RIGIDITY OF COXETER GROUPS AND ARTIN GROUPS

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Abstract. A Coxeter group is rigid if it cannot be defined by two nonisomorphic diagrams. There have been a number of recent results showing that various classes of Coxeter groups are rigid, and a particularly interesting example of a nonrigid Coxeter group has been given in [17]. We show that this example belongs to a general operation of "diagram twisting". We show that the Coxeter groups defined by twisted diagrams are isomorphic, and, moreover, that the Artin groups they define are also isomorphic, thus answering a question posed by Charney. Finally, we show a number of Coxeter groups are reflection rigid once twisting is taken into account.

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A Coxeter group W is rigid if it cannot be defined by two nonisomorphic Coxeter diagrams and strongly rigid if any two Coxeter systems for the group are conjugate. There have been a number of recent results showing that various classes of Coxeter groups are strongly rigid or rigid ([9], [16], [19]), and a particularly interesting example of a nonrigid Coxeter group has been given in [17]. We show that Mühlherr's example belongs to a general operation of "diagram twisting". We show that the Coxeter groups defined by twisted diagrams are isomorphic, and, moreover, that the Artin groups they define are also isomorphic, thus answering a question posed by Charney

[1]Supported under NSF grant no. DMS-9996342
[2]Supported under NSF grant no. DMS-9971682
[3]Supported under NSF grant no. DMS-0083097
Date: November 14, 2000.

in [2]. Finally, we show a number of Coxeter groups are reflection rigid once twisting is taken into account.

Section 1 contains the basic definitions and Section 2 reviews the recent results on rigidity. In Section 3 we discuss the weaker notion of reflection rigidity. Then in Section 4 we present our main construction. Section 5 shows how these modifications are the only ones possible for tree diagrams and Section 6 gives an example where deciding what twisting is possible can be difficult. Section 7 shows that reflection rigidity results always extend from Coxeter groups to Artin groups. The final section, Section 8, presents two natural conjectures about the extent to which general Coxeter groups and Artin groups are rigid.

1. Basic definitions

Coxeter groups and Artin groups are defined by presentations which can be concisely summarized in diagram form. The question of rigidity for these groups in essence asks when these defining diagrams can be recovered from the groups themselves. In this section we review the relations between the Coxeter and Artin groups, their generating sets, and the diagrams used to define them.

Definition 1.1 (Diagrams)**.** A *diagram* is an undirected graph Γ without loops or multiple edges with a map $\text{EDGES}(\Gamma) \to \{2,3,4,\dots\}$ which assigns an integer greater than 1 to each of its edges. Since such diagrams are used to define Artin groups and Coxeter groups they are often called *Artin diagrams* or *Coxeter diagrams.*

Remark 1.2 (Other conventions)**.** The reader should note that the Dynkin diagrams traditionally drawn to summarize the presentations of the finite Coxeter groups use a different convention about which edges to include in the diagram. For a finite Coxeter group every pair of generators satisfies a nontrivial relation and the diagram defined above would always be a complete graph. To simplify the picture, Dynkin diagrams do not draw the edges labeled 2, i.e. the ones which indicate a commuting pairs of generators. We will not use this convention.

Definition 1.3 (Artin groups)**.** Let $(a,b)^j$ denote the alternating string of a's and b's of length j, starting with a. Thus, $(a,b)^2 = ab$, $(a,b)^3 = aba$, and $(b,a)^3 = bab$. Let Γ be a diagram and let S be its vertex set. The *Artin group defined by* Γ is the group A_Γ generated by S subject to the relations: $(s,t)^j = (t,s)^j$ whenever s and t are vertices in Γ connected by an edge labeled j. More generally, a group A with generating set S is called an *Artin group* with *Artin generators* S if there exists a diagram Γ with vertex set S such that $A = A_\Gamma$. Example 1.5 shows that not every generating set of an Artin group is a set of Artin generators. If S is a set of Artin generators for A then the pair (A,S) is called an *Artin system.*

Remark 1.4 (Recovering diagrams from Artin systems)**.** The process of defining an Artin system from a diagram can be reversed. Let A be an arbitrary group generated by a set S. A diagram $\Gamma = \Gamma_S$ can be defined from the pair (A, S) as follows. Let the vertices of Γ correspond to the elements of S, and draw an edge between distinct s and t in S if and only if there is an integer $j \geq 1$ such that $(s,t)^j = (t,s)^j$ in A. The set of j for which this relation holds is the set of multiples of the smallest such j; we label the edge by this smallest j. There is clearly a natural map from the Artin group defined by this diagram onto the original group A. This map is an isomorphism if and only if the original group A was an Artin group with Artin generators S, and in this case the construction recovers the diagram for the group. As a result, the Artin system (A, S) and the diagram Γ contain essentially the same information about the group A. In fact, the procedure recovers the diagram even after adding the relations $s^2 = 1$ for $s \in S$ to turn A into a Coxeter group. See Remark 1.7 below.

