Quantum groups and
Quantum K-theory

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in collaboration with M. Aganagic, D. Maulik,
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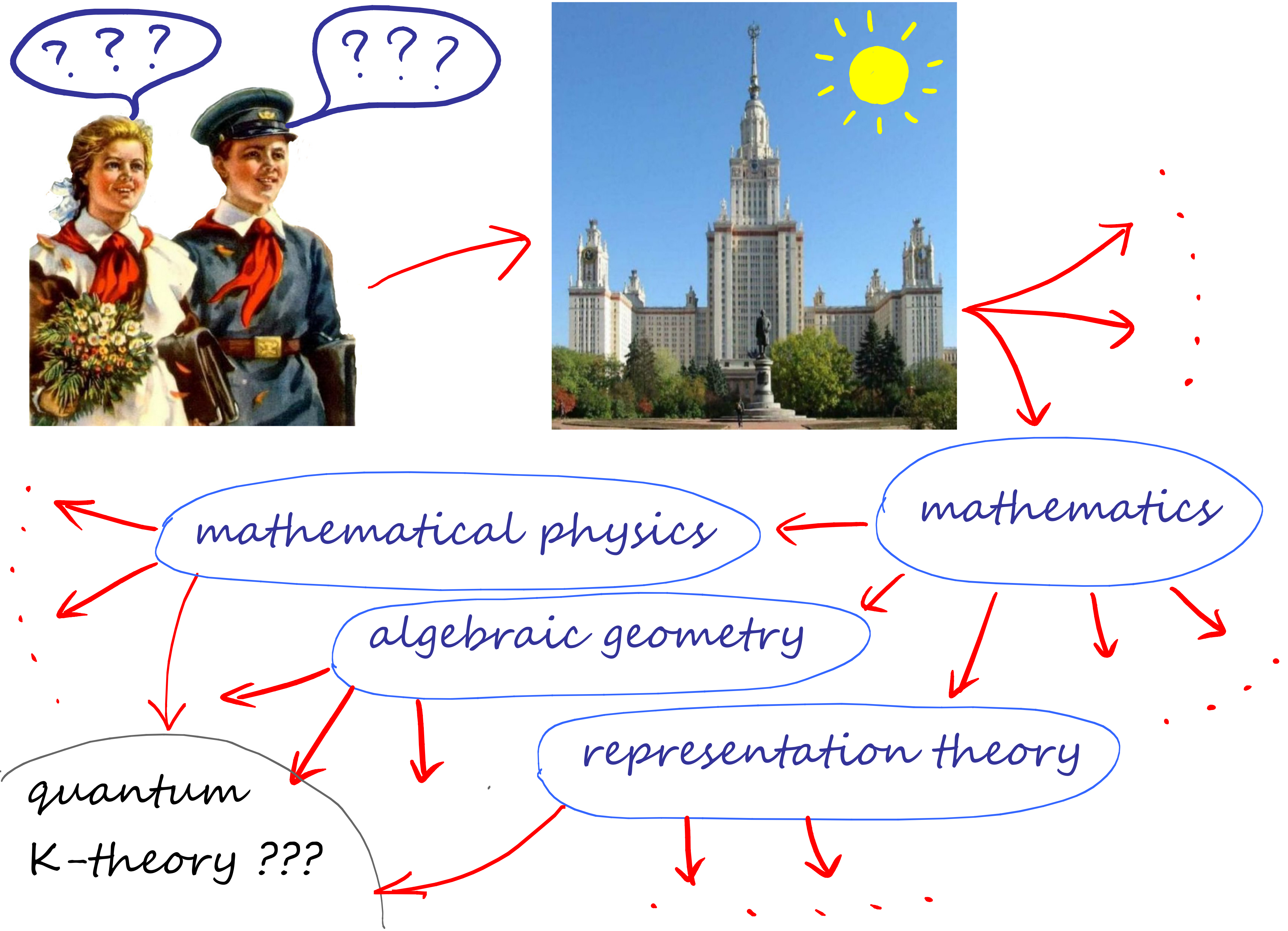
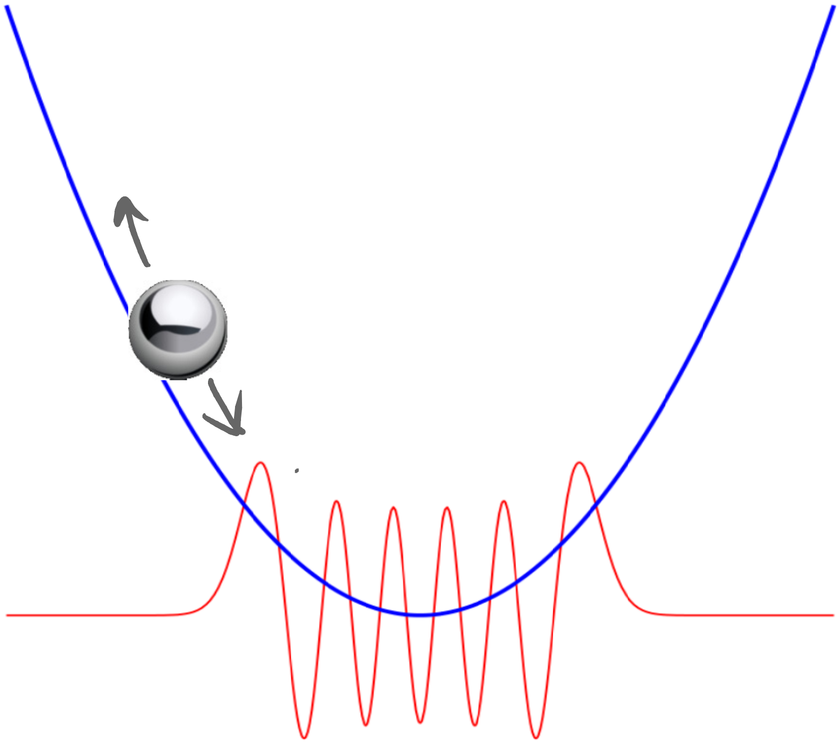
1. Why Quantum K-theory ?


???
???
mathematics
mathematical physics
algebraic geometry
representation theory
quantum
K-theory ???



A central problem in quantum mechanics is to find the eigenspaces of an operator of the form

$$-\Delta + V$$

from metric on the configuration space

potential

It is interesting and important to know these eigenspaces as representations of the group of symmetries

With fermions, one can study Dirac operators

even
fermion #

$$\begin{tikzcd} \mathcal{H}_0 \arrow[r, bend left, "\not{\partial}"] & \mathcal{H}_1 \arrow[l, bend left, "\not{\partial}"] \end{tikzcd}$$

odd
fermion #

and of special importance is their index

$$\text{Index } \not{\partial} = \left(\operatorname{Ker} \not{\partial}\right)_0 - \left(\operatorname{Ker} \not{\partial}\right)_1$$

as a virtual representation of symmetries

this index does not change under continuous
equivariant deformations

whole branches of mathematics were
developed to study it

in particular, if the metric on the configuration space M is Kähler then

$$\not{\partial} = \bar{\partial} + \bar{\partial}^* \hookrightarrow K_M^{1/2} \otimes \bigoplus_{n=0}^{\dim_{\mathbb{C}} M} \Omega_M^{0,n}$$

