Problem set 4

Due Wednesday, February 22

1. Let $\{e_\alpha^{(i)}\}$ and $\{e_{-\alpha}^{(i)}\}$ be dual bases of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, that is, suppose

$$(e_\alpha^{(i)}, e_{-\alpha}^{(j)}) = \delta_{ij},$$

with respect to the invariant symmetric form on a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$. Show that

$$\sum_i e_{-\alpha}^{(i)} \otimes \left[x, e_\alpha^{(i)}\right] = \sum_i \left[x, e_{-\beta}^{(i)}\right] \otimes e_\beta^{(i)}$$

for any $x \in \mathfrak{g}_{\beta-\alpha}$.

2. In the notation of Problem 1, show that

$$\sum_i \left[e_{-\alpha}^{(i)}, \left[x, e_\alpha^{(i)}\right]\right] = \sum_i \left[\left[x, e_{-\beta}^{(i)}\right], e_\beta^{(i)}\right] \in \mathfrak{g}_{\beta-\alpha}$$

and, similarly,

$$\sum_i e_{-\alpha}^{(i)} \left[x, e_\alpha^{(i)}\right] = \sum_i \left[x, e_{-\beta}^{(i)}\right] e_\beta^{(i)} \in \mathcal{U}(\mathfrak{g})_{\beta-\alpha}\,.$$

3. Let M be a $\mathfrak{g}_\alpha$-module such that for any $m \in M$,

$$\mathfrak{g}_\alpha\, m = 0$$

for all but finitely many positive α. (For example, if $\mathfrak{h}$-action on M is diagonalizable and all weights are bounded from above.) Show that the operator

$$\Omega' = 2 \sum_{\alpha>0,i} e_{-\alpha}^{(i)} e_\alpha^{(i)} + \sum_i e_0^{(i)} e_0^{(i)}$$

is well-defined on M and commutes with $\mathfrak{h}$.

4. Let M be as above and let $\rho \in \mathfrak{h}$ be such that

$$(\rho, h_i) = \frac{1}{2} a_{ii}\,.$$

Show that the operator

$$\Omega = \rho + \Omega'$$

commutes with the action of $\mathfrak{g}$.

5. Let M be a Verma module for $\mathfrak{g}$, that is a module freely generated by a vector $|\lambda\rangle$, $\lambda \in \mathfrak{h}^*$, such that

$$h|\lambda\rangle = \lambda(h)|\lambda\rangle$$

and

$$e_i|\lambda\rangle = 0\,.$$

Show that Ω acts by a scalar operator in M and compute that scalar.