Comparison theorem for improper integrals
(with explicit examples)

1 Intro

This is a complement to the comparison theorem for improper integrals in the textbook. The vanilla version presented in the textbook is good enough to solve some very easy examples and it becomes exponentially gory with the complexity of the integral. Fortunately it is not hard to refine the statement in the book, and turn it into a powerful tool to estimate the convergence of arbitrarily complicated integrals.

2 Comparison (vanilla version)

In words: suppose that a continuous positive function \( h(x) \) is smaller than another continuous function \( f(x) \) in a neighborhood of \( +\infty \); if the integral of the bigger function is convergent, then the integral of the smaller function is convergent; vice versa, if the smaller function has a divergent integral then the integral of the bigger function must diverge too. More precisely:

**Proposition 2.1.** Let \( f(x) \) be a non negative continuous function on \( [a, +\infty) \). Let \( h(x) \) be another continuous function on the same interval such that \( f(x) \geq h(x) \) for all \( x \geq a \). The following implications hold:

\[
\int_{a}^{+\infty} f(x) \, dx < +\infty \Rightarrow \int_{a}^{+\infty} h(x) \, dx < +\infty
\]

and

\[
\int_{a}^{+\infty} h(x) \, dx = +\infty \Rightarrow \int_{a}^{+\infty} f(x) \, dx = +\infty
\]

and an analogous statement holds for the interval \((-\infty, a]\) and for negative functions.

A similar statement holds in the neighborhood of a vertical asymptote.

3 Comparison (enhanced version)

The idea that we try to implement is the following: the improper integral of a power function, an exponential, or a logarithm are very easy to study; we would like to reduce any given function \( f(x) \) to one of these functions or a product of them by comparison. This can be made precise with the definition of limit.

Consider the example from the book:

\[
\int_{1}^{+\infty} \frac{x + 1}{\sqrt{x^4 - x}}
\]
To determine if the integral converges or not we need to study the behavior of the function at infinity and for \( x \to 1 \) where it has an asymptote. To study the behavior at 1 we rewrite the function in the following way:

\[
\frac{x + 1}{\sqrt{x^4 - x}} = \frac{x + 1}{\sqrt{x(x - 1)(x^2 + x + 1)}} = \frac{x + 1}{\sqrt{x^2 + x + 1}} \cdot \frac{1}{\sqrt{x - 1}}
\]

In this writing we have separated the part of this function responsible for the divergence \( \frac{1}{\sqrt{x - 1}} \) from everything else. Using limits:

\[
\lim_{x \to 1^+} \frac{x + 1}{\sqrt{x^2 + x + 1}} = \frac{2}{\sqrt{3}}
\]

\[
\lim_{x \to 1^+} \frac{1}{\sqrt{x - 1}} = +\infty
\]

The enhanced version of the comparison theorem will tell us that in order study the integral in a neighborhood of 1 we can ignore everything that converges to a number or more precisely:

\[
\int_1^{1+\epsilon} \frac{x + 1}{\sqrt{x^4 - x}} \, dx < +\infty \Leftrightarrow \int_1^{1+\epsilon} \frac{1}{\sqrt{x - 1}} \, dx < +\infty
\]

where \( \epsilon \) is some positive number.

We can repeat the same procedure at infinity, but in this case we want to isolate the part of the function which is responsible for the convergence to zero at infinity:

\[
\frac{x + 1}{\sqrt{x^4 - x}} = \frac{x(1 + \frac{1}{x})}{x^2(\sqrt{1 - \frac{1}{x^2}})} = \frac{1 + \frac{1}{x}}{x} \cdot \frac{1}{\sqrt{1 - \frac{1}{x^2}}}
\]

As in the previous case we can make this precise using limits:

\[
\lim_{x \to +\infty} \frac{1 + \frac{1}{x}}{\sqrt{1 - \frac{1}{x^2}}} = 1
\]

\[
\lim_{x \to +\infty} \frac{1}{x} = 0
\]

This pair of limits tells us that the function goes to zero like \( \frac{1}{x} \) and with the comparison theorem we will be able to conclude that:

\[
\int_1^{+\infty} \frac{x + 1}{\sqrt{x^4 - x}} \, dx < +\infty \Leftrightarrow \int_1^{+\infty} \frac{1}{x} \, dx < +\infty
\]

In this example the integral converges in a neighborhood of 1 but it diverges at infinity, so it is divergent.