Canonical bundle

and so, by Hodge and Dolbeault,

$$\text{Index } \not{\partial} = \chi\left(K_M^{1/2}\right)$$

holomorphic Euler characteristic

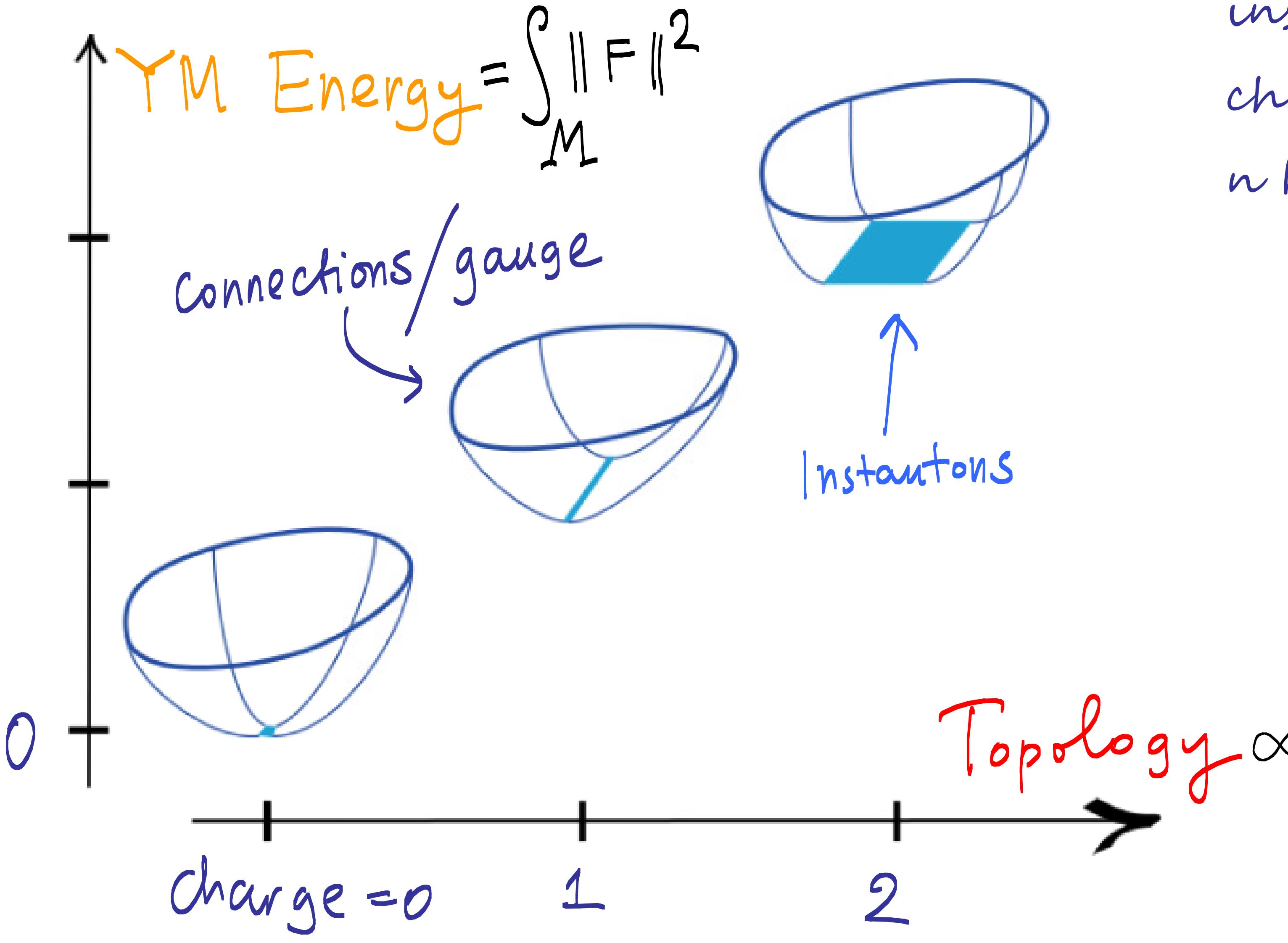
other bundles can appear here, too

Quantum field theory is like quantum mechanics with an infinite-dimensinal configuration space.

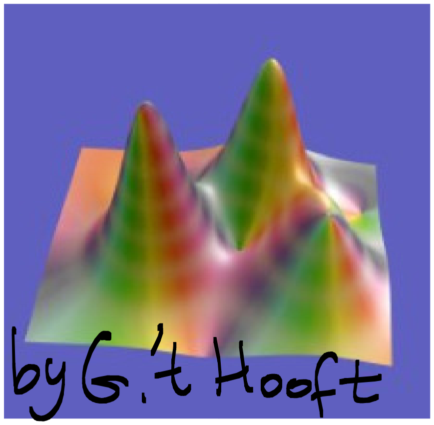
A key feature of many supersymmetric theories is that the vacua (≈critical points of the potential), on which the index is always localized, form a countable union of finite-dimensional algebraic varieties.

Thus the index becomes an object of algebraic geometry

For example, in Yang-Mills theory, ASD connections, or instantons minimize energy given topology



instantons of
charge n look like
n bumps on M


by G. 't Hooft

The importance of instantons was recognized in the 70's by Polyakov and others.

Index computations reduce to computations on the moduli spaces of instantons.

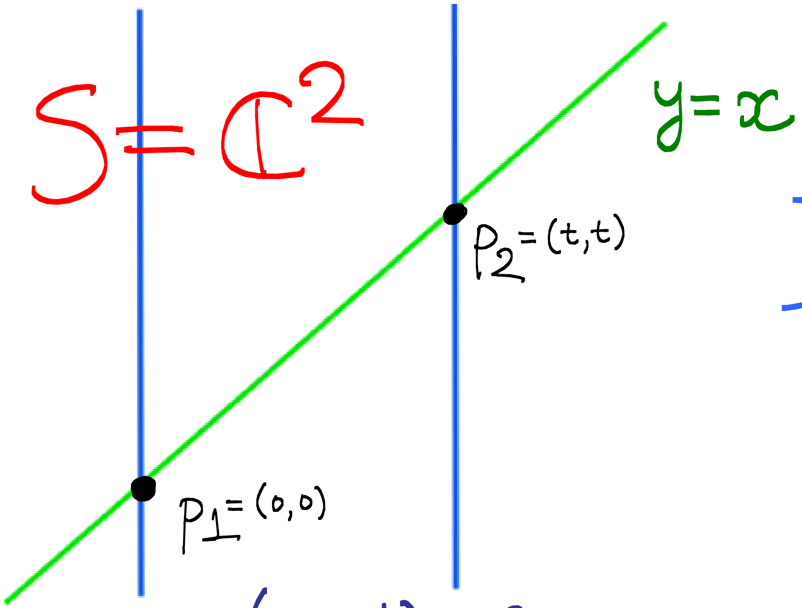
By Donaldson, instantons are the same as holomorphic vector bundles for Kähler surfaces $S^2=M^4$

Instanton calculus has been an area of dramatic progress in mathematical physics. In particular, Nekrasov & Co. derived the challenging predictions of Seiberg and Witten from it, and much more ...

For an algebraic surface S, the computations in this Donaldson-Witten-Nekrasov-... theory are computations in K-theory (or cohomology) of moduli of bundles and more general sheaves on S.

