Problem set 11

Due Wednesday, April 19

Let X be a smooth complex projective variety and let $\Omega^k = \Lambda^k T^* X$ be the kth exterior power of the cotangent bundle of X. By Hodge theory,

$$H^q(X, \Omega^p) \subset H^{p+q}(X, \mathbb{C}),$$

and therefore any connected algebraic group G acting on X acts trivially on $H^q(X, \Omega^p)$. In the following problems, we explore this and similar phenomena from the point of view of localization formulas.

1. Compute

$$\chi(X, \Omega^k) \in K_G(\mathrm{pt})$$

by localization and show that the trace of this virtual representation is a bounded function on G. Conclude that it is a multiple of the trivial representation of G.

2. Suppose that $G = \mathbb{C}^\times \ni z$ and that G acts on X with isolated fixed points $\{p_i\}$. Compute the asymptotics of $z \to 0$ in the localization formula from problem 1 and interpret the results in terms of the attracting manifolds

$$\mathrm{Attr}(p_i) = \{x \in X, \lim_{z \to 0} z \cdot x = p_i\}$$

of the fixed points.

3. Generalize to the case when fixed points are not isolated.

4. Suppose that the action of $G = \mathbb{C}^\times$ is nontrivial and suppose that a fractional power $\mathcal{K}^s$, $0 < s < 1$, of the canonical bundle $\mathcal{K}$ of X exists in $\mathrm{Pic}(X)$. Show that

$$\chi(X, \mathcal{K}^s) = 0.$$

What does this say for projective spaces ?

5. Compute the canonical bundle of $X = G/P$, where P is a parabolic subgroup.