Example 1.5 (Non-Artin generating sets)**.** Let Γ be the trivial graph with a single vertex and no edges. The Artin group it defines is isomorphic to the integers. The set $S = \{2, 3\}$ is also a set of generators for the integers. The diagram that S determines is a graph with two vertices connected by an edge labeled 2 (since 2 and 3 commute in $\mathbb{Z}$), and the Artin group defined by this diagram is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Since $\mathbb{Z} \oplus \mathbb{Z}$ maps onto $\mathbb{Z}$ but is not isomorphic to it, the set S is a set of generators for $\mathbb{Z}$ which is not a set of Artin generators.

The situation for Coxeter groups is very similar.

Definition 1.6 (Coxeter groups)**.** Let Γ be a diagram and let S be its vertex set. The *Coxeter group defined by* Γ is the group W_Γ obtained as a quotient of A_Γ by adding the relations $s^2 = 1$ for each $s \in S$. Notice that in the Coxeter group W_Γ, the relation $(s,t)^j = (t,s)^j$ can be rewritten as $(st)^j = 1$, but in the Artin group A_Γ, it cannot.

The other definitions are similar. A group W is called a *Coxeter group* with *Coxeter generators* S if there exists a diagram Γ with vertex set S such that $W = W_\Gamma$. As is the case for Artin groups, not every generating set is a set of Coxeter generators. If S is a set of Coxeter generators for W then the pair (W, S) is called an *Coxeter system.*

Remark 1.7 (Recovering diagrams from Coxeter systems)**.** The procedure for recovering diagrams from Coxeter systems is nearly identical to that described in Remark 1.4. Let W be an arbitrary group generated by a set of involutions S. A diagram $\Gamma = \Gamma_S$ can be defined from the pair (W, S) as follows. Let the vertices of Γ correspond to the elements of S, and draw an edge between distinct s and t in S if and only if the element st is of finite order in W. The label of such an edge is the order of the element st. There is clearly a natural map from the Coxeter group defined by this diagram onto the original group W. This map is an isomorphism if and only if the original group W was a Coxeter group with Coxeter generators S, and in this case the construction recovers the diagram for the group. The reason for this is that the order of an element st in a Coxeter group W_Γ is indeed the label of the corresponding edge of Γ. More generally, the Coxeter group defined by any full subdiagram of Γ is a subgroup of W_Γ. See, e.g., [15, Theorem 8.2].

We will need the following basic facts about a Coxeter system (W, S). For each subset $J \subseteq S$, let W_J denote the subgroup of W generated by the elements in J. If W_J is finite, then J is called a *spherical subset* of S.

Theorem 1.8. *Let H be a finite subgroup of W. Then there is a spherical subset $J \subseteq S$ such that a conjugate of H is contained in W_J.*

Proof. See, for instance, section 4 of [3]. □

Theorem 1.9. *The conjugacy classes of the maximal finite subgroups of W are in one-to-one correspondence with the maximal spherical subsets of S.*

Proof. This theorem is probably well-known to the experts, since its proof is implicit in the literature (see for instance [7], [11] and [20]). The conceptually most insightful approach might be via the following observation.

Lemma 1.10. *Suppose a group W acts properly discontinuously on a space Σ. Let A and B be maximal finite subgroups of W. If the images in Σ/W of the fixed point sets Σ^A and Σ^B are not disjoint then they are equal and A is conjugate to B. Thus, if all finite subgroups have non-empty fixed point sets then maximal finite subgroups are classified by the images of their fixed sets in Σ/W.*

Indeed, if there is some point $x \in \Sigma^A$ with a translate $gx \in \Sigma^B$, then, since A and B are maximal finite, $A = G_x$ and $B = G_{gx}$ so $B = gAg^{-1}$ and $\Sigma^B = g\Sigma^A$.

This lemma can be applied to any one of several complexes that a Coxeter group W acts on. For instance, the Vinberg-Davis complex $\Sigma = \Sigma(W, S)$ (defined in [10]) is a building-like complex built out of chambers, each of which is a copy of the simplicial complex K_S obtained by taking the geometric realisation of the poset of spherical subsets of S. This K_S is also the quotient Σ/W. If $T \subset S$ is a spherical subset then the image of the fixed point set of W_T is the subcomplex determined by spherical subsets containing T. In particular, if we have a maximal finite subgroup, it is a conjugate of W_J for some maximal spherical $J \subset S$, and the image of its fixed set in K_S is the vertex determined by J. □

2. Prior results on rigidity

A Coxeter group W is *rigid* if any two Coxeter generating sets S and S' determine the same diagram, that is, the diagrams Γ_S and $\Gamma_{S'}$ are isomorphic. Equivalently, for any two Coxeter generating sets S and S', there is an automorphism $\rho : W \to W$ which carries S to S'. The Coxeter group W is *strongly rigid* if any two Coxeter generating sets for W are conjugate, that is, ρ can always be chosen inner. This implies that, up to inner automorphisms, any automorphism of W must come from an automorphism of the diagram Γ. Hence the natural map $\mathrm{Aut}(\Gamma) \to \mathrm{Aut}(W)$ induces a surjection $\mathrm{Aut}(\Gamma) \to \mathrm{Out}(W)$. Conversely, if W is rigid and the latter map is onto then W is strongly rigid.