In a certain precise technical sense, all of these reduce to computations with the Hilbert schemes of S, which parametrize ideals of functions of finite codimension n = pointlike intantons of rank 1 and charge n

$S = \mathbb{C}^2$



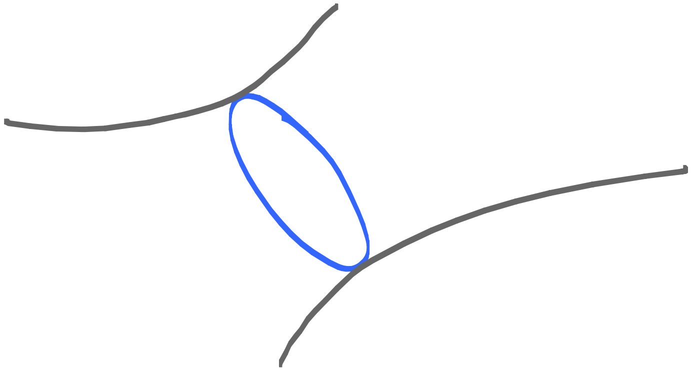
$x(x-t)=0$

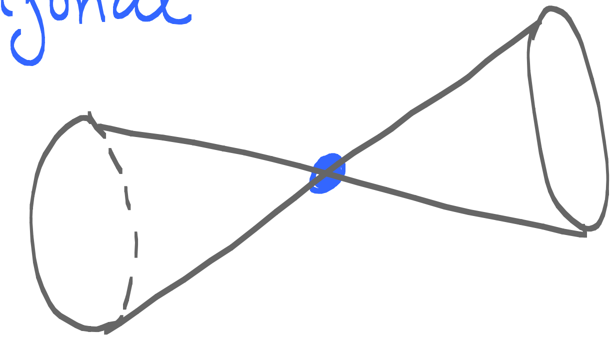
$$I_{\{P_1,P_2\}} = (x-y,\ x^2-tx) \subset \mathbb{C}[x,y]$$

$$\uparrow$$

remains there even as $t \to 0$,
so remembers the direction

$$\mathrm{Hilb}(S,2) = \text{Blow-up}_{\text{diagonal}}\ \mathrm{Sym}^2 S$$





There is a generalization of Donaldson theory to complex 3-folds X, known as the Donaldson-Thomas theory.

It deals with moduli of sheaves on X, and among those the Hilbert schemes play again the central role.

They are now Hilbert schemes of curves C in X, i.e. instantons remain defects of complex codimension 2.

In place of instanton charge, connected components of Hilb(X,curves) are now indexed by

$$[C] = (\text{degree}, \chi(\mathcal{O}_C)) \in K_{Top}(X)$$

Hilbert schemes of 3-folds, and other DT moduli spaces, are badly singular. However, by the work of Thomas, they have a good virtual structure sheaf, which is what's needed in index computations. Further, Nekrasov and AO proved DT moduli spaces are spin, so we can define

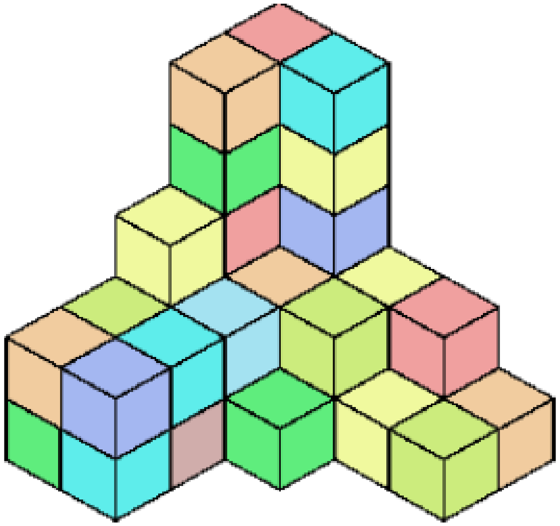
$$\text{Index} = \chi\left(\mathrm{Hilb}\left(X, \substack{\text{curves} \\ C\subset X}\right), \mathcal{O}_{\mathrm{vir}} \otimes K_{\mathrm{vir}}^{1/2} \; z^{[C]}\right)$$

disconnected

variable

We will see these generating functions have many deep and surprizing properties and connect to many branches of math

II. My first DT moduli space



For a first taste of the theory, take the simplest 3-fold

$$X = \mathbb{C}^3$$

There are no (complete) curves in it, only points, so

$$\mathrm{Hilb}(X, \underset{\text{points}}{n}) = \text{ideals } I \subset \mathcal{O} = \underset{\underset{GL(3)}{\circlearrowleft}}{\mathbb{C}[x_1, x_2, x_3]}$$

of codimension n

$$= \text{matrices } X_1, X_2, X_3 \in \mathrm{Mat}(\mathbb{C}^n) \Big/ GL(\underset{\underset{\mathcal{O}/I}{\cong}}{\mathbb{C}^n})$$

+ cyclic vector

Such that $[X_i, X_j] = 0$

Smooth

Let $F \overset{\text{def}}{=}$ matrices $X_1, X_2, X_3 \in \mathrm{Mat} \big/ GL$
+ cyclic vector

and note

$$\left\{ [X_i, X_j] = 0 \right\} \Longleftrightarrow \nabla \varphi = 0$$

where

$$\varphi = \mathrm{tr}\left(X_1 X_2 X_3 - X_1 X_3 X_2 \right)$$

is a function of weight $\det(\mathbb{C}^3)$ w.r.t. $GL(3)$

This gives $\mathrm{Hilb} \subset F$ its virtual structure sheaf

$$\mathcal{O}_{vir} \overset{def}{=} \Big[\cdots \xrightarrow{\nabla\varphi} \underset{\det(\mathbb{C}^3)^{-2}}{\underset{\otimes}{\Lambda^2 T_F}} \xrightarrow{\nabla\varphi} \underset{\det(\mathbb{C}^3)^{-1}}{\underset{\otimes}{T_F}} \xrightarrow{\nabla\varphi} \mathcal{O}_F \Big]$$

with

$$K_{vir} = \det(T_F)^{-2} \otimes \det(\mathbb{C}^3)^{\dim F}$$

Now our goal is to compute the trace of

$$g = \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \in T \subset GL(3)$$

acting in $\chi\left(\mathrm{Hilb}, \mathcal{O}_{\mathrm{vir}} \otimes K_{\mathrm{vir}}^{1/2}\, z^{n}\right)$

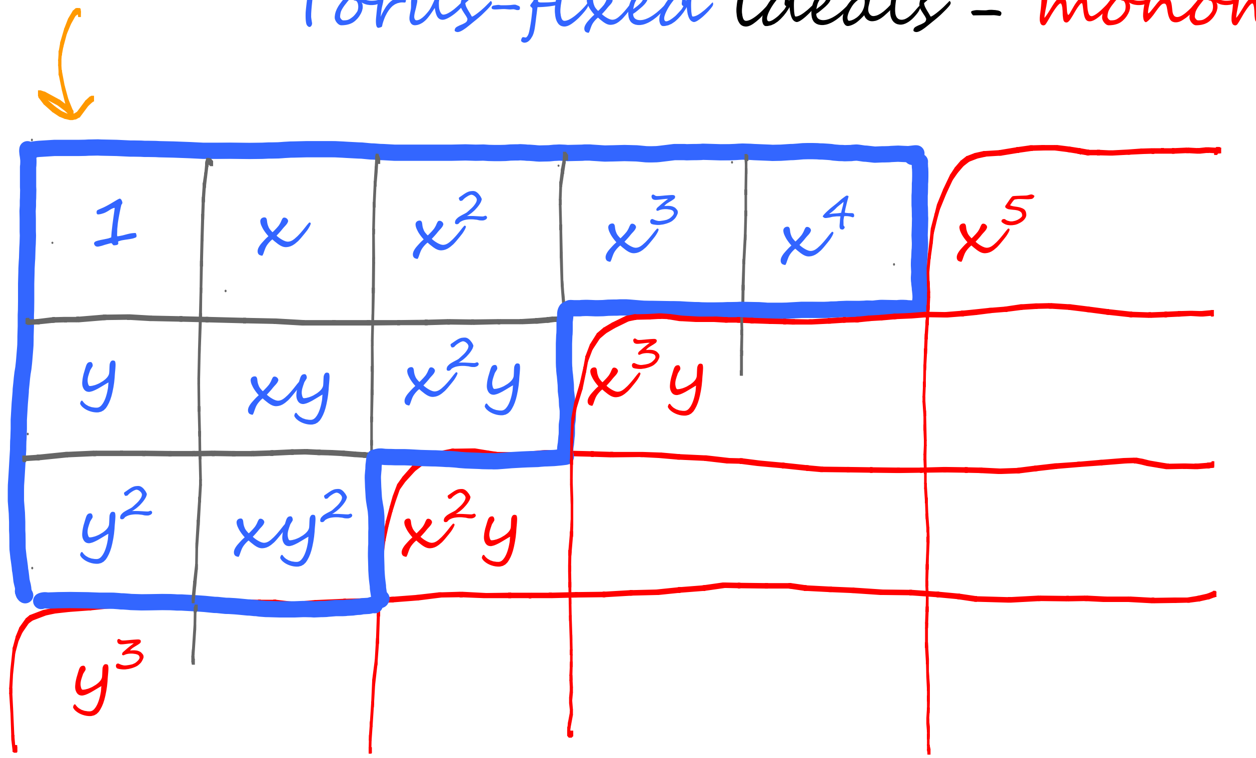
by a fundamental principle, a.k.a. localization, only

