Flexibility in Contact and Symplectic Geometry

Oleg Lazarev
Michael Zhao Memorial Student Colloquium

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Rolling without slipping

Consider a car in $\mathbb{R}^2$ with position $x, y$ and angle $\theta$ with the $x$-axis; configuration space is $\{(x, y, \theta)\} = \mathbb{R}^2 \times S^1$.

If the car slips, its path $(x(t), y(x(t)), \theta(t))$ can be arbitrary; for example $(t, 0, \pi/4)$. 

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- **Question:** can any path in $\mathbb{R}^3$ be $C^0$-approximated by the motion of a non-slipping car?
Formal/genuine functions

- Graph of function $z(x)$ with its derivative: $(x, \frac{dz}{dx}, z(x)) \subset \mathbb{R}^3$
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**Example:** replace ODE $\left(\frac{df}{dx}\right)^2 + f(x)^2 \frac{df}{dx} = x^5$ with *algebraic* equation $y^2 + yz^2 = x^5$; curves in this hypersurface tangent to $\xi$ are solutions to the ODE
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- **Question:** can any formal function approximated by a genuine function?
Contact distribution

The contact distribution $\xi$ and submanifolds tangent to it are the key objects.
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Figure: The contact distribution $\xi_{std} = \ker(dz - ydx) \subset T\mathbb{R}^3$, image due to Patrick Massot
Definition: a contact structure $\xi$ on a manifold $Y^{2n+1}$ is a hyperplane distribution $\xi^{2n} = \ker(\alpha)$ for a 1-form $\alpha$ with $\alpha \wedge (d\alpha)^n \neq 0$, maximally non-integrable.
Contact geometry

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- **Examples:** 1-jet space $J^1(M) = T^*M \times \mathbb{R}$, $(S^{2n-1}, \xi_{\text{std}})$
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The (universal cover of the) previous two examples are contactomorphic: exists a diffeomorphism $\phi : (M, \xi_M) \to (N, \xi_N)$ such that $\phi^*\xi_N = \xi_M$
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Non-slipping car and graph of a genuine function are isotropics

Basic but important linear algebra fact: if $\Lambda^k \subset (\mathbb{Y}^{2n+1}, \xi)$ is isotropic, then $k \leq n$ (called Legendrian if $k = n$).

Intuition: contact distribution is maximally non-integrable.
Classical flexibility results

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- **Gray stability theorem:** if \((Y, \xi_t)\) is isotopy of contact structures on a closed manifold \(Y\), then all contactomorphic, i.e. exists diffeotopy \(\phi_t\) of \(Y\) such that \(\phi_t^* \xi_t = \xi_0\)
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- **Weinstein neighborhood theorem:** any Legendrian \(\Lambda^n \subset (Y^{2n+1}, \xi)\) has a neighborhood that is contactomorphic to neighborhood of \(\Lambda\) in \(J^1(\Lambda)\)
Partial Differential Relations

- Many geometric problems given by a PDE, e.g. existence of contact structure, contactomorphism, isotropic embedding

- Definition: A formal contact structure is a 1-form $\alpha$ and a non-degenerate 2-form $\omega$ on $\ker \alpha$ (but $\omega \neq d\alpha$)

- Definition: A formal isotropic embedding is smooth embedding of $L_k$ and a homotopy $E_k^t$ of $k$-planes in $TY^{2n+1}$ over $L$ such that $E_0 = TL$ and $E_1 \subset \xi$ (but $E_t \neq TL$)

- Consider $i: \text{Solutions} \rightarrow \text{Formal Solutions}; h$-principle holds when $i$ is a (weak) homotopy equivalence, i.e. geometric problem reduces to algebraic topology

- Question: does $h$-principle hold for contact structures or isotropic submanifolds?
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Rigidity in contact geometry

- There are non-local, deformation stable invariants of contact manifolds, Legendrians called *contact homology* and *Legendrian contact homology*, Gromov-Witten type invariant defined using J-holomorphic curves. Related to wrapped Fukaya category, mirror symmetry...
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Many Legendrian knots in \((\mathbb{R}^3, \xi_{\text{std}})\) are formally isotopic but not Legendrian isotopic, distinguished by Legendrian contact homology.

**Figure:** Chekanov Legendrians in \(\mathbb{R}^2_{xz}\); images due to John Etnyre
Rigidity in contact geometry, II

- Similarly, many contact structures are formally contactomorphic but not contactomorphic.

**Figure**: Standard and overtwisted structures; images due to Patrick Massot

- h-principle fails for contact manifolds, isotropic submanifolds! $i_*$ is not injective on $\pi_0$; for Legendrian knots, $i_*$ is not surjective on $\pi_0$.

- **Question**: what is the boundary between rigidity and flexibility?
Flexibility for isotropics

- Gromov's h-principle for subcritical isotropics: two formally isotopic $\Lambda_1^k, \Lambda_2^k \subset (Y^{2n+1}, \xi)$ with $k < n$ are genuinely isotopic
Contact Manifolds and Legendrian Submanifolds
Symplectic manifolds and Weinstein domains

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Flexibility for isotropics

- **Gromov’s h-principle for subcritical isotropics**: two formally isotopic $\Lambda^k_1, \Lambda^k_2 \subset (Y^{2n+1}, \xi)$ with $k < n$ are genuinely isotopic.

