Integral of a rational function

1 Intro

Most calculus textbooks present this topic as the method of partial fractions. They usually contain a bunch of apparently unrelated examples (sometimes not exhaustive) and do not focus on the purpose of this method. For this reason I have decided to write a few notes about “partial fractions” and I want to focus on what the method is good for and not on examples. Everything that follows is devoted to a “proof” (with numerous and fundamental omissions) of the following statement:

**Proposition 1.1.** Every rational function admits a primitive that can be expressed using elementary functions.

This result is proved by presenting a very explicit algorithm for the construction of such a primitive. Finding the primitive of a generic function is a mystery, but the primitive of a rational function is the output of an algorithm!!! Having an algorithm means that we have some kind of machine; we can put a function in it, turn the crank and the output is the result. This specific algorithm has a few drawbacks: first of all it’s very time consuming, integrals that can be solved in a few steps by some other method may require pages of calculations, finally it relies on the knowledge of a factorization of the denominator, but such a factorization might not be so easy to find. In synthesis:

If you can reduce an integral to the integral of a rational function you have solved the integral (just turn the crank).

If you can choose between this method and another one, choose the other one.

2 Factorization of a polynomial over the real numbers

**Definition 2.1.** A polynomial $P(x)$ is a function of this kind:

$$P(x) = \sum_{i=0}^{n} a_i x^i$$

If we assume that $a_n$ is not zero than $n$ is called the degree of the polynomial.

**Definition 2.2.** A function $f(x)$ is called rational if it can be written as a quotient of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

The problem that we would like to consider is the following: *is it possible to write a polynomial $P(x)$ as the product of polynomials of lower degree?* Ideally we would like to write $P(x)$ as the
product of polynomials of the lowest degree possible\(^1\); if the degree of \(P(x)\) is \(n\), we would like to express \(P(x)\) as the product of \(n\) polynomials of degree 1:

\[
P(x) = \prod_{i=0}^{n} (c_i x + d_i)
\]

Unfortunately there are examples of polynomials that cannot be written in this way:

\[
x^2 + 1 = x^2 - (-1) = x^2 - (\sqrt{-1})^2 = (x + \sqrt{-1})(x - \sqrt{-1})
\]

We can write this polynomial as the product of two linear factors only if we have a number \(i\) such that \(i^2 = -1\), but no real number satisfies this condition. In general every degree two polynomial \(ax^2 + bx + c\) whose discriminant \(b^2 - 4ac\) is negative cannot be written as the product of two linear factors. Even if we cannot write a polynomial as the product of linear factors we can still say something about its factorization:

**Proposition 2.3.** Every polynomial \(P(x)\) of degree \(n\) can be written as the product of linear and quadratic factors:

\[
P(x) = \left( \prod_{i=0}^{l} (c_i x + d_i) \right) \left( \prod_{j=0}^{m} (e_j x^2 + f_j x + g_j) \right)
\]

(1)

where \(n = l + 2m\) and \(c_i, d_i, e_j, f_j, g_j\) are real numbers.

The proof of this statement requires some knowledge of the complex numbers and for this reason it’s omitted. This proposition tells us that there is a set of numbers \(c_i, d_i, e_j, f_j, g_j\) such that \(P(x)\) is factorized like in equation (1); unfortunately it doesn’t say how we can find these numbers.

If the polynomial has degree two we have a formula to find its roots:

\[
x_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

if the polynomial has degree three there are more complicated formulas due to Cardano to find the factorization; if the polynomial has degree four there are much more complicated formulas due to Galois to find the factorization of the polynomial; if the polynomial has degree 5 or higher, Galois proved that there is no general formula to express the roots of \(P(x)\) using the four operations and radicals. This is the main weak spot of the algorithm that we are going to present: even if a factorization is always possible, we might not be able to find it exactly\(^2\). Even when it’s possible to find an exact factorization it might not be easy. Consider for instance the polynomial \(x^4 + 1\). It doesn’t have any root, still the proposition tells us that we can write it as the product of two quadratic polynomials. Since we don’t know the complex numbers we have to use some trick:

\[
x^4 + 1 = x^4 + 2x^2 + 1 - 2x^2 = (x^2 + 1)^2 - 2x^2 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)
\]

Another option is to make an ansatz. We know from proposition 2.3 that we can write the polynomial in the following way:

\[
x^4 + 1 = (x^2 + bx + c)(x^2 + dx + e)
\]

\(^1\)Remember that the degree of the product of two polynomials is the sum of the degrees

\(^2\)It’s always possible and actually easy to find an approximated factorization at an arbitrarily high precision.
We can compare the two sides and we obtain the following non-linear system of four equations in four unknowns:

\[
\begin{align*}
    ce &= 1 \\
    d + b &= 0 \\
    c + e + db &= 0 \\
    be + cd &= 0
\end{align*}
\]

We don’t know a general method to solve non-linear systems, however we can use equation one to find \( c = 1 \) and equation two to find \( b = -d \) and plug these into equations three and four. After some more algebra we find that \( c = 1 \) and \( d = \pm \sqrt{2} \) which is the solution we found with the previous calculation.