$$\boxed{\text{fixed points}}$$

contribute to the trace

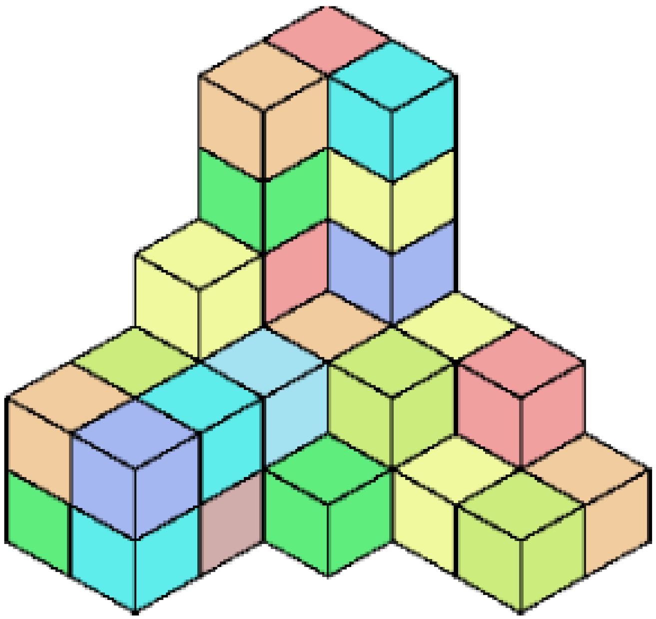
in 2D

Torus-fixed ideals = monomial ideals



Monomials in red are the generators of an ideal I. Monomials in blue form a basis of C[x,y]/I.

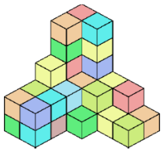
Same in any dimension



McMahon

so

$$\text{Index} = \sum_{\pi =} (-z)^{|\pi|} \prod_{w \,\in\, \text{weights of } T_\pi F} \frac{\hat{a}(t_1 t_2 t_3 / w)}{\hat{a}(w)}$$

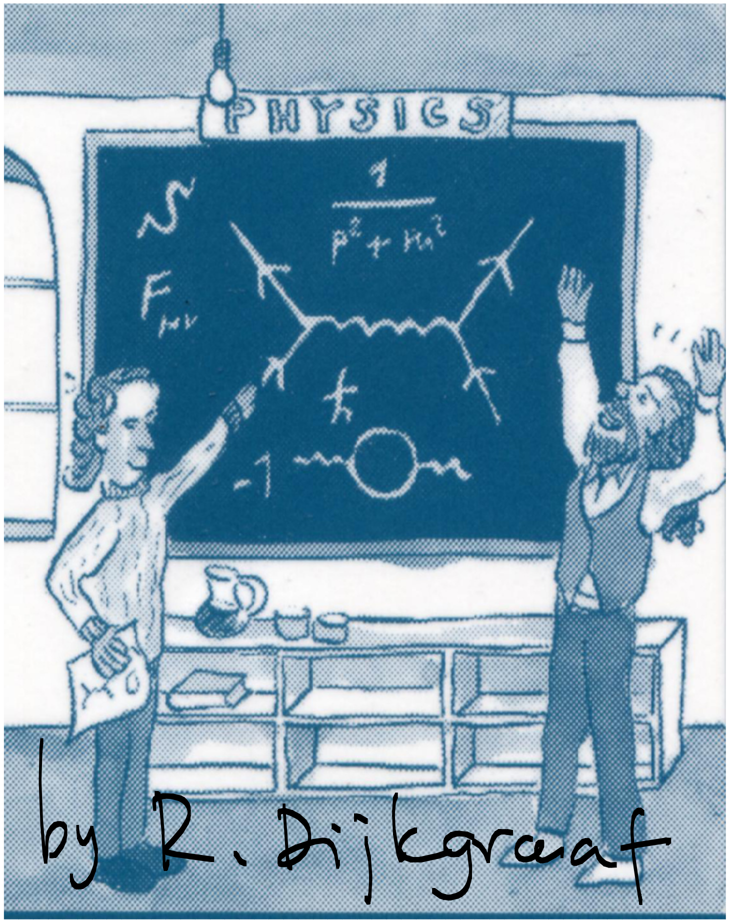


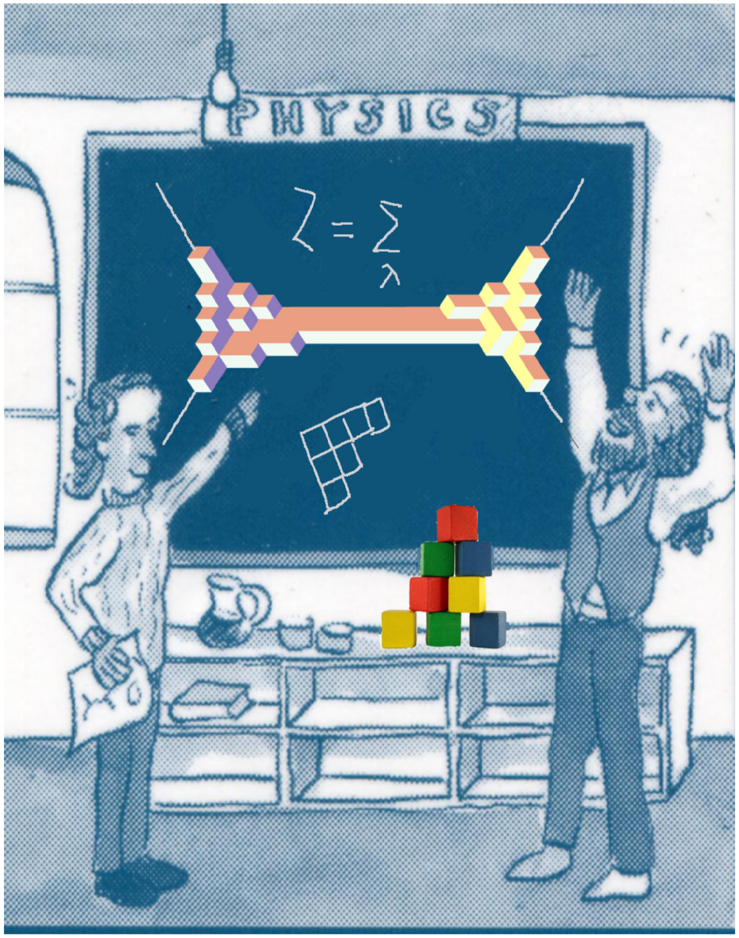
quadratic in $[\mathcal{O}/I]$ as a T-module

where $\hat{a}(t) = t^{1/2} - t^{-1/2}$

a completely algorithmic, if not very compact, infinite series

The role of such localization series in modern mathematical physics is not unlike the role of Feynman diagrams





Can we sum the series ?