- **h-principle fails for general Legendrians ($k = n$) by LCH**

- **Definition**: a Legendrian $\Lambda^n \subset Y^{2n+1}$ is *loose* if $n \geq 2$ and it has a ‘zig-zag’ in its $xz$-projection.

**Figure**: Loose chart, i.e. zig-zag, pictured in $\mathbb{R}^2_{xz}$ and in $\mathbb{R}^2_{xy}$.
Loose Legendrians

- **Murphy’s h-principle for loose Legendrians:** formally isotopic loose Legendrians are Legendrian isotopic; any smooth embedding can be $C^0$-approximated by a loose Legendrian.
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  ![Approximating slipping path](image)

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- **Open problem:** If $\Lambda$ has vanishing LCH, is it loose?
Loose Legendrians, II

Loose Chekanov knots (in high-dimensions) are Legendrian isotopic
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- **Open problems:** Is $i_*$ injective on $\pi_0$ in dimension 4? Is $i_*$ surjective on $\pi_0$ in dimensions $> 4$?
Weinstein domains

- An exact symplectic manifold \((M^{2n}, d\alpha)\) has contact boundary if \((\partial M, \ker \alpha)\) is a contact manifold
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- **Weinstein:** can attach a handle to an isotropic sphere \(\Lambda^{k-1} \subset \partial M^{2n}\) and get a new symplectic manifold with contact boundary \(M^{2n} \cup H^k_{\Lambda}\)
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![Weinstein handle attachment](image)

**Figure:** Weinstein handle attachment
Definition: a Weinstein domain $W^{2n}$ is iterated Weinstein handle attachment to $(B^{2n}, \omega_{\text{standard}})$, i.e. symplectic handlebody
Weinstein domains, II

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Rigidity for Weinstein domains

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- **McLean:** infinitely many different formally symplectomorphic Weinstein structures on $B^{2n}$, $n \geq 4$ (distinguished by symplectic homology)
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Figure: Sketch of an exotic Weinstein ball.
Definition: a Weinstein domain $W^{2n}$, $n \geq 3$ is flexible if all $n$-handles are attached along loose Legendrians.
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Figure: $T^*S^n$ and $T^*S^n_{\text{flex}}$
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Figure: $T^*S^n$ and $T^*S^n_{flex}$

Cieliebak-Eliashberg: A formal Weinstein manifold $W^{2n}, n \geq 3$, has a genuine Weinstein structure. Two formally symplectomorphic flexible structures are symplectomorphic.
Flexibility for Weinstein domains

- **Definition**: a Weinstein domain $W^{2n}, n \geq 3$ is *flexible* if all $n$-handles are attached along loose Legendrians.

![Diagram](image)

**Figure**: $T^* S^n$ and $T^* S^n_{\text{flex}}$

- **Cieliebak-Eliashberg**: A formal Weinstein manifold $W^{2n}, n \geq 3$, has a genuine Weinstein structure. Two formally symplectomorphic flexible structures are symplectomorphic.

- **Question**: can this result be used to construct symplectic structures on closed manifolds?
Modifying Weinstein presentations

- Can modify Weinstein presentation by doing handle-slides and create/cancel handles; easy to create more handles
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![Figure: Handle-slides and handle cancellation/creation]
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- $WCrit(W) : = \text{minimum number of Weinstein handles for } W$
- $Crit(W) : = \text{minimum number of smooth handles}$
Modifying Weinstein presentations

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\[ WCrit(M) \geq Crit(M) \geq \text{rank } H^*(M; \mathbb{Z}) \]
Smale’s h-cobordism theorem: if \( \dim M \geq 5, \pi_1(M) = 0, \) then \( \text{Crit}(M) = \text{rank } H^*(M; \mathbb{Z}); \) key is Whitney trick
Modifying Weinstein presentations, II

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Cieliebak-Eliashberg: \( WCrit(W_{\text{flex}}) = \text{Crit}(W) \)

McLean: exist \( W \) with \( WCrit(W) \geq \text{Crit}(W) + 2 \); Whitney trick fails!
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L. any Weinstein \( W^{2n}, n \geq 3 \), has \( WC\text{rit}(W) \leq \text{Crit}(W) + 2 \)
Modifying Weinstein presentations, II

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- **L.** any Weinstein \( W^{2n} \), \( n \geq 3 \), has \( WCrit(W) \leq \text{Crit}(W) + 2 \)
- Implies restrictions on J-holomorphic curve invariants: there is no Weinstein structure on the ball \( B^{2n} \) whose wrapped Fukaya category is that of \( T^*S^n_{\text{std}} \), i.e. modules over \( C_*(\Omega S^n) \)
Modifying Weinstein presentations, II

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- **L.** any Weinstein $W^{2n}$, $n \geq 3$, has $WCrit(W) \leq \text{Crit}(W) + 2$

- Implies restrictions on J-holomorphic curve invariants: there is no Weinstein structure on the ball $B^{2n}$ whose wrapped Fukaya category is that of $T^*S^n_{\text{std}}$, i.e. modules over $\mathcal{C}_*(\Omega S^n)$

- **Question:** what is the interaction between symplectic flexibility and rigidity?