MODERN ANALYSIS HOMEWORK 1

HAND IN ON TUESDAY JANUARY 29 IN CLASS.

Notation: Recall that if $A \subset X$ then the *complement of* A *in* X is defined as

$$X \setminus A := \{x \in X : x \notin A\},$$

and we also use the notation A' for this if it is clear what set X is meant.

A fairly obvious notation that we didn't introduce in class is: $A \supset B$ means the same thing as $B \subset A$ (B is a subset of A).

Some notes on logic. The symbol $\Rightarrow$ means "implies" and $\Leftarrow$ means "is implied by." So if A and B are two statements then $A \Rightarrow B$ means "A implies B" or "if A is true then B is true".

Contrapositives. The sentences "$A \Rightarrow B$" and "(not B) $\Rightarrow$ (not A)" are called *contrapositives* of each other but they mean exactly the same thing. If this is not clear to you, think about it a bit. For example "If I'm hot I sweat" and "if I don't sweat I'm not hot" are different ways of saying the same thing.

If you are asked to prove a statement of the form "$A \Rightarrow B$" it is sometimes easier to prove the contrapositive "(not B) $\Rightarrow$ (not A)". (This is also sometimes called "proof by contradiction" since you can think of it as trying to show that if you assume the conclusion B is not true, then this contradicts the assumtion that A is true.)

(1) Let X be a set, I be some index set, and $A_i \subset X$ for each $i \in I$. For $A \subset X$, we use the shorthand A' for the complement of A in X. Prove "DeMorgan's laws":
  (a) $\left(\bigcup_{i \in I} A_i\right)' = \bigcap_{i \in I} A'_I$
  (b) $\left(\bigcap_{i \in I} A_i\right)' = \bigcup_{i \in I} A'_I$

(2) Prove that if X is a finite set then $\#(\mathcal{S}(X)) = 2^{\#X}$. (Recall $\mathcal{S}(X)$ is the set of all subsets of X and $\#X$ means the size of X, i.e., the number of elements in X.

(3) Recall that if $f\colon X \to Y$ is a map and $B \subset Y$ then $f^{-1}B := \{x \in X : f(x) \in B\}$. Prove:
  (a) $A \subset f^{-1}fA$ for any $A \subset X$.
  (b) $ff^{-1}B \subset B$ for any $B \subset Y$.
  (c) Give examples to show that neither of the above inclusions has to be an equality. (I.e. Find sets $X, Y, A \subset X$ and a map $f\colon X \to Y$ with $A \neq f^{-1}fA$ and sets $X, Y, B \subset Y$ and a map $f\colon X \to Y$ with $ff^{-1}B \neq B$.)

(4) For a map $f\colon X \to Y$ prove
  (a) f is injective $\Leftrightarrow \forall A \subset X\ fA' \subset (fA)'$.
  (b) f is surjective $\Leftrightarrow \forall A \subset X\ fA' \supset (fA)'$.
  (Here, for a subset $A \subset X$ A' is short for the the complement $X \setminus A$

while for a subset $B \subset Y$ B' means the complement $Y \setminus B$. Also, remember that the notation $\forall$ is short for "for all.")

Hint: In each case split the proof into two parts: the implication from left to right, and the implication from right to left. In both cases, I recommend proving the implication from right to left by proving its contrapositive (see above). So, for example, the contrapositive of the $\Leftarrow$ part of part (a) is:

$$f \text{ is not injective} \Rightarrow \text{it is not true that } fA' \subset (fA)' \text{ for every } A \subset X.$$

To prove this you assume f is not injective and try to use this fact to find a subset $A \subset X$ for which $fA' \not\subset (fA)'$.