recall that if

V = a representation of G

$S^\bullet V$ = its symmetric algebra

then

$$\mathrm{tr}_{S^\bullet V}\, g = \exp \sum_{n=1}^{\infty} \frac{1}{n}\, \mathrm{tr}_V\, g^n$$

Conjectured by Nekrasov ~10 years ago

Theorem*

$$\text{Index} = S^{\bullet}\, \frac{\prod_{i<j\leq 3} \hat{a}(t_i t_j)}{\prod_{i=1}^{5} \hat{a}(t_i)} =: V$$

with $t_4 = z \big/ \sqrt{t_1 t_2 t_3}$ $\quad t_5 = 1 \big/ z\sqrt{t_1 t_2 t_3}$

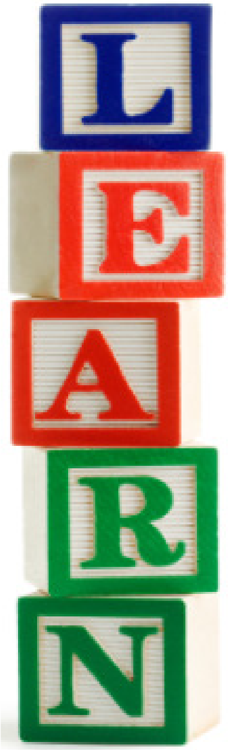
and so $\operatorname{diag}(t_1, \dots, t_5) \in SL(5)$

With a suitable prefactor

$$V \rightsquigarrow \frac{\sum_{i=1}^{5}\left(t_i^{-1} - t_i\right)}{\prod_{i=1}^{5} \hat{a}(t_i)}$$

complete SL(5) symmetry

III. What can we LEARN from this ?



visibly, V from the theorem is χ of some sheaf on

$$\begin{array}{ccccc} Z = & \mathbb{C}^5 & = & \mathbb{R}^{10} \\ & \circlearrowright & & \circlearrowright \\ & SU(5) & \subset & SO(10) \end{array}$$

and, in fact, the point of Nekrasov's conjecture is that this is the index of fields of M-theory/gauge on Z

M-theory is a supergravity theory in dim=10+1, singled out by its many remarkable features. Its fields are:

- graviton, the metric
- its superpartner, gravitino
- 3-form A, analog of vector potential in EM