An Artin group A is *rigid* if any two Artin generating sets for A determine the same diagram, and there is a similar reformulation in terms of automorphisms. The Artin group A is *strongly rigid* if for any two Artin generating sets S and S' the set S' is conjugate to either S or $S^{-1} := \{s^{-1} | s \in S\}$.

The main open question is the following:

Problem 2.1. *Which Coxeter groups and which Artin groups are rigid? Which are strongly rigid?*

There have been three recent positive results on the rigidity for Coxeter groups.

Theorem 2.2 (Radcliffe [19]). *Right-angled Coxeter groups are rigid.*

An Artin group or Coxeter group is called *right-angled* when all of the edges in the Coxeter diagram are labeled 2. Radcliffe has also extended his results to include Coxeter diagrams in which each edge label is either 2 or a multiple of 4. On the other hand, right-angled Coxeter groups in general are not strongly rigid. See Theorem 4.10.

Theorem 2.3 (Kaul [16]). *Let S and S' be Coxeter generating sets for a Coxeter group W. If (W, S) is of type K_n, then (W, S') is also of type K_n and the multiset of edge labels in the corresponding diagrams will be preserved. In particular, if (W, S) is of type K_n and all but one of the edge labels in Γ_S are identical, then W is rigid.*

A Coxeter system (W, S) is of *type* K_n if $|S| = n$, the diagram Γ_S is a complete graph on n vertices, and all of its edge labels are odd. A multiset is a collection in which the order of the entries does not matter, but multiplicities do. Thus the multisets $\{1, 1, 2\}$ and $\{1, 2, 2\}$ are different. For an extension of this result see Lemma 5.3.

Theorem 2.4 (Charney and Davis [9]). *If a Coxeter group is capable of acting effectively, properly, and cocompactly on some contractible manifold, then it is strongly rigid.*

The dihedral group of order 12, on the other hand, shows that not all Coxeter groups are rigid.

Example 2.5 ($\mathbf{D}_6$). The group $\mathbf{D}_6$ can be presented as a Coxeter group in two distinct ways as shown in Figure 1. The group $\mathbf{D}_6$ can also be viewed as the group of symmetries of a regular hexagon. The generators on the left in this view correspond to a pair of reflections. The generators on the right include the element which acts on the hexagon by a half-rotation. A similar

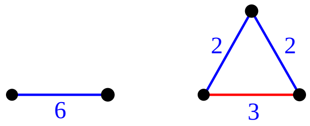

FIGURE 1. Two distinct Coxeter diagrams for $\mathbf{D}_6$

ambiguity in presentation exists for the dihedral groups $\mathbf{D}_k$ whenever k is twice an odd number.

Remark 2.6. Even though the Coxeter groups defined by the diagrams in Figure 1 are isomorphic, the Artin groups defined by these diagrams are distinct since they have distinct geometric dimensions. The geometric dimension of a group is the minimal dimension of an Eilenberg-MacLane space for the group. The Artin group defined by the diagram on the left has a 2-dimensional Eilenberg-MacLane space and it contains $\mathbb{Z} \oplus \mathbb{Z}$ (see, for instance, [4]). Thus it has geometric dimension 2 (see [8]). Similarly, the Artin group defined by the diagram on the right has a 3-dimensional Eilenberg-MacLane space and it contains $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ ([5]). Thus it has geometric dimension 3.

3. Reflection rigidity

As a result of Example 2.5, the rigidity question for Coxeter groups is sometimes modified to require that the members of the new generating set are reflections.

Definition 3.1 (Reflections)**.** In a Coxeter system (W, S) the conjugates of elements of S will be called *reflections* and the set of all reflections will be denoted R_S. Notice that being a reflection depends on the set S and not just on the group W. By analogy the conjugates of the elements S in an Artin system (A, S) will be called "reflections", even though these elements are not even involutions. The set of all reflections in (A, S) will again be denoted R_S.

The reflection terminology is derived from the geometric representation of W. More specifically, every Coxeter system (W, S) has a faithful representation as a set of linear transformations of a vector space V over $\mathbb{R}$, having a basis in one-to-one correspondence with S. The action of W on V will preserve a nondegenerate symmetric bilinear form, and the elements of S will act as reflections. A reflection in this context is an element of W which fixes a hyperplane of V and sends some nonzero vector to its negative. An element of W will act as a reflection in this sense if and only if it lies in R_S. See [15, Section 5.3] for details. The terminology for Artin groups is merely by analogy.