### 3 Partial fractions

The first part of the algorithm consists in writing a given rational function in a way that we can easily integrate.

**Step 1:** Let \( P(x) \) be a rational function. If the degree of \( P \) is greater or equal than the degree of \( Q \) we perform the long division of polynomials. Using Euclid’s algorithm we can find polynomials \( D \) and \( R \) such that:

\[
\frac{P(x)}{Q(x)} = D(x) + \frac{R(x)}{Q(x)}
\]

and the degree of \( R \) (the remainder) is strictly less than the degree of \( Q \).

\[
\int \frac{P(x)}{Q(x)} \, dx = \int D(x) \, dx + \int \frac{R(x)}{Q(x)} \, dx
\]

The function \( D(x) \) is a polynomial and we know how to integrate it. We still don’t know how to integrate \( \frac{R(x)}{Q(x)} \) but at least we know that the degree of \( R \) is strictly less than the degree of \( Q \). In the next step we will learn how to deal with this case.

**Step 2:** We need to factorize the denominator \( Q(x) \) as in proposition 2.3. To keep the notation as simple as possible we will deal with the linear factors and the quadratic factors separately.

**The linear case:** assume that \( Q(x) \) is the product of linear factors:

\[
Q(x) = \prod_{i=1}^{n} (a_i x + b_i)
\]

We collect \( a_i \) for each one of the linear factors and rewrite the factorization in this way:

\[
Q(x) = C \prod_{i=1}^{n} (x + b'_i)
\]

where \( C = \prod_{i=1}^{n} a_i \) and \( b'_i = \frac{b_i}{a_i} \). Notice that while \( b_i \) can be zero, \( a_i \) is certainly non-zero. At this point of the algorithm we notice that some of the linear factors might be repeated\(^3\), which means that some of the \( b'_i \) might be equal for different values of \( i \). We collect all the factors with the same \( b'_i \) and rewrite the factorization in the following way:

\[
Q(x) = C \prod_{i=1}^{m} (x + b'_i)^{d_i} \tag{2}
\]

\(^3\)If we don’t collect \( a_i \) it can happen that two pairs \((a_i, b_i)\) are different, and still the two linear factors they represent have the same root and should be regarded as equal.
and now we can assume that all the \( b'_i \) are distinct for all \( i = 1 \ldots m \), and \( m \) is the number of distinct \( b'_i \) and it satisfies the following condition: \( \sum_{i=1}^{m} d_i = n \). Now that the denominator is written in a suitable way, it is possible to prove with techniques of commutative algebra the following result:

**Proposition 3.1.** For all polynomials \( R, Q \) such that the degree of \( R \) is less than the degree of \( Q \), and \( Q \) can be factorized as in (2), there exist real numbers \( e_{ij} \), with \( 1 \leq i \leq m \) and \( 1 \leq j \leq d_i \) such that:

\[
\frac{R(x)}{Q(x)} = C \sum_{i=1}^{m} \sum_{j=1}^{d_i} \frac{e_{ij}}{(x + b'_i)^j} \tag{3}
\]

The quadratic case: assume that \( Q(x) \) is the product of irreducible quadratic factors only:

\[
Q(x) = \prod_{i=1}^{n} (a_i x^2 + b_i x + c_i)
\]

As in the linear case we collect \( a_i \) first:

\[
Q(x) = C \prod_{i=1}^{n} \left( x^2 + b'_i x + c'_i \right)
\]

and \( C \) is the product \( \prod_{i=1}^{n} a_i \) and \( b'_i = \frac{b_i}{a_i} \), \( c'_i = \frac{c_i}{a_i} \). Since the quadratic factors are irreducible they have a negative discriminant:

\[
b'_i^2 - 4c'_i < 0; \ \forall i
\]

Some of the factors might be repeated, which means that some of the ordered couples \( (b'_i, c'_i) \) might be equal for different values of \( i \). We collect the repeated factors and write the polynomial in the following way:

\[
Q(x) = C \prod_{i=1}^{m} \left( x^2 + b'_i x + c'_i \right)^{d_i} \tag{4}
\]

and now we can assume that all the ordered couples \( (b'_i, c'_i) \) are distinct for \( i = 1, \ldots, m \), and \( m \) is the number of distinct factors. As in the linear case the following statement holds:

**Proposition 3.2.** For all polynomials \( R, Q \) such that the degree of \( R \) is less than the degree of \( Q \), and \( Q \) can be factorized as in (4), there exist ordered pairs of real numbers \( (e_{ij}, f_{ij}) \) with \( 1 \leq i \leq m \) and \( 1 \leq j \leq d_i \) such that:

\[
\frac{R(x)}{Q(x)} = C \sum_{i=1}^{m} \sum_{j=1}^{d_i} \frac{e_{ij} x + f_{ij}}{(x^2 + b'_i x + c'_i)^j} \tag{5}
\]