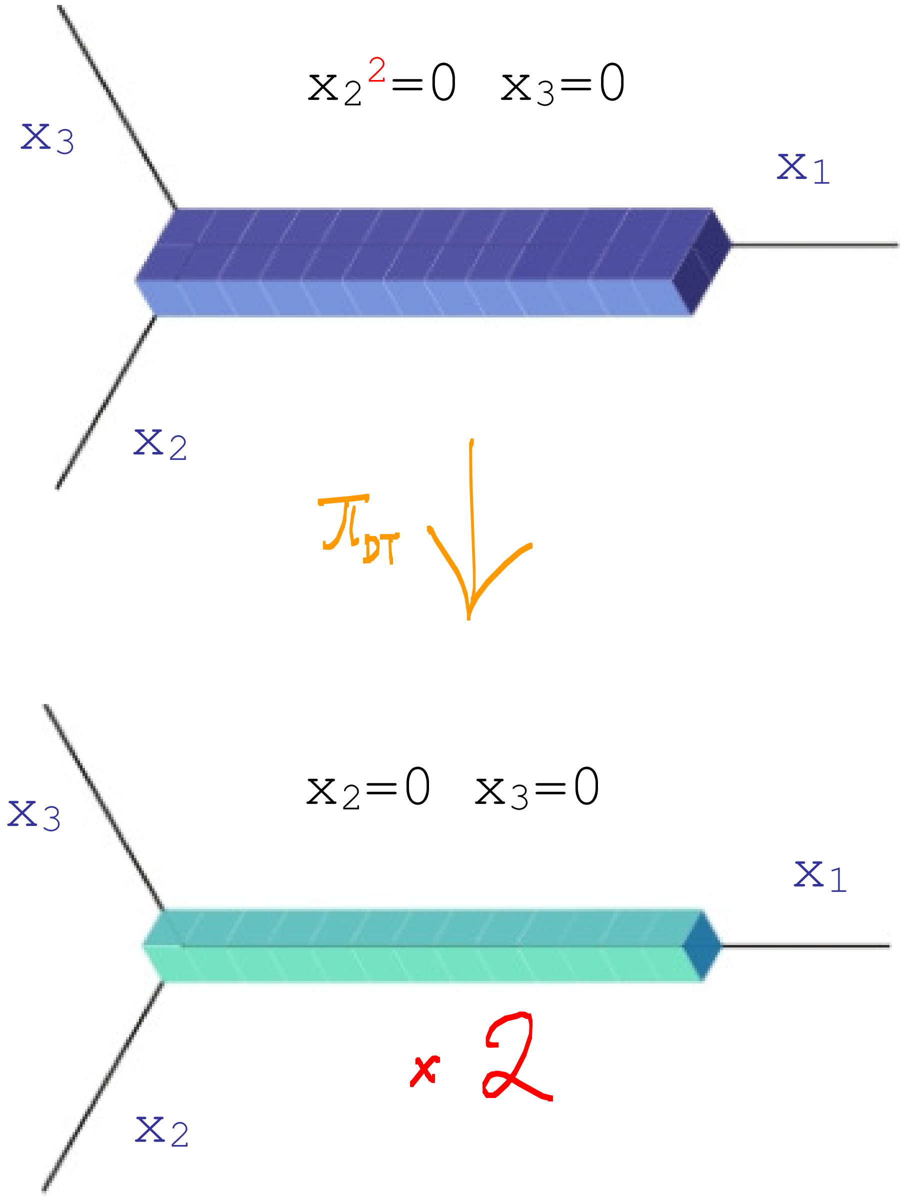
in addition it has

- membranes, charged under A
- M5 branes, magnetically charged under A

not today

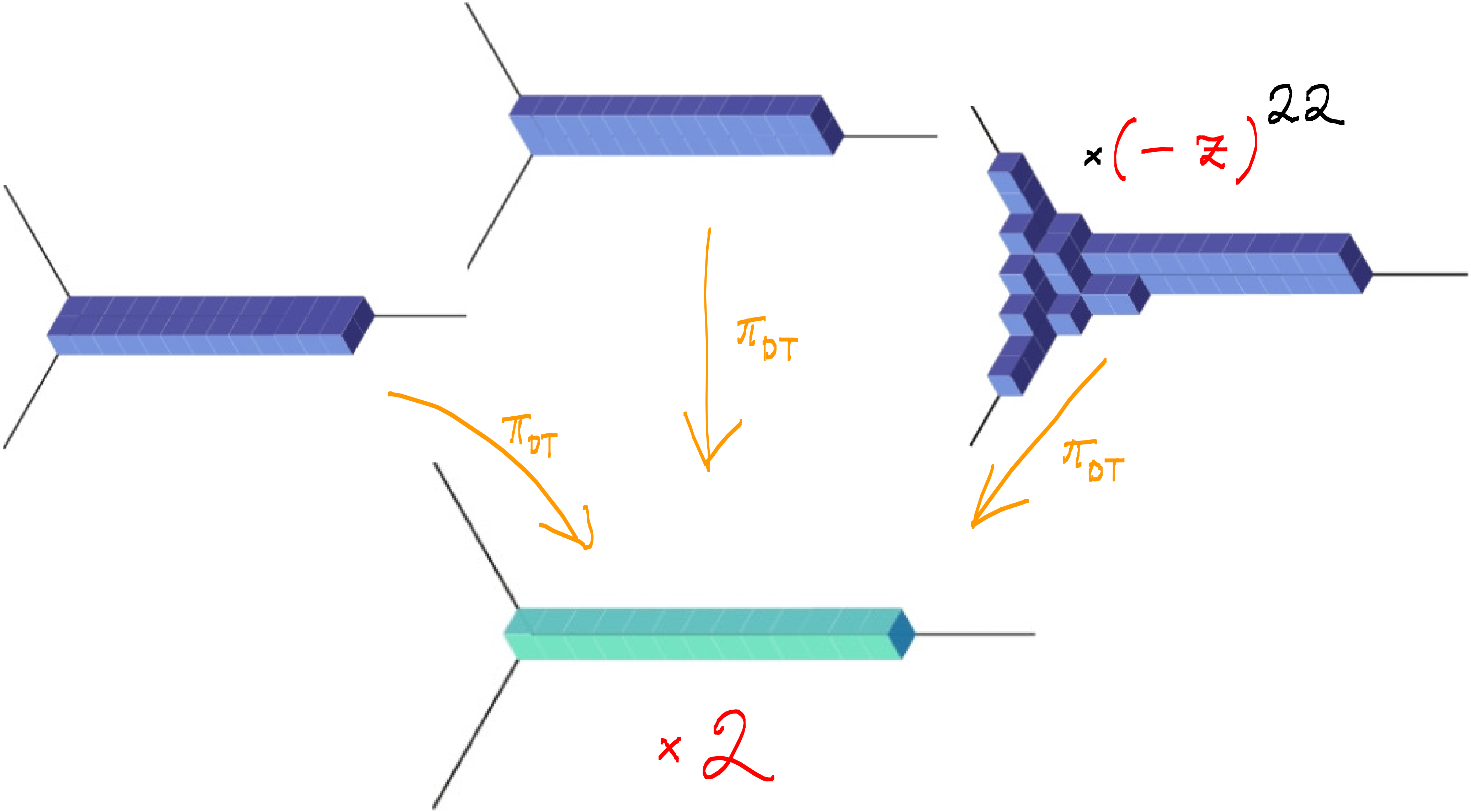
Generalizing the above formula, Nekrasov and AO conjectured that the curve contributions may similarly be summed up using membranes of M-theory.

In fact, this should work curve-by-curve, or rather point-wise on the Chow variety of 1-cycles in X

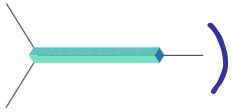


there is a map from Hilb(X),
which parametrizes
subschemes of X, to Chow(X),
which parametrizes cycles in
X, that is, sums of reduced
irreducible subvarieties of X
with multiplicities

Many points of Hilb(X) map to the same point in Chow(X)



While

$\pi_{DT}{}^{-1}($  $)$ = countable $\cup$ of alg varieties,

the boxcounting parameter z makes

$$\pi_{DT,*}\left(\mathcal{O}_{vir}\otimes K_{vir}^{1/2}\,(-z)^{\chi(\cdot)}\right)$$

a well-defined sheaf on Chow(X). To compute this sheaf means to sum up all boxes, and membranes, conjecturally, give the answer

The setup is as follows ...

Let Z be a 5-fold with trivial K_Z

Let $z \in \mathbb{C}^*$ act on Z preserving the 5-form and so that

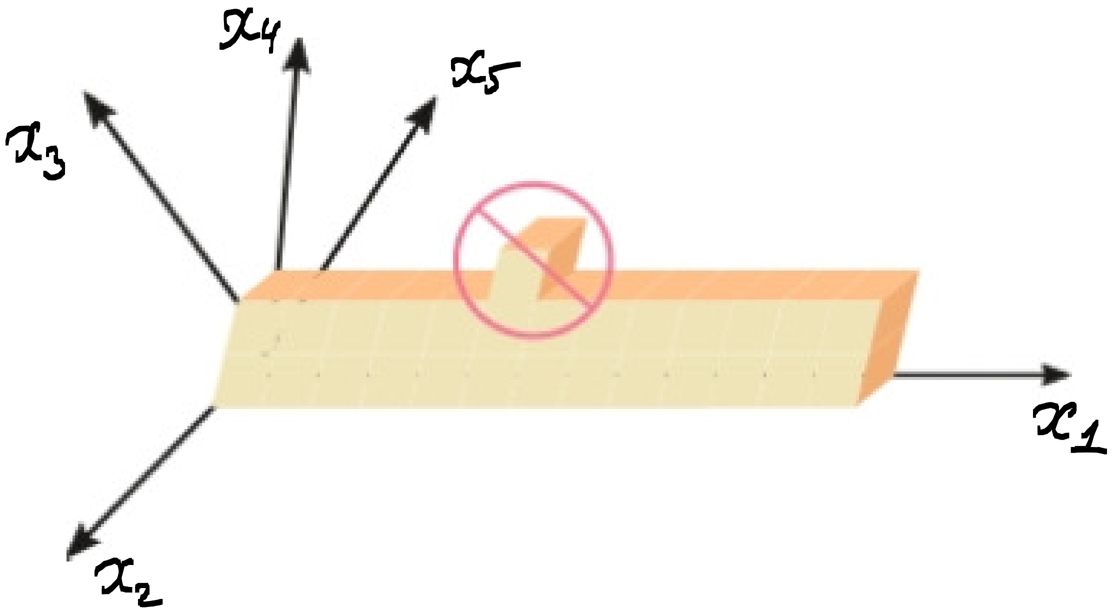
X = fixed locus

is purely 3-dimensional. For simplicity, in this talk, let's assume X is connected, e.g.

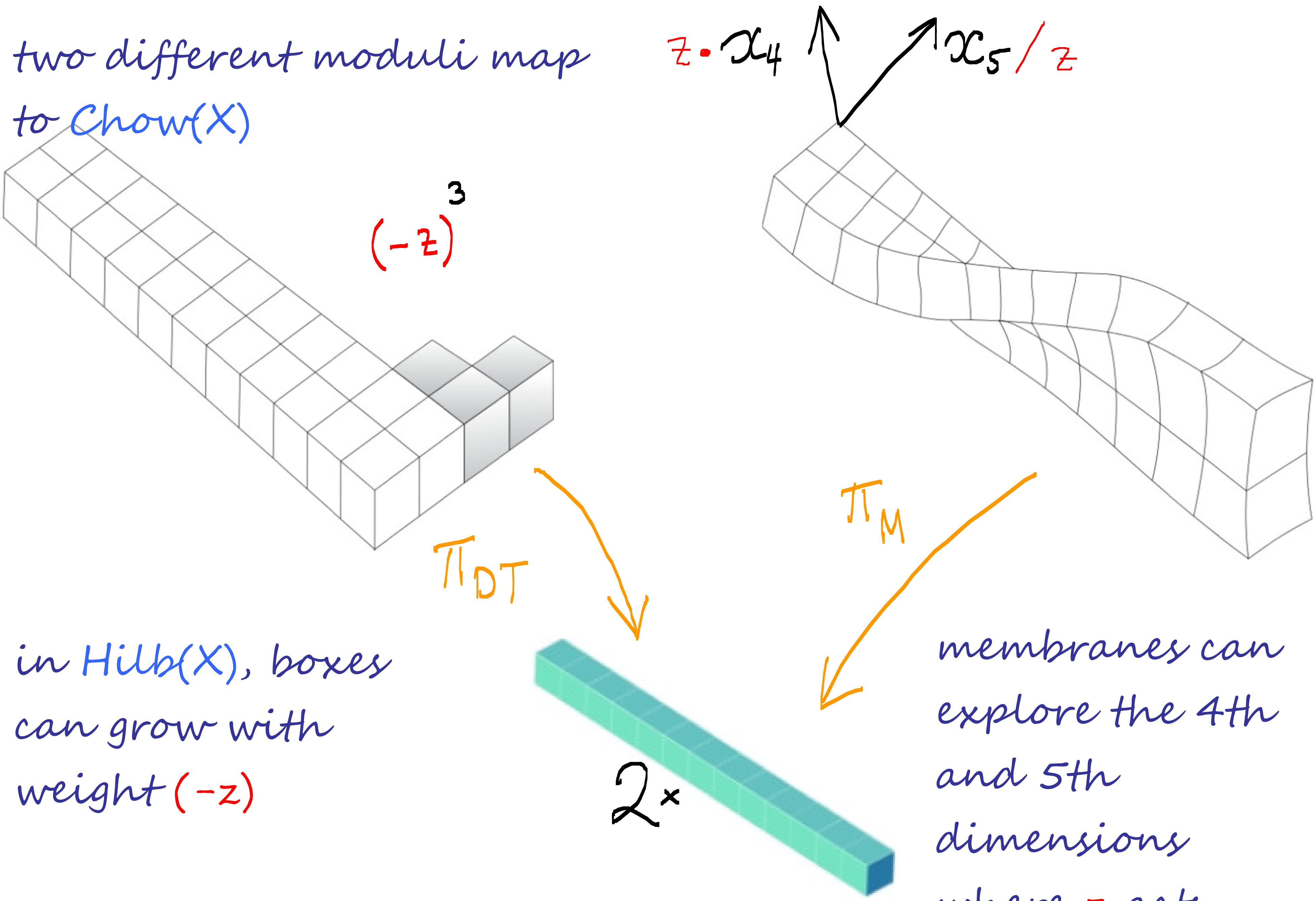
$$\begin{tikzcd} Z & = & \underset{z}{\mathcal{L}_1 \circlearrowleft} \oplus \mathcal{L}_2 \overset{z^{-1}}{\circlearrowright} \arrow[d] \\ & & X \end{tikzcd}$$

