MODERN ALGEBRA 1 HOMEWORK 01, DUE WED JAN 29

Homework is due before 1pm each Wednesday in the Modern Algebra 1 homework box on the 4th floor of the Math Building.

About 50% of homeworks will include optional problems labelled "extra credit." Extra credit problems must be handed in directly to me, in class or by email, and NOT put in the homework box. The (rare) grade of A+ is for exceptional work, and cannot be earned without some extra credit work.

If $f\colon A \to B$ is a map and $A' \subseteq A$ one writes $f(A')$ for the set

$$f(A') := \{b \in B : b = f(a) \text{ for some } a \in A'\}\,.$$

It is called the *image of* A' *under* f. The set $f(A)$ is simply called the *image of* f.

If $B' \subseteq B$ one writes $f^{-1}(B')$ for the set

$$f^{-1}(B') := \{a \in A : f(a) \in B'\}$$

(note that f^{-1} itself is not a map unless f is bijective—the notation $f^{-1}(B')$ should be seen as a whole). $f^{-1}(B')$ is called the *inverse image of* B' *by* f.

For $b \in B$ one usually abbreviates $f^{-1}(\{b\})$ as $f^{-1}(b)$. It is often called the *fiber of* f *over* b.

(1) For the map $f\colon \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ what are the following:
   (a) $f(2)$
   (b) $f(\{2\})$
   (c) $f(\{-1, 0, 1\})$
   (d) $f^{-1}(1)$
   (e) $f^{-1}(0)$
   (f) $f^{-1}(-1)$
   (g) $f^{-1}(\{0, 1, 2\})$
(2) For each of the following sets, say how many elements it has:
   $\emptyset$, $\{\emptyset\}$, $\{1, 2, 3, 4, \{\{5, 6\}, 7, 8\}\}$, $\{\{1, 2, 3\}, \{3, 2, 1\}\}$.
(3) Prove that if $f\colon A \to B$ and $g\colon B \to C$ and $g\circ f$ is surjective, then g is surjective. ($g\circ f$ is the *composition* of the maps f and g, defined by $g\circ f(a) := g(f(a))$.)
(4) Prove that if $f\colon A \to B$ and $g\colon B \to C$ and $g\circ f$ is injective, then f is injective.
(5) Recall that the power set $\mathcal{P}(A)$ is the set of subsets of of A, i.e., $\mathcal{P}(A) = \{B : B \subseteq A\}$. Let $P(A)$ be the set of all maps $f\colon A \to \{0, 1\}$. Show that the map $b\colon P(A) \to \mathcal{P}(A)$ defined by $b(f) = f^{-1}(1)$ is bijective.
(6) (Extra credit) Recall that $|A| \le |B|$ is defined to mean that there exists an injective map $A \rightarrowtail B$. Prove carefully that this is also equivalent to the existence of a surjective map $B \twoheadrightarrow A$, so long as $A \neq \emptyset$. (Your proof should make clear that it uses the "Axiom of choice", which you can google, but we will probably also say what it is on Monday.)