The general case: in general we can only assume that \( Q(x) \) can be factorized as in (1) which is a mixture of linear and quadratic terms or if you prefer \( Q = Q_{\text{lin}} Q_{\text{quad}} \) where \( Q_{\text{lin}} \) is the product of all the linear factors and \( Q_{\text{quad}} \) is the product of all the quadratic factors. The final formula for \( \frac{R(x)}{Q(x)} \) is a summation where the contribution of the factors in \( Q_{\text{lin}} \) is like in formula (3) and the contribution from the quadratic factors in \( Q_{\text{quad}} \) is like in formula (5). In the next section we will see that every term in this summation can be explicitly integrated.
4 Integration

To complete the proof of proposition 1.1 we only have to prove that each summand in (3) and in (5) can be integrated.

The linear case: we have to integrate the function \( \frac{1}{(x+b)^j} \) where \( j \geq 1 \). This case is particularly easy and the result is the following:

\[
\int \frac{1}{(x+b)^j} \, dx = \begin{cases} 
\ln |x+b| & \text{if } j = 1 \\
\frac{1}{1-j} \left( \frac{1}{(x+b)^{j-1}} \right) & \text{if } j > 1
\end{cases}
\]

The quadratic case: we have to integrate the function \( \frac{e^{x+f}}{(x^2+bx+c)^j} \) where \( j \geq 1 \) and \( b^2 - 4c < 0 \).

The first step is to complete the square in the denominator:

\[
(x^2 + bx + c) = \left( x^2 + bx + \frac{b^2}{4} - \frac{b^2}{4} + c - \frac{b^2}{4} \right) = \left( \left( x + \frac{b}{2} \right)^2 + c - \frac{b^2}{4} \right)
\]

Notice that \( b^2 - 4c < 0 \) implies \( c - \frac{b^2}{4} > 0 \), for this reason we can rewrite the integral as:

\[
\int \frac{e^{x+f}}{(x^2+bx+c)^j} \, dx
\]

where \( m^2 = c - \frac{b^2}{4} \). After a shift we can rewrite the integral in this way:

\[
\int \frac{ny+l}{(y^2+m^2)^j} \, dy = n \int \frac{y}{(y^2+m^2)^j} \, dy + l \int \frac{1}{(y^2+m^2)^j} \, dy
\]

We need to treat the two summands separately. The first one is easy and can be solved with a substitution \( u = y^2 \):

\[
\int \frac{y}{(y^2+m^2)^j} \, dy = \frac{1}{2} \int \frac{d(y^2+m^2)}{(y^2+m^2)^j} = \begin{cases} 
\frac{1}{2} \ln (y^2+m^2) & \text{if } j = 1 \\
\frac{1}{2(1-j)} \left( \frac{1}{(y^2+m^2)^{j-1}} \right) & \text{if } j > 1
\end{cases}
\]

The second one is slightly trickier and we distinguish two further cases. If \( j = 1 \) the integral is just a rescaled arc-tangent:

\[
\int \frac{1}{y^2+m^2} \, dy = \frac{m}{m^2} \int \frac{d\left( \frac{y}{m} \right)}{\left( \frac{y}{m} \right)^2 + 1} = \frac{1}{m} \arctan \left( \frac{y}{m} \right)
\]

If \( j > 1 \) we need the trig. substitution \( y = m \tan \theta \):

\[
\int \frac{1}{(y^2+m^2)^j} \, dy = \frac{m}{m^{2j}} \int \frac{\sec^2 \theta \, d\theta}{\sec^{2j} \theta} = \frac{1}{m^{2j-1}} \int \frac{d\theta}{\sec^{2(j-1)} \theta}
\]

Since \( j - 1 > 0 \) we can rewrite the integral in this way:

\[
\frac{1}{m^{2j-1}} \int \cos^{2(j-1)} \theta \, d\theta
\]

The integral of a power of cosine can be solved recursively by parts:

\[
\int \cos^{2(j-1)} \theta \, d\theta = \frac{1}{2(j-1)} \cos^{2j-3} \theta \sin \theta + \frac{2j-3}{2(j-1)} \int \cos^{2(j-2)} \theta \, d\theta
\]
Equivalently we can use the substitution \( y = m \sinh \theta \) which produces the following result:

\[
\frac{1}{m^{2j-1}} \int \frac{\cosh \theta}{\cosh^{2j} \theta} d\theta = \frac{1}{m^{2j-1}} \int \text{sech}^{2j-1} \theta \ d\theta
\]

A power of hyperbolic secant can be solved recursively by parts just like cosine:

\[
\int \text{sech}^{2j-1} \theta \ d\theta = \frac{1}{2(j - 1)} \text{sech}^{2j-3} \theta \tanh \theta + \frac{2j - 3}{2(j - 1)} \int \text{sech}^{2j-3} \theta \ d\theta
\]