Definition 3.2 (Reflection rigidity)**.** Let (W, S) be a Coxeter system. If every Coxeter generating set S' contained in R_S determines the same diagram as (W, S), then the system (W, S) is called *reflection rigid*. Equivalently, reflection rigidity means that the Coxeter group W and the set of reflections R_S uniquely determines the Coxeter diagram. The equivalence of these two definitions will be shown in Lemma 3.7. As above, there is also the notion of *strong reflection rigidity*. We call a Coxeter system (W, S) *strongly reflection rigid* if for each set of Coxeter generators $S' \subseteq R_S$ there is an element of W which conjugates S' to S. Analogous definitions apply to Artin systems.

This leads to the modified version of Problem 2.1.

Problem 3.3. *Which Coxeter systems and which Artin systems are reflection rigid? Which are strongly reflection rigid?*

Remark 3.4 (Types of rigidity)**.** The relationships between the above concepts are summarized in Figure 2. All four types of rigidity are distinct. In Example 2.5, we noted that the dihedral group $\mathbf{D}_6$ is not rigid. With its standard 2-generator Coxeter presentation it is, however, strongly reflection rigid. The dihedral group $\mathbf{D}_5$, on the other hand, is rigid but not strongly reflection rigid.

$$\begin{array}{ccc} W \text{ is strongly rigid} & \Longrightarrow & (W,S) \text{ is strongly reflection rigid} \\ \Downarrow & & \Downarrow \\ W \text{ is rigid} & \Longrightarrow & (W,S) \text{ is reflection rigid} \end{array}$$

FIGURE 2. Types of rigidity

The remainder of the section will be devoted to preliminary observations on reflection rigidity. We begin by examining the conjugacy classes contained in R_S.

Definition 3.5 (Γ_{odd})**.** Let Γ be a diagram. The *odd part* of Γ is the subdiagram Γ_{odd} with the same vertex set but only containing those edges whose label is odd. This subdiagram is intimately related to the conjugacy classes of reflections. See Figure 3.

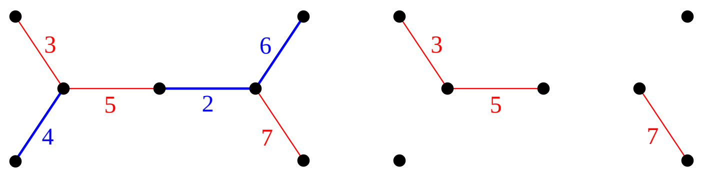

FIGURE 3. A diagram Γ and its odd part Γ_{odd}.

Lemma 3.6. *Let Γ be diagram and let (W, S) be the corresponding Coxeter system. The conjugacy classes in $R_S \subset W$ are in one-to-one correspondence with the connected components of Γ_{odd}. Similarly, if (A, S) is the corresponding Artin system, then the conjugacy classes in $R_S \subset A$ are also in one-to-one correspondence with the connected components of Γ_{odd}.*

Proof. In a dihedral group $\mathbf{D}_m$, m odd, the two standard generators are conjugate to each other. Thus, any pair of vertices in Γ connected by a path consisting of edges with odd labels will correspond to elements of W which are in the same conjugacy class. On the other hand, consider the presentation with generators S and relations of the form $s^2 = 1$ for each $s \in S$, $s = t$ for all pairs s and t in the same connected component of Γ_{odd}, and $st = ts$ for all other pairs s and t in S. Note that all of these relations must hold in the abelianization of W and that the original relations defining W are derivable from these relations. Thus, this is a presentation of the abelianization of W. Since elements from distinct connected components of Γ_{odd} are sent to distinct elements in the abelianization, they are not conjugate to each other in W. This completes the proof in the Coxeter case. The proof in the Artin case is identical except that the relations $s^2 = 1$ are not used. □

The next result uses Lemma 3.6 to show that the two definitions of reflection rigidity are equivalent.

Lemma 3.7. *If S and S' are two Coxeter generating sets for a Coxeter group W and $S' \subset R_S$, then $R_S = R'_S$. Similarly, if S and S' are two Artin generating sets for an Artin group A and $S' \subset R_S$, then $R_S = R'_S$.*

Proof. By Lemma 3.6 there is a one-to-one correspondence between the conjugacy classes in R_S and the the connected components of Γ_{odd}. In particular, if S' generates the Coxeter group W (or the Artin group A), then the image of S' must generate the abelianization of W (or A). In particular, S' must contain at least one reflection in each of the conjugacy classes which correspond to the connected components of Γ_{odd}. But if S' contains one such reflection, then $R_{S'}$ will contain the entire conjugacy class. This shows that $R_{S'}$ contains each of the conjugacy classes whose union is R_S, and this completes the proof. □

Theorem 3.8. *Let W be a Coxeter group and let S and S' be two Coxeter generating sets for W. If $R_S = R_{S'}$, then $|S| = |S'|$.*

