Calculus III, section 7/8.
2nd Midterm 4/6/2010

Solve as many exercises as you can. Exercises marked with $\heartsuit$ are slightly more difficult and meant for people who are running for an A+; I suggest you to solve the standard part of the test first.

1) Find and sketch the domain of the following two functions:
   i) $\sqrt{2 - y \ln (y^2 - x^2)}$
   ii) $\ln (2 - x^2 - y^2 - 2z^2)$

2) Reduce the following equation to the standard form and classify the surface (if you don’t remember the name you can help yourself with a graph):
   
   $$-x^2 - y^2 + 2z^2 + 4x = 7$$

   Find the tangent plane and the normal line at the point $(1, 2, -2)$
   (I suggest you to use implicit differentiation).

3) Let $C$ be the curve given by parametric equations:

   $$\mathbf{r}(t) = \begin{cases} 
   e^{-2t} \cos t \\
   e^{-2t} \sin t \\
   3e^{-2t} 
   \end{cases}$$

   i) Calculate tangent line and normal plane at the point $t = 0$.
   ii) The curve $C$ lies on a cone; find its equation.
   iii) Calculate the limit:

   $$\lim_{t \to +\infty} \mathbf{r}(t)$$

   $\heartsuit$) Calculate the length of the arc between the point $(0, 0, 0)$ and the point $(1, 0, 3)$ (Hint: if you have understood the spirit of the previous question you probably know how to do this).
5) Calculate two of the following limits (you can do all of them if you have enough time):

i) \[ \lim_{{(x,y) \to (0,0)}} \frac{x^3 - y^3}{x - y} \]

ii) \[ \lim_{{(x,y) \to (0,0)}} \frac{x^2 y^3}{2x^4 + y^6} \]

iii) \[ \lim_{{(x,y) \to (0,0)}} \frac{x^4 + 2y^4}{x^2 + y^2} \]

b) \[ \lim_{{(x,y) \to (1,1)}} \frac{x - y}{2 - x^2 - y^2} \]

6) An archer wants to hit a target located 100 m from him and at the same height from the ground as his bow. Knowing that he shoots the arrow at a velocity of 100 m/s, which angle of elevation should he use to hit the target? What is the maximal height reached by the arrow (with respect to the bow)? Solve the problem by solving the equation \( F = ma \), you are not allowed to use higher level formulas! You can approximate \( g \) with \( 10 \text{ m/s}^2 \).

7) Calculate first order partial derivatives of the following functions:

i) \( \ln \left( x + \sqrt{x^3 + xy^2} \right) \)

ii) \( \frac{xyz}{x^2 + y^2 + z^2} \)

iii) Let \( f(x, y) \) be a differentiable function with continuous second order partial derivatives. Let \( x = s \cos t \) and \( y = s \sin t \). Calculate \( \frac{\partial f}{\partial t} \) and \( \frac{\partial^2 f}{\partial t^2} \).

b) Study the family of level curves of the following function of two variables:

\[ z = \frac{x - y}{2 + x^2 + y^2} \]

Are there lines on this surface? What is the range of the function?