$$\mathcal{L}_1 \otimes \mathcal{L}_2 = K_X$$

Membranes in Z are, roughly, immersed 1-dimensional schemes satisfying a certain stability condition.



This stability condition prohibits unnecessary boxes and makes the moduli spaces of membranes bounded for fixed degree



two different moduli map to Chow(X)

in Hilb(X), boxes can grow with weight (-z)

membranes can explore the 4th and 5th dimensions where z acts

Conjecture [NO], approximate form

$$\mathbb{J}_{DT,*}\left(\text{DT index}\right) = S^{\cdot}_{\text{Chow}}\ \mathbb{J}_{M,*}\left(\begin{matrix}\text{membrane}\\ \text{index}\end{matrix}\right)$$

semigroup

rational function of z

predicts many striking identities, including identities ("dualities") between very different DT indices, because different X may sit in the same Z

the general strategy for proving these sort of statements is, in my experience, to

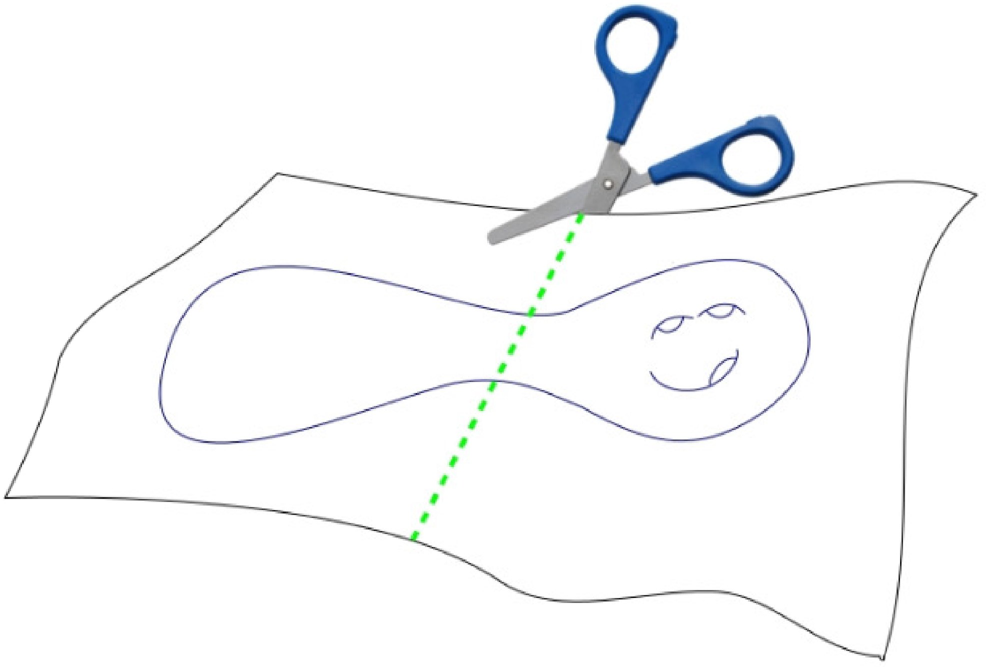
- use formal properties to reduce to special cases
- in which one can determine each side separately

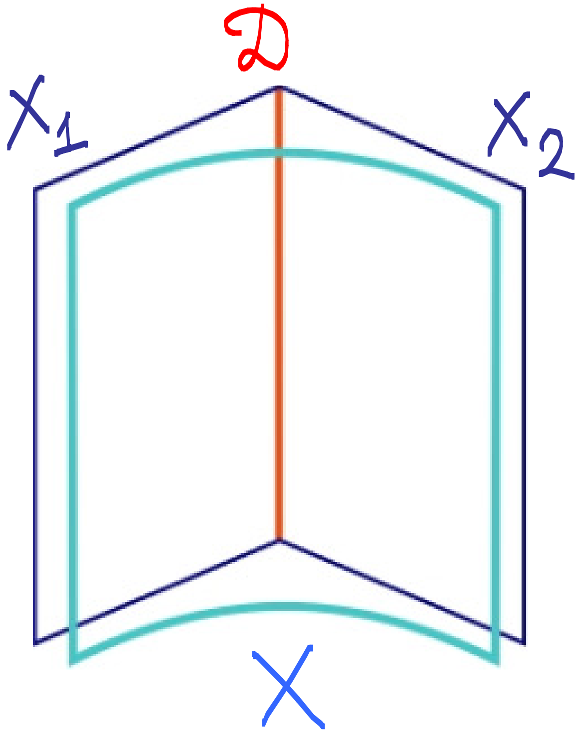
the proof thus comes with an algorithm, which may be quite involved, to compute either side. In particular, one can see dualities without proving the whole conjecture

formal properties bring us to a discussion of ....

IV. DT Theory of 3-folds

By definition, manifolds are glued from pieces and one way to understand manifolds is to cut them back into those pieces. This does not work so well with curves which never fit into affine charts.