Proof. This is an immediate consequence of a result of Dyer [13]. In [13] he proves that given a Coxeter system (W, S) and a subset S' of R_S, the subgroup generated by S' is a Coxeter group with a canonical generating set whose size is at most that of S'. Deodhar [12] also proved that such a subgroup is a Coxeter group, but the size estimate is harder to derive from his results. In the special case where S' generates all of W, Dyer's canonical generating set is simply S itself. Thus, $|S| \leq |S'|$. Since S' is also a Coxeter generating set and $R_S = R_{S'}$, the roles of S and S' can be reversed, thereby proving the opposite inequality. □

Theorem 3.9. *If Γ is a diagram in which every edge label is even, then its Coxeter system is reflection rigid.*

Proof. Let W be a Coxeter group, let S and S' be Coxeter generating sets for W with $R_S = R_{S'}$, and let Γ and Γ' be the the corresponding diagrams. Suppose in addition that every label in Γ is even. By Lemma 3.6 the elements in S (and the connected components of Γ'_{odd}) are in one-to-one correspondence with the conjugacy classes in R_S. On the other hand, by Theorem 3.8 $|S| = |S'|$. Thus each connected component of Γ'_{odd} must consist of a single vertex, every edge in Γ' must be even, and the vertex set of Γ' can be recovered from W and R_S alone.

Next, let s and t be vertices in S, let C and D be the conjugacy classes of W containing s and t, and let s' and t' be the unique vertices in S' which belong to C and D, respectively. If there are elements $c \in C$ and $d \in D$ which generate a finite dihedral group H, then by Theorem 1.8, H can be conjugated into one of the finite subgroups generated by a subset of S. Since conjugating c and d will leave these generators inside C and D, H can be conjugated inside the subgroup generated by s and t. Similarly, H can be conjugated inside the subgroup generated by s' and t'. This shows that s and t are connected by an edge if and only if s' and t' are connected by an edge and that when both edges exist the labels must be the same. Since the entire diagram Γ' can be recovered from W and R_S alone, the system (W, S) is reflection rigid. □

Theorem 3.10. *If Γ is the diagram of a finite Coxeter group, then Γ is reflection rigid.*

Proof. Let (W, S) Coxeter system and let R_S denote the set of reflections in W. Then the decomposition of W into irreducible (Coxeter)-factors can be recovered from the set R_S as follows: take the maximal partition of R_S such that any two reflections commute whenever they are contained in different elements of the partition; the groups generated by the elements of the partition are precisely the desired irreducible factors. It follows that if S' is another Coxeter generating set for W and $R_S = R_{S'}$ then there are isomorphisms between the irreducible factors. Hence the problem of reflection-rigidity is reduced to the irreducible case. So suppose (W, S) and (W, S') are both irreducible Coxeter systems for W, $|W|$ is finite, and $R_S = R_{S'}$. We know $|S| = |S'|$ by Theorem 3.8, and the claim now follows from the classification of the finite Coxeter groups. □

4. Diagram twisting

This section contains our main result, Theorem 4.5. Before stating and proving the theorem, we begin by illustrating its application.

Example 4.1. In [17] it is shown that the Coxeter groups defined by the diagrams in Figure 4 are isomorphic even though the diagrams themselves are not. Another example of the same phenomenon is given in Figure 5. Again, the Coxeter groups are isomorphic but the Coxeter diagrams are not. In each case, the top of the diagram has been "twisted" while the lower portion remains fixed.

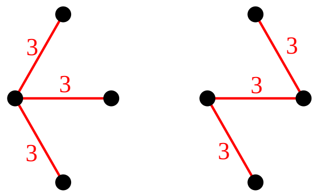

FIGURE 4. Diagrams for isomorphic Coxeter groups

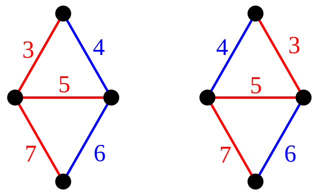

FIGURE 5. Additional diagrams for isomorphic Coxeter groups

In order to define twisting we will need a few more standard definitions.

Definition 4.2 (Γ_U and W_U)**.** Let Γ be a diagram with vertex set S. If U is a subset of S, let Γ_U be the full subdiagram on U, that is the diagram with vertex set U and all edges with both ends in U. This subdiagram defines a Coxeter group (actually a subgroup of the original Coxeter group) which we will denote by W_U.

Definition 4.3 (Longest element)**.** Let a^b denote the conjugation $b^{-1}ab$. Suppose (W, S) is a finite Coxeter system (that is, W is finite). Then there is a unique element $\Delta_W \in W$, called its *longest element*, whose word length with respect to S is maximal. When an element of $S \subseteq W$ is conjugated by Δ_W, the result is always another element of S. Moreover, the map $s \mapsto s^{\Delta_W}$ is a permutation of S whose square is the identity (see, e.g., [15]). If (A, S) is the Artin system corresponding to (W, S), then A is called an *Artin group of finite type* and it also contains an element Δ_A with properties similar to those of the element Δ_W. In particular, conjugation by Δ_A gives the same involutive permutation of S as the one described above ([14]).