**Proposition 3.1.** Let \( f(x) \) be a continuous function on \([a, \infty)\); suppose also that \( f(x) \geq 0 \) for all \( x \geq a \) and that it can be written as the product of two non-negative functions \( f(x) = g(x)h(x) \). Suppose that the behavior of \( g \) and \( h \) at infinity is prescribed by the following limits:

\[
\begin{align*}
&\lim_{x \to +\infty} h(x) = 0 \\
&\lim_{x \to +\infty} g(x) = L \neq 0 ; \quad L \in \mathbb{R}
\end{align*}
\]
then we say that \( f(x) \) is approximated by \( h(x) \) at infinity (in symbols \( f(x) \approx_{x \to \infty} h(x) \)) and we can conclude that:
\[
\int_{a}^{+\infty} f(x) \, dx < +\infty \iff \int_{a}^{\infty} h(x) \, dx < +\infty
\]
An analogous statement holds for \((−\infty, a]\) and for negative functions.

**Proposition 3.2.** Let \( f(x) \) be a continuous function on \((a, b]\); suppose also that \( f(x) \geq 0 \) for all \( x \in (a, b] \) and that it can be written as the product of two non-negative functions \( f(x) = g(x)h(x) \). Suppose that the behavior of \( g \) and \( h \) at \( a \) is prescribed by the following limits:
\[
\begin{align*}
\lim_{x \to a^+} h(x) \, dx & = +\infty \\
\lim_{x \to a^+} g(x) \, dx & = L \neq 0 ; \quad L \in \mathbb{R}
\end{align*}
\]
then we say that \( f(x) \) is approximated by \( h(x) \) at \( a \) (in symbols \( f(x) \approx_{x \to a^+} h(x) \)) and we can conclude that:
\[
\int_{a}^{b} f(x) \, dx < +\infty \iff \int_{a}^{b} h(x) \, dx < +\infty
\]
An analogous statement holds for a left neighborhood of \( a \) and for negative functions.

(proof of prop. 3.1). The proof of 3.2 is more or less the same and it is left to the reader. Since \( L \) is strictly positive (if \( L = 0 \) the statement is false!) we can find a positive number \( \epsilon \) such that \( L - \epsilon > 0 \). According to the definition of limit, there exists a number \( M(\epsilon) \) such that \( \forall x > M(\epsilon) \) the following inequality is satisfied:
\[
0 < L - \epsilon < g(x) < L + \epsilon
\]
We can multiply the inequality by the non negative function \( h(x) \) and we have the following:
\[
0 \leq (L - \epsilon)h(x) \leq g(x)h(x) \leq (L + \epsilon)h(x)
\]
Denote with \( M_a \) the maximum between \( a \) and \( M(\epsilon) \). For all \( x \geq M_a \) the previous inequality is satisfied and according to the comparison theorem 2.1 we have:
\[
(L + \epsilon) \int_{M_a}^{+\infty} h(x) \, dx < +\infty \Rightarrow \int_{M_a}^{+\infty} g(x)h(x) \, dx < +\infty
\]
and also:
\[
(L - \epsilon) \int_{M_a}^{+\infty} h(x) \, dx = +\infty \Rightarrow \int_{M_a}^{+\infty} g(x)h(x) \, dx = +\infty
\]
Notice that if \( L - \epsilon \) were negative the last implication wouldn’t hold! To complete the proof we just observe that the function is continuous on \([a, M_a]\).

