Problem set 6

Due Wednesday, March 8

For $G = GL(n)$, Weyl character formula gives symmetric Laurent polynomials

$$s_\lambda(x_1, \dots, x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)}{\prod_{i<j}(x_i - x_j)}$$

that go back to at least Jacobi and are called *Schur functions*. Here

$$\lambda = (\lambda_1 \geq \dots \geq \lambda_n) \in \mathbb{Z}^n$$

is a highest weights of $GL(n)$. Since

$$s_{\lambda+(1,\dots,1)}(x) = \left(\prod x_i\right) s_\lambda(x)$$

we may assume $\lambda_n \geq 0$ without loss of generality. In this case, λ is called a partition.

1. Show that

$$s_{(\lambda_1,\dots,\lambda_n,0)}(x_1, \dots, x_n, 0) = s_\lambda(x_1, \dots, x_n)\,,$$

and that Schur functions form a $\mathbb{Z}$-basis in the ring

$$\Lambda = \varprojlim \mathbb{Z}[x_1, \dots, x_n]^{S(n)}$$

an element of which is a sequence of symmetric polynomials $f_n(x_1, \dots, x_n)$ of bounded degree defined for $n \gg 0$ such that

$$f_{n+1}(x_1, \dots, x_n, 0) = f_n(x_1, \dots, x_n)\,.$$

2. If

$$\lambda = (\underbrace{1, 1, \dots, 1}_{k})$$

then $s_\lambda(x)$ is the elementary symmetric function e_k. What is the representation-theoretic meaning of this ? Similarly, if

$$\lambda = (k, 0, 0, \dots)$$

then $s_\lambda(x)$ is the sum of all monomials of degree k, normally called the complete homogeneous symmetric function and denoted h_k.

3. Show that $e_1, e_2, \ldots$ are free commutative generators of Λ. Similarly for $h_1, h_2, \ldots$.

4. Clear denominators to find a formula for the expansion of $e_k\, s_\lambda$ in Schur functions.

5. Similarly, clear denominators to find a formula for the expansion of $p_k\, s_\lambda$ in Schur functions, where

$$p_k = \sum x_i^k\,.$$