Definition 4.4 (Twisting)**.** Let Γ be a diagram with vertex set S and let U and V be disjoint subsets of S such that

1. W_V is a finite Coxeter group, and
2. every vertex in $S \setminus (U \cup V)$ which is connected to a vertex of U by an edge is also connected to each $v \in V$ by an edge labeled 2.

These conditions are shown schematically in Figure 6. When U and V satisfy these conditions, let Δ be the longest element of W_V. We define a new diagram Γ' by changing each edge of Γ that connects a vertex $u \in U$ to a vertex $v \in V$ to connect instead from u to $v^{\Delta} \in V$, and leaving other edges unchanged. We also define a new generating set S' of W by replacing

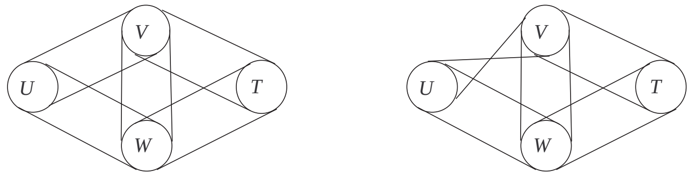

FIGURE 6. Schematic picture of the effect of twisting. The full diagram on V is finite type. The strips denote bundles of edges. The vertical bundle includes every edge from V to W and all are weighted 2.

each element $s \in U$ by s^Δ and leaving the rest of S unchanged. That is, $S' = U^\Delta \cup (S \setminus U)$. We call this operation *twisting U by Δ* (or *twisting U around V* if we want to stress V). To stress the effect on the diagram we sometimes say *diagram twisting.*

We make the same definitions for the Artin group defined by Γ, twisting by the longest element Δ of A_V or its inverse.

Theorem 4.5. *If S' and Γ' are obtained as above by twisting by Δ, then S' is a Coxeter generating set for W_Γ and its diagram is Γ'. The analogous statement holds also for the Artin group A_Γ.*

Proof. We will prove the Artin version of the theorem. The Coxeter version is identical. We must show that the generating set $S' = U^\Delta \cup (S \setminus U)$ satisfies the Artin relations corresponding to Γ'. The only cases where there is something to prove is if one or both of the generators in question is in U^Δ. If they both are, then the desired relation $(s^\Delta, t^\Delta)^j = (t^\Delta, s^\Delta)^j$ is just the conjugate by Δ of the relation $(s,t)^j = (t,s)^j$ coming from Γ. If $s^\Delta \in U^\Delta$ and $t \in S \setminus (U \cup V)$ then the desired relation $(s^\Delta, t)^j = (t, s^\Delta)^j$ is the conjugate by Δ of the relation $(s,t)^j = (t,s)^j$, since $t = t^\Delta$ by condition 2 of Definition 4.4. Finally, if $s^\Delta \in U^\Delta$ and $t \in V$ the desired relation $(s^\Delta, t^\Delta)^j = (t^\Delta, s^\Delta)^j$ for Γ' is again the conjugate by Δ of the relation $(s,t)^j = (t,s)^j$ for Γ.

Thus the Artin relations for S' given by Γ' follow from the Artin relations for S given by Γ. Since S results from S' by twisting by Δ^{-1}, the same argument shows that the relations for S follow from the ones for S'. Thus S' is an Artin generating set with diagram Γ'. □

Definition 4.6 (Twist equivalence of diagrams)**.** Twisting can be used to define an equivalence relations on diagrams. If Γ and Γ' are diagrams and Γ' can be obtained from Γ by a finite sequence of diagram twists, then we say Γ and Γ' are *twist equivalent* and we will write $\Gamma \sim \Gamma'$. If for every pair of Coxeter generating sets S and S' of a Coxeter group W, the diagrams Γ_S and $\Gamma_{S'}$ are twist equivalent, then we say that W is *rigid up to diagram twisting.* If (W, S) is a Coxeter system and for every Coxeter generating set

$S' \subseteq R_S$, the diagram Γ_S is twist equivalent to $\Gamma_{S'}$, then we say that (W, S) is *reflection rigid up to diagram twisting.*

Remark 4.7 (Nontrivial twists)**.** If W is a finite Coxeter group in which the element Δ is central, then conjugating by Δ will leave the generating set unchanged, the permutation of V will be the identity, and Γ will equal Γ'. Thus the only diagram twistings will can alter the diagram occur when Δ is not central in W_V.

The finite Coxeter groups in which the element Δ is not central are those of type A_n for $n \geq 2$, D_n for n odd, E_6, or $I_2(m)$ for m odd (see [6]). As a result, all of the nontrivial diagram twistings described in Definition 4.4 can be generated by twisting over a subdiagram of one of these types. There are two important special cases to note: if all of the edge labels are even, then no nontrivial diagram twisting is possible, and if all of the edge labels are odd, then only dihedral twisting is involved, i.e. twisting around subdiagrams of type $I_2(m)$ where m is odd.