**4 Explicit examples**

**Example 0:** Determine for which real numbers \( \alpha \) the following integral is finite:
\[
\int_{0}^{\pi} \frac{\sin^3 x}{x^\alpha} \, dx
\]
First of all I would like to remind you that the limit of \(x^\alpha\) can be solved as follows:

\[
\lim_{x \to 0^+} x^\alpha = \begin{cases} 
0 & \text{if } \alpha > 0 \\
1 & \text{if } \alpha = 0 \\
+\infty & \text{if } \alpha < 0 
\end{cases}
\]

The integrand is not continuous at zero where the numerator is zero and the denominator can be 1, 0 or \(\infty\). We need to compare \(\sin x\) with a power function; in order to do this we recall the fundamental limit:

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1
\]

In order to use the proposition we multiply and divide the integrand by \(x^3\):

\[
\frac{x^3 \sin^3 x}{x^3 x^\alpha}
\]

Now we notice that:

\[
\lim_{x \to 0} \frac{\sin^3 x}{x^3} = 1 \Rightarrow \int_0^\pi \frac{\sin^3 x}{x^\alpha} \, dx \approx \int_0^\pi x^{3-\alpha} \, dx
\]

This last integral is convergent if and only if \(4 - \alpha > 0 \iff \alpha < 4\)

**Example 1:** Determine for which real numbers \(\alpha\) the following integral is finite:

\[
\int_0^\infty \frac{1 - \cos x}{\sqrt{x^\alpha + x^6}} \, dx
\]

The function can have a vertical asymptote at 0 and we have to split the calculation in two pieces:

\[
\int_0^{+\infty} = \int_0^\epsilon + \int_\epsilon^{+\infty} = I_0 + I_\infty
\]

We start with \(I_0\). To determine the behavior at 0 we have to compare \(x^\alpha\) with \(x^6\), so it is convenient to distinguish two cases: \(\alpha < 6\) and \(\alpha \geq 6\). In the first case we collect \(x^\alpha\).

\[
\frac{1 - \cos x}{\sqrt{x^\alpha + x^6}} = \frac{1 - \cos x}{x^{\frac{\alpha}{2}} \sqrt{1 + x^{6-\alpha}}} \approx \frac{1 - \cos x}{x^{\frac{\alpha}{2}}}
\]

Observe that the numerator goes to zero as \(x\) goes to zero, so we need a way to compare it with \(x^{\frac{\alpha}{2}}\). In order to do this we use the fundamental limit:

\[
\lim_{x \to +0} \frac{1 - \cos x}{x^2} = \frac{1}{2}
\]

The limit tells us that \(1 - \cos x \approx \frac{x^2}{2}\). By comparison we can replace the function with a simpler one:

\[
\frac{1 - \cos x}{x^\frac{\alpha}{2}} \approx \frac{x^2}{x^\frac{\alpha}{2}} = x^{2-\frac{\alpha}{2}}
\]

and the integral of this last function converges when \(\alpha < 6\). In the second case \(\alpha \geq 6\) and we collect \(x^6\):

\[
\frac{1 - \cos x}{\sqrt{x^\alpha + x^6}} = \frac{1 - \cos x}{x^3 \sqrt{x^\alpha - 6} + 1} \approx \frac{1 - \cos x}{x^3}
\]
Using the same approximation for cosine we obtain:

\[
\frac{1 - \cos x}{x^3} \approx \frac{x^2}{x^3} = x^{-1}
\]

and the integral of this function is divergent. Putting the two cases together we conclude that the integral \(I_0\) converges for \(\alpha < 6\). Now we study \(I_\infty\). First of all we observe that \(\cos x\) is a bounded function:

\[
0 \leq 1 - \cos x \leq 2
\]

For this reason we can approximate in this way:

\[
\frac{1 - \cos x}{\sqrt{x^\alpha + x^6}} \leq \frac{2}{\sqrt{x^\alpha + x^6}}
\]

Notice that the approximation from below is not good since we should multiply everything by zero. We still need to compare \(x^\alpha\) with \(x^6\). If \(\alpha < 6\):

\[
\frac{2}{\sqrt{x^\alpha + x^6}} = \frac{2}{x^3 \sqrt{\frac{1}{x^\alpha} + 1}} \approx \frac{1}{x^3}
\]

and the integral of this function is convergent. If \(\alpha \geq 6\):

\[
\frac{2}{\sqrt{x^\alpha + x^6}} = \frac{2}{x^{\frac{3\alpha}{2}} \sqrt{\frac{1}{1 + \frac{1}{x^\alpha - 6}}} \approx \frac{2}{x^{\frac{3\alpha}{2}}}
\]

and this is convergent for \(\alpha > 2\). The integral \(I_\infty\) converges for every value of \(\alpha\). Since both \(I_0\) and \(I_\infty\) must converge the solution is the intersection \(\alpha < 6\).

