and $c^{-1}$. Similarly, the second attachment goes in the order of $b, c, f_*(b)^{-1}$ and $c^{-1}$. Thus, we have the following presentation for $\pi_1(T_f)$:

$$\pi_1(T_f) = \langle a, b, c \mid acf_*(a)^{-1}c^{-1}, bcf_*(b)^{-1}c^{-1} \rangle.$$ 

For $X = S^1 \times S^1$, again let $a$ and $b$ denote the circles in the 1-skeleton of $X$. $T_f$ now has the CW structure with the 2-skeleton as above, plus one more 2-cell corresponding to the torus (thus is attached along $aba^{-1}b^{-1}$), and one more 3-cell. Since attaching a 3-cell doesn’t change $\pi_1$ by Proposition 1.26, we only care about the 2-skeleton of $T_f$ in computing $\pi_1$. Similar to above, we obtain the following presentation of $\pi_1(T_f)$:

$$\pi_1(T_f) = \langle a, b, c \mid acf_*(a)^{-1}c^{-1}, bcf_*(b)^{-1}c^{-1}, [a, b] \rangle.$$ 

\[\square\]

**Exercise 1.2.22.** Compute the *Wirtinger presentation* of a piecewise linear knot $K$ in $\mathbb{R}^3$ according to Hatcher’s instruction on p.55.

(a) Show that $\pi_1(\mathbb{R}^3 - K)$ has a presentation with one generator $x_i$ for each strip $R_i$ and one relation of the form $x_ix_jx_i^{-1} = x_k$ for each square $S_i$.

(b) Show that the abelianization of $\pi_1(\mathbb{R}^3 - K)$ is $\mathbb{Z}$. 

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References:


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*Note: The diagrams are not transcribed here for brevity. They illustrate the transformations and attachments described in the text.*
(c) Compute $\pi_1(\mathbb{R}^3 - K)$ for the knots in the homework.

Proof. (a) We need to compute $\pi_1(X)$, where $X$ is the space constructed in Hatcher. Pick an orientation of $K$ and a point $a$ in the plane. We construct a loop $x_i$ based at $a$ around each strip, and make it so that the direction of the loop is consistent with the orientation of $K$ (say, by the right hand rule). Now, since the plane is contractible, we can contract the plane to the point $a$, and also contract each strip $R_i$ longitudinally to a circle based at $a$ (see picture).

The new space $Y$ is homotopy equivalent to $X$, and has a CW complex structure of one 0-cell $a$, a 1-cell $x_i$ for each strip $R_i$, and 2-cells corresponding to the squares $S_l$ at each crossing. We also observe that the attaching map of each square $S_l$ is of the form $x_i x_j x_i^{-1} x_k^{-1}$, so $\pi_1(Y) \simeq \pi_1(X)$ is indeed the desired group.

(b) From part (a), we have $\pi_1(\mathbb{R}^3 - K) = \langle x_1, x_2, \cdots | x_i x_j x_i^{-1} x_k^{-1} \text{ for each square } S_l \rangle$. Abelianizing $\pi_1(\mathbb{R}^3 - K)$ would imply that the relations become

$$x_i x_j x_i^{-1} x_k^{-1} = x_i x_i^{-1} x_j x_k^{-1} = x_j x_k^{-1},$$

which is the same as identifying generator $x_j$ with $x_k$. Since $K$ is homeomorphic to $S^1$, when we go around $K$ once we would encounter all the strips, hence all generators are identified with that of a single strip $x_1$. Thus, $\pi_1(\mathbb{R}^3 - K) \simeq \langle x_1 \rangle \simeq \mathbb{Z}$. 


(c) The first knot has one strand $a$ and no crossing, hence $\pi_1(\mathbb{R}^3 - K) = \langle a \rangle \simeq \mathbb{Z}$.

The second knot has one strand $a$ and one crossing, and the relation at the crossing is $aaa^{-1} = a$ which is trivial, hence $\pi_1(\mathbb{R}^3 - K) = \langle a \rangle \simeq \mathbb{Z}$.

The third knot has two strands $a, b$ (with $a$ the left strand and $b$ the right strand), and two crossings. The relation at the left crossing is $aaa^{-1} = b$ and at the right crossing is $bbb^{-1} = a$. In either case, we get $a = b$, hence $\pi_1(\mathbb{R}^3 - K) \simeq \langle a \rangle \simeq \mathbb{Z}$. 

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