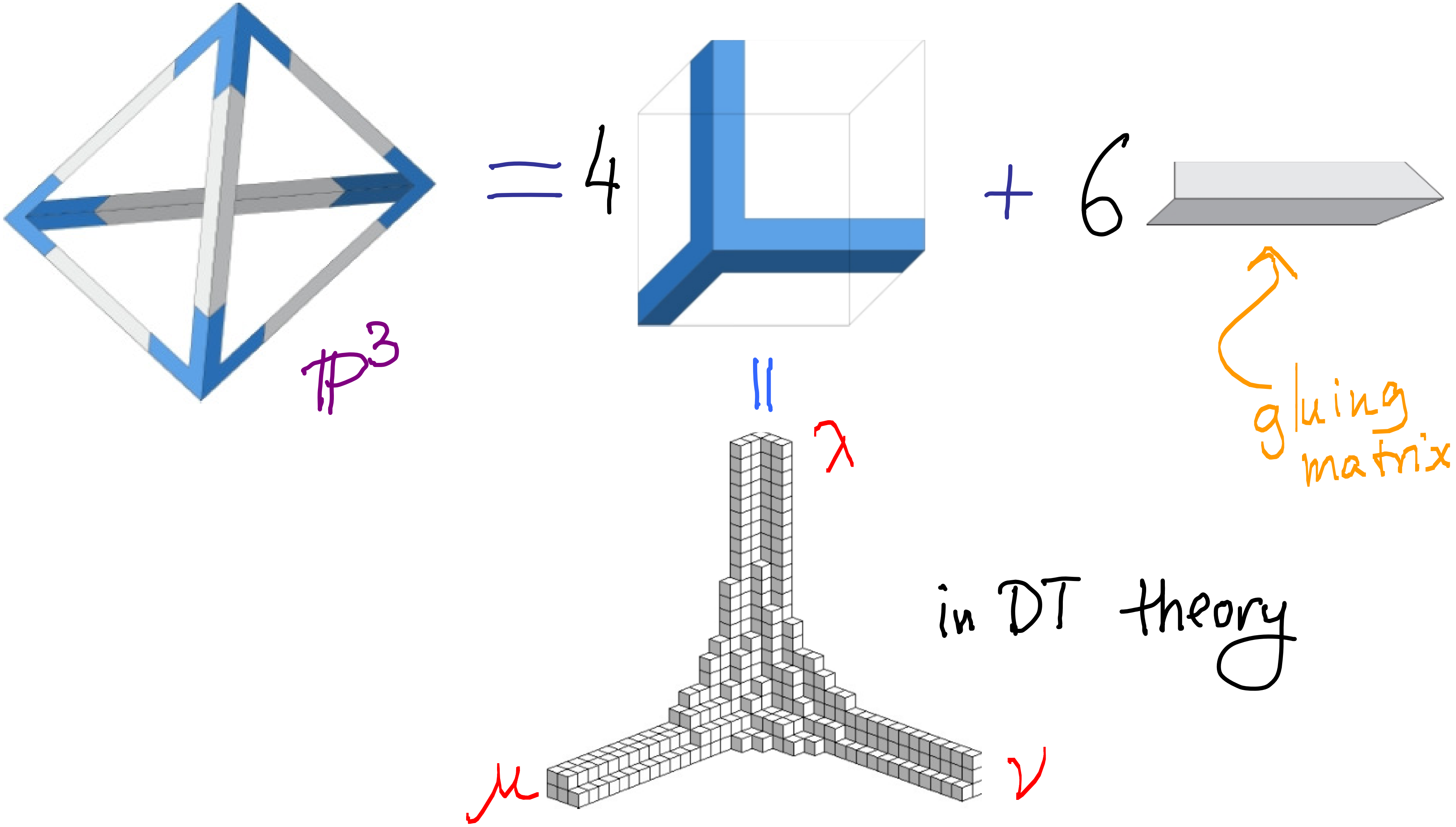


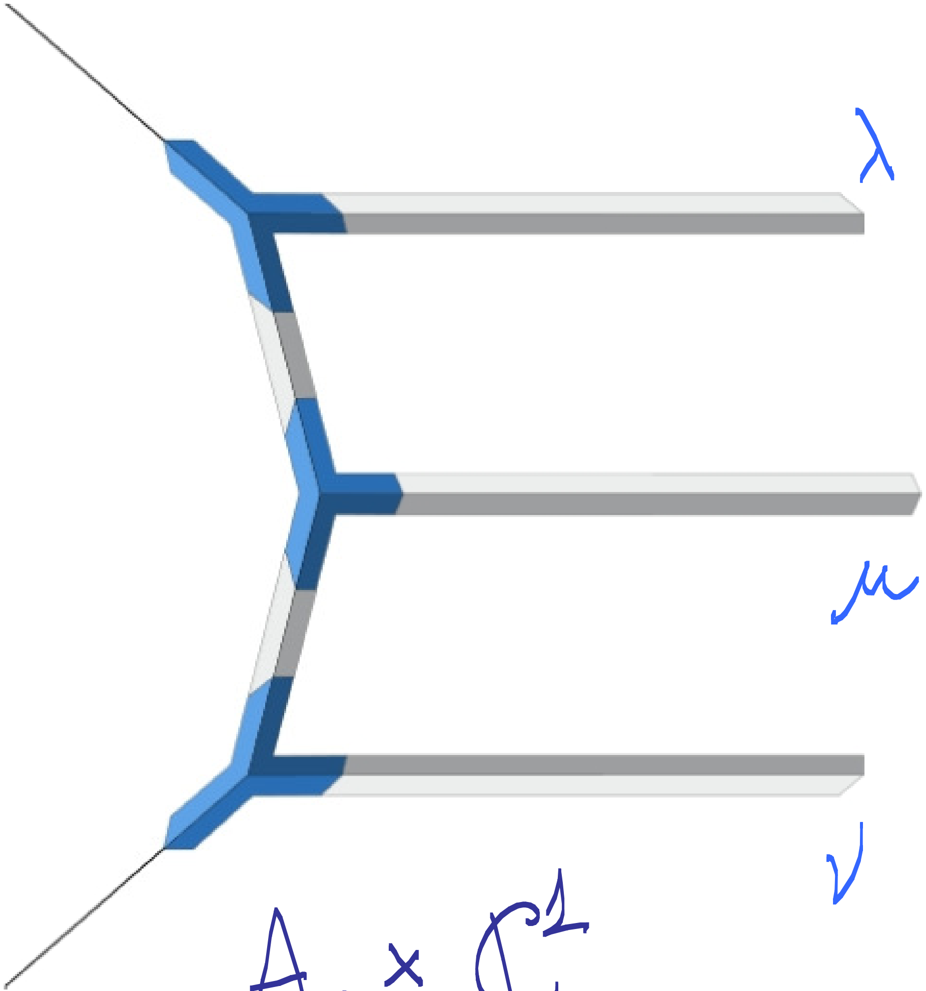
a good way to break up a variety X is to degenerate it to a union of two smooth varieties X_1 and X_2 along a smooth divisor D

In their work on MNOP conjecture, Pandharipande and Pixton have perfected the art of reconstructing curve counts in X from those in X_1 and X_2. Their work is in cohomology, but its key elements may be lifted to K-theory

These moves reduce, in a cerain sense, all possible 3-folds to toric 3-folds

Localization allows to break up toric 3-folds into further pieces - vertices & edges, which are certain tensors with 3 & 2 partition labels, respectively

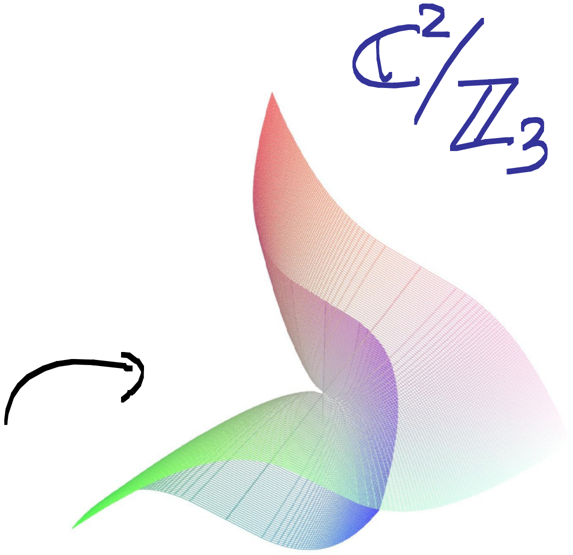




$A_2 \times \mathbb{C}^1$

Equivalently, one can work with $(n+1)$-valent tensors that appear when X is a bundle of A_n-surfaces over a curve.

A_2 surface singularity



These tensors may be characterized in terms of

$$\mathcal{U}_q\left(\widehat{\widehat{\mathfrak{gl}(n)}}\right) \qquad q \approx t_2 t_3$$

A very interesting quantum group!

We arrived at the end of the title of these lectures ...