**Example 3:** Determine if the following integral is finite:

\[
\int_{-\infty}^{+\infty} \frac{e^{3x} + 3e^{2x}}{e^{4x} - 2e^{2x} + 1} \, dx
\]

First of all we notice that this function has a primitive that can be calculated explicitly. If we substitute \(e^x\) with \(y\) the integrand becomes a rational function and it can be integrated by partial fractions. However the calculation might be very very long and tedious. Since we are only interested in the convergence of the integral we can use the comparison theorem. Both numerator and denominator are polynomials in \(e^x\) and it might be convenient to find their factors:

\[
\frac{e^{3x} + 3e^{2x}}{e^{4x} - 2e^{2x} + 1} = \frac{e^{2x}(e^x + 3)}{(e^x - 1)^2(e^x + 1)^2}
\]

We need to study the behavior of the function at \(+\infty\) but also at \(x = 0\) where \((e^x - 1)\) becomes zero. We split the integral in three pieces:

\[
\int_{-\infty}^{+\infty} = \int_{-\infty}^{-\epsilon} + \int_{-\epsilon}^{\epsilon} + \int_{\epsilon}^{+\infty} = I_{-\infty} + I_0 + I_{+\infty}
\]

We start with \(I_\infty\). To study the behavior at infinity we collect the biggest power of \(e^x\) from both the numerator and the denominator:

\[
\frac{e^{3x} + 3e^{2x}}{e^{4x} - 2e^{2x} + 1} \approx \frac{1}{e^x}
\]
and this last function is certainly integrable between $\epsilon$ and $+\infty$. To study the behavior at $-\infty$ we collect the smallest power of $e^x$ from both numerator and denominator:

$$
\frac{e^{3x} + 3e^{2x}}{e^{4x} - 2e^{2x} + 1} = 2 \frac{(e^x + 3)}{e^{2x} - 2e^{2x} + 1} \approx e^{2x}
$$

and this last function is integrable from $-\infty$ to $-\epsilon$. To study the behavior in a neighborhood of 0 we need to compare $e^x - 1$ with a power function; in order to do it we use the fundamental limit:

$$
\lim_{x \to 0} \frac{e^x - 1}{x} = 1
$$

Using this limit we have the following approximation:

$$
\frac{e^{2x}(e^x + 3)}{(e^x - 1)^2(e^x + 1)^2} \approx \frac{e^{2x}(e^x + 3)}{x^2(e^x + 1)^2} \approx \frac{1}{x^2}
$$

and the integral of this last function is certainly divergent in a neighborhood of 0. Putting the three cases together we can conclude that the integral is divergent.

**Example 4:** Determine for which real numbers $\alpha$ the following integral is finite:

$$
\int_{0}^{+\infty} \left( \sqrt{1 + x^2 + x^\alpha} - \sqrt{1 + x^2} \right) x^{-\frac{3}{2}} \, dx
$$

The function might have an asymptote at 0 and it is continuous on $(0, +\infty)$. We have to split the integral in two pieces:

$$
\int_{0}^{+\infty} = \int_{0}^{\epsilon} + \int_{\epsilon}^{+\infty} = I_0 + I_\infty
$$

We start with the vertical asymptote at 0. It is convenient to distinguish two cases: $\alpha \geq 0$ and $\alpha < 0$.

If $\alpha \geq 0$ the integrand is a 0/0 in a neighborhood of zero and because of this, it is convenient to rationalize it:

$$
\left( \sqrt{1 + x^2 + x^\alpha} - \sqrt{1 + x^2} \right) x^{-\frac{3}{2}} = \frac{x^{\frac{\alpha - 3}{2}}}{\sqrt{1 + x^2 + x^\alpha + \sqrt{1 + x^2}}}
$$

The behavior of the function at zero is now apparent:

$$
\lim_{x \to 0^+} \frac{1}{\sqrt{1 + x^2 + x^\alpha + \sqrt{1 + x^2}}} = \begin{cases} 
\frac{1}{2} & \text{if } \alpha > 0 \\
\frac{1}{\sqrt{2+1}} & \text{if } \alpha = 0 
\end{cases}
$$

By comparison $I_0 < \infty$ if and only if:

$$
\int_{0}^{\epsilon} x^{\frac{\alpha - 3}{2}} \, dx < +\infty
$$

and this is true if $\alpha - \frac{3}{2} + 1 > 0 \iff \alpha > \frac{1}{2}$.