The operation of twisting is still interesting even when it does not change the diagram up to isomorphism, for instance when Δ is central in W_V. Except in trivial cases the generating sets S and S' will then be non-conjugate, the group W will not be strongly rigid, and the automorphism induced by mapping S to S' will be an "exotic" automorphism of the Coxeter group. We therefore make the following definitions.

Definition 4.8 (Automorphisms generated by twists)**.** If a sequence of twists takes the Coxeter generating set S to a set S' with Γ_S isomorphic to $\Gamma_{S'}$ then a map $S \to S'$ that realizes this isomorphism gives an automorphism of W that we say is *generated by twists.* For fixed S, the set $\mathrm{Aut}_{\mathrm{twist}}(W)$ of all such automorphisms is an interesting subgroup of $\mathrm{Aut}(W)$. It includes the group of inner automorphisms (since conjugation by a generator v is the twist of $S \setminus \{v\}$ about $\{v\}$) as well as automorphisms induced by symmetries of Γ_S. It is a subgroup of the group $\mathrm{Aut}(W, R_S)$ of reflection-preserving automorphisms. It depends up to conjugation in $\mathrm{Aut}(W)$ only on the equivalence class of the diagram Γ_S.

The question as to whether $\mathrm{Aut}_{\mathrm{twist}}(W)$ equals $\mathrm{Aut}(W)$ or $\mathrm{Aut}(W, R_S)$ relates to rigidity concepts analogous to those of Definition 4.6, but based on twisting Coxeter generating sets rather than diagrams.

For example, for a right-angled Coxeter group W (which is rigid by Theorem 2.2) the full automorphism group is known in some detail, see [18] and [21].

Theorem 4.9 ([18])**.** $\mathrm{Aut}_{twist}(W) = \mathrm{Aut}(W, R_S)$ *for any right-angled Coxeter group.* $\square$

In fact, the main theorem of [18] gives a presentation in terms of twists of the subgroup of $\mathrm{Aut}(W, R_S)$ that fixes Γ_S ($\mathrm{Aut}(W, R_S)$ acts on Γ_S by Lemma 3.6). Combining this with a result of Patrick Bahls [1] gives:

Theorem 4.10. *A right-angled Coxeter group is strongly reflection rigid if and only if:*

- *for each vertex v of the Coxeter graph Γ the full subgraph on the set of vertices not connected to v by an edge is connected;*

and is strongly rigid if and only if in addition

- *each v is the intersection of all complete subgraphs of Γ containing v.*

Proof. If the first condition fails the automorphism induced by twisting a component of the full subgraph in question about $\{v\}$ will take S to a non-conjugate set S'. Conversely, if the condition holds then the Corollary on page 631 of [18] implies that any reflection preserving automorphism that induces the identity automorphism of Γ_S is inner, so (W, S) is strongly reflection rigid. The second part follows because the condition is Bahls' necessary and sufficient condition for the automorphism group of a right-angled Coxeter group to preserve the set of reflections. □

The analogous issues are interesting for Artin groups but need modification since Artin groups have more automorphisms. For instance, there is an automorphism of an Artin group that inverts all elements of a set of Artin generators. This automorphism does not preserve reflections. A less trivial example of an automorphism that does not preserve reflections is the automorphism of the Artin group $A_{I_2(m)}$ for m even given by $a \mapsto bab$, $b \mapsto b^{-1}$.

5. Rigidity of trees

In this section we use Theorem 4.5 to show that Coxeter groups defined by diagrams which are trees are reflection rigid up to diagram twisting (Theorem 5.7). We begin with some preliminary remarks about twisting and trees.

Example 5.1 (Trees)**.** Let Γ and Γ' be trees, all of whose edge labels are odd. If the multiset of edge labels for Γ is the same as the multiset of edge labels for Γ', then $\Gamma \sim \Gamma'$. As a result the Coxeter groups defined by Γ and Γ' are isomorphic and the Artin groups defined by Γ and Γ' are isomorphic. More specifically, by repeatedly applying Theorem 4.5 to diagrams of this type, it is possible to reduce the diameter of the tree until it has a single, special vertex which is connected to every other vertex by an edge. When both Γ and Γ' are reduced in this way, the resulting diagrams will be the identical if and only if the two trees use the same multiset of edge labels. Thus the Coxeter group defined by Γ is determined by its multiset of edge labels.

We will show below that, conversely, Γ also determines this multiset of edge labels. But first, a quick general remark about odd-labeled diagrams.

