Euler Characteristic
(and the Unity of Mathematics)

Oleg Lazarev

SPLASH

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Graphs

Collection of dots (vertices) connected by lines (edges)
Planar Graphs

Definition: A graph is *planar* if there are no edge crossings

Let $V =$ number of vertices, $E =$ number of edges, $F =$ number of faces (including outside face)

For Graph A: $V = 5$, $E = 6$, $F = 3$
Euler characteristic

**Definition:** the Euler characteristic of a planar graph $G$ is

$$\chi(G) = V - E + F$$

- Ex 1. $\chi(G) = 4 - 6 + 4 = 2$.

- Ex 2. $\chi(G) = 7 - 9 + 4 = 2$.

- Ex 3. $\chi(G) = 8 - 7 + 1 = 2$. 
Euler’s Theorem

**Euler’s theorem:** All planar graphs have Euler characteristic 2!

- Ex. when subdivide an edge by adding a vertex, 
  \[ V' - V = 1, E' - E = 1, F' - F = 0 \]  
  so \( \chi(G') = \chi(G) \)

- Ex: when add an edge connected to two vertices, 
  \[ V' - V = 0, E' - E = 1, F' - F = 1 \]  
  so \( \chi(G') = \chi(G) \)

- Ex: when add an edge connected to one vertex, 
  \[ V' - V = 1, E' - E = 1, F' - F = 0 \]  
  so \( \chi(G') = \chi(G) \)

- So \( \chi(G) = \chi(\text{single vertex}) = 1 + 1 = 2 \)
Planarity is really an intrinsic property: a graph is planar if it can be drawn so that no edge crossings.

Example:

These all have the same underlying graph, with edges redrawn. Hence the graph is planar.
Non-planar graphs

Planar graphs have $e \leq 3v - 6$

- For any planar graph, $3f \leq 2e$
- Euler characteristic $2 = v - e + f$
- So $6 = 3v - 3e + 3f \leq 3v - 3e + 2e = 3v - e$
- Hence $e \leq 3v - 6$ for planar graphs

**Ex 1.** $K_5$ graph

$V = 5, E = 10$ so $10 \not\leq 3 \cdot 5 - 6$ so $K_5$ is not planar!
Ex 2. $K_{3,3}$ graph

By similar argument, $K_{3,3}$ is non-planar.

Kuratowski’s Theorem: a graph is planar if and only if it has no subgraphs that are (expansions of) $K_{3,3}$ or $K_5$
Topology: “the study of geometric properties and spatial relations unaffected by the continuous change of shape or size of figures.”
All (orientable) surfaces without boundary are i.e. spheres and multi-holed donuts.

Example of non-orientable surface: Mobius strip
Triangulation of surfaces:

**Definition:** Euler $\chi(S, T)$ of a surface $S$ is $V - E + F$ for some triangulation $T$ of $S$
Euler characteristic of Surfaces

- Euler’s Theorem implies that all spheres have Euler characteristic 2!

- Similar to proof of Euler’s Theorem: $\chi(S)$ is independent of triangulation, i.e. Euler characteristic is a topological invariant!

- HW: what is Euler characteristic for donuts with $g$ holes?

- HW: define Euler characteristic for non-planar graphs
Platonic Solids

Constructed from congruent regular polygons (equal side length and angles) with the same number of faces meeting at each vertex.

- Plato: each is associated to "classical elements" - Earth, air, water, and fire (fifth one?)
- Ex. Euler characteristic of tetrahedron: $4 - 6 + 4 = 2$
- Ex. Euler characteristic of octahedron: $6 - 12 + 8 = 2$
Classification of Platonic Solids

Theorem: These are the only Platonic solids!

▶ Suppose faces are $n$-gons and $m$ edges meet at each vertex
▶ So $nF = 2E$ and $mV = 2E$
▶ Euler characteristic becomes $V - E + F = \frac{2E}{m} - E + \frac{2E}{n} = 2$
or $\frac{1}{m} + \frac{1}{n} = \frac{1}{E} + \frac{1}{2}$
▶ There are only five such pairs satisfying this equation
  $(m, n) = (3, 3), (3, 4), (3, 5), (4, 3), (5, 3)$ and these all give Platonic solids
  (tetrahedron, hexahedron, dodecahedron, octahedron, icosahedron)
Hairy Ball Theorem

Theorem: Can’t comb a hairy ball flat (two hairs will always be sticking out).

Or there is always a point on Earth where wind velocity is exactly zero (actually two points).
Hairy Donuts can be combed.

"Combability" depends on topology.
Mathematicians use vector fields $\nu$

At each point on a

Ex. $I(S^2, \nu) = 2$

Ex. $I(T^2, \nu) = 0$
Poincare-Hopf Theorem: for any vector field $\nu$

- Links topology and geometry; a priori vector fields has nothing to do with topology; if don’t take alternating signs
Proposition: exists a vector field \( \nu \) on \( X \) with \( \iota(X, \nu) = \chi(X) \)
Geometry

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Not a topological invariant!
$X$ is a surface Gauss-Bonnet Theorem: $\int_X K(x) = \chi(X)$.

- Links geometry and topology!
- Right-hand-side seems like it depends on how put surface into space but actually independent!
Betti Numbers

- Euler characteristic are a shadow of more invariants
- To any space $X$, we can assign sequence of ‘Betti numbers’ $b_i(X) \geq 0$, where $i \geq 0$ is an integer; that are invariants
- Then $\chi(X) = \sum_{i \geq 0} (-1)^i b_i(X)$
Thank You!