If $\alpha < 0$ we have to collect it from the denominator since $x^\alpha$ becomes divergent:

$$
\frac{x^{\frac{\alpha - 3}{2}}}{\sqrt{1 + x^2 + x^\alpha + \sqrt{1 + x^2}}} = \frac{x^{\frac{\alpha - 3}{2}}}{\sqrt{x^{-\alpha} + x^{2-\alpha} + 1 + x^{-\frac{3}{2}}} \sqrt{1 + x^2}}
$$
The behavior at zero is determined by the following limit:

\[ \lim_{x \to 0^+} \frac{1}{\sqrt{x^{-\alpha} + x^{2-\alpha} + 1 + x^{-\frac{\alpha}{2}} \sqrt{1 + x^2}} = 1} \]

The function behaves like \( x^{\frac{\alpha}{2} - \frac{3}{2}} \) and by comparison:

\[ I_0 < +\infty \iff \int_0^\epsilon x^{\frac{\alpha}{2} - \frac{3}{2}} \, dx < +\infty \]

and this is fulfilled for \( \frac{\alpha}{2} - \frac{3}{2} + 1 > 0 \iff \alpha > 1 \). The integral \( I_0 \) doesn’t converge for negative values of \( \alpha \).

Now we study \( I_\infty \). In this case we notice that in order to determine the magnitude of the denominator we have to compare \( x^\alpha \) with \( x^2 \). For this reason we distinguish two cases: \( \alpha \leq 2 \) and \( \alpha > 2 \). In the first case we collect \( x^2 \) from the denominator and we have:

\[ \frac{x^{\alpha - \frac{3}{2}}}{\sqrt{1 + x^2 + x^\alpha + \sqrt{1 + x^2}}} = \frac{x^{\alpha - \frac{3}{2}}}{\sqrt{x^{-2} + 1 + x^{2-\alpha} + \sqrt{x^{-2} + 1}}} \]

The asymptotic behavior is determined by the following limit:

\[ \lim_{x \to +\infty} \frac{1}{\sqrt{x^{-\alpha} + x^{2-\alpha} + 1 + x^{-\frac{\alpha}{2}} \sqrt{x^{-2} + 1}}} = \frac{1}{2} \]

and \( I_\infty \) is approximated by:

\[ \int_\epsilon^{+\infty} x^{\alpha - \frac{3}{2}} \, dx \]

which is convergent when \( \alpha - \frac{5}{2} + 1 < 0 \iff \alpha < \frac{3}{2} < 2 \). If \( \alpha > 2 \) we collect \( x^\alpha \) instead:

\[ \frac{x^{\alpha - \frac{3}{2}}}{\sqrt{1 + x^2 + x^\alpha + \sqrt{1 + x^2}}} = \frac{x^{\frac{\alpha}{2} - \frac{3}{2}}}{\sqrt{x^{-2} + 1 + x^{2-\alpha} + \sqrt{x^{-2} + 1}}} \]

and the asymptotic behavior is determined by:

\[ \lim_{x \to +\infty} \frac{1}{\sqrt{x^{-\alpha} + x^{2-\alpha} + 1 + x^{2-\alpha} + x^{-\alpha}}} = 1 \]

The integral \( I_\infty \) is approximated by:

\[ \int_\epsilon^{+\infty} x^{\frac{\alpha}{2} - \frac{3}{2}} \, dx \]

which is convergent for \( \frac{\alpha}{2} - \frac{3}{2} + 1 < 0 \iff \alpha < 1 \); but since \( \alpha > 2 \) we don’t have any new solution.

Since we want both \( I_0 \) and \( I_\infty \) to converge, the solution is the intersection of the two solutions: \( \frac{1}{2} < \alpha < \frac{3}{2} \).