Remark 5.2 (Odd labels)**.** Let Γ be a diagram in which all of the edge labels are odd and let (W, S) be the Coxeter system it defines. Reflection rigidity for (W, S) and rigidity of W are equivalent in this case because the set $R_S \subseteq W$ is precisely the collection of involutions in W. One way to see this is to note that every involution in W is conjugate to an involution in one of the finite subgroups of W generated by a subset of S (Theorem 1.8). When all of the edge labels are odd, the only finite subgroups of W are dihedral groups $I_2(m)$ where m is odd. Finally, it is easy to check that each of the involutions in these dihedral subgroups are themselves conjugate to elements of S. As a result, every Coxeter generating set S' for W is contained in R_S.

Lemma 5.3. *Let W be a Coxeter group, let S and S' be Coxeter generating sets for W, and let Γ and Γ' be the corresponding diagrams. If Γ is a finite connected diagram and all of its edge labels are odd, then Γ' is also a finite connected diagram with the same number of vertices, the same number of edges, and the same multiset of edge-labels.*

Proof. By Remark 5.2, $S' \subseteq R_S$ and there is only one conjugacy class of involutions in W. This shows that the diagram Γ' must be connected (Lemma 3.6). Moreover, by Theorem 3.8, $|S| = |S'|$ and Γ and Γ' have the same number of vertices.

To see that Γ and Γ' have the same multiset of edge-labels (and hence the same number of edges) we first note that by Theorem 1.9 there is a bijection between the conjugacy classes of maximal finite subgroups in W and the edges of Γ since dihedral groups are the only finite Coxeter groups in which all edge-labels are odd.

Let H be a dihedral subgroup $\mathbf{D}_{m'}$ of W corresponding to a particular edge of Γ'. By Theorem 1.8, H can be conjugated into a maximal finite subgroup of W generated by a subset of S. But the only subsets of S which generate a finite subgroup are the dihedral groups corresponding to the edges of Γ. Since dihedral groups $\mathbf{D}_m$, m odd, do not contain any dihedral groups $\mathbf{D}_{m'}$ with m' even, this shows that all of the edge-labels in Γ' are odd. At this point, the preceding argument shows that there is a bijection from the edges of Γ' to the conjugacy classes of finite subgroups in W. Combining the two bijections shows that Γ and Γ' have the same number of edges and the same multiset of edge-labels. $\square$

Theorem 5.4. *If Γ is a diagram whose graph is a finite tree and whose edge labels are odd, then its Coxeter system is rigid up to diagram twisting.*

Proof. Let (W, S) be the Coxeter system corresponding to Γ and let $S' \subseteq W$ be another Coxeter generating set for W with diagram Γ'. By Lemma 5.3 Γ and Γ' have the same number of vertices, the same number of edges and Γ' is connected. Under these three restricts, Γ' must be a tree. Since Lemma 5.3 also shows that Γ and Γ' have the same multiset of edge-labels, by Example 5.1, $\Gamma \sim \Gamma'$. $\square$

In the following we indicate how the theorem above can be generalized to arbitrary trees. In order to do that we define the *even derivation* of a

Coxeter diagram which is a tree; this is an even labeled Coxeter diagram which is a tree.

Definition 5.5 (Γ_{even})**.** Let Γ be a Coxeter diagram which is a tree. The *even part* of Γ, denoted Γ_{even}, is the tree which results when each of the edges with an odd label is retracted to a point. See Figure 7 for an illustration. Note that the even part of Γ is formed by contractions, whereas the odd part of Γ is a subdiagram (Definition 3.5). The latter determines the conjugacy classes in R_S, while the former keeps track of the relationships between these conjugacy classes. Notice in particular, that there is a natural bijection between the connected components of Γ and the vertices of Γ_{even} and a group homomorphism $\phi : \Gamma \to \Gamma_{\text{even}}$ which sends each $s \in S$ to the element of S_{even} which is associated with the conjugacy class of s.

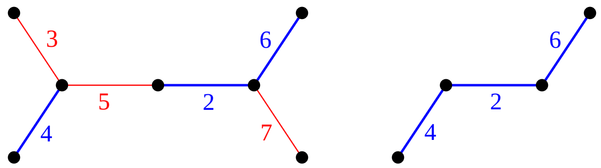

FIGURE 7. A tree diagram Γ and its even part Γ_{even}.

Example 5.6. Let Γ and Γ' be two finite tree diagrams, and suppose that there is a diagram isomorphism $\phi : \Gamma_{\text{even}} \to \Gamma'_{\text{even}}$ such that for every vertex v in Γ_{even} the multiset of edge-labels in the connected component of Γ_{odd} associated with v is the same as the multiset of edge-labels in the connected component of Γ'_{odd} associated with $\phi(v)$. Under these conditions, Γ and Γ' are twist equivalent. To see this, note that both diagrams are twist equivalent to a diagram Γ'' in which Γ''_{even} is a subdiagram of Γ''. To do this, use twisting to modify each connected component of Γ_{odd} as described in Example 5.1. Then more twisting can be used so that each edge with an even label has endpoints which are the special, central vertices in their respective connected components of the odd part of the diagram. This process has been illustrated in Figure 8.

