Problem set 7

Due Wednesday, March 22

Recall from class that the Kac-Moody Lie algebra corresponding to the Cartan matrix

$$C = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

is related to the algebra of Laurent polynomials

$$g(t) \in \mathfrak{sl}_3 \otimes \mathbb{C}[t^{\pm 1}]$$

such that

$$g(-t) = -g(t)^T .$$

Here T denotes the transposition, it is convenient to transpose along the other diagonal of a matrix.

1. Show that the denominator formula for this Lie algebra is equivalent to the quintuple product identity

$$\prod_{n\geq 1}(1-q^n)(1-q^{n-1}z)(1-q^n/z)(1-q^{2n-1}z^2)(1-q^{2n-1}/z^2) = \\ = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(3n^2+n)}(z^{3n}-z^{3n-1}) .$$

2. Deduce Euler's pentagonal number theorem

$$\prod_{n\geq 1}(1-q^n) = \sum_n (-1)^n q^{\frac{1}{2}(3n^2-n)/2}$$

from the triple product identity. What does it say about the number $p(n)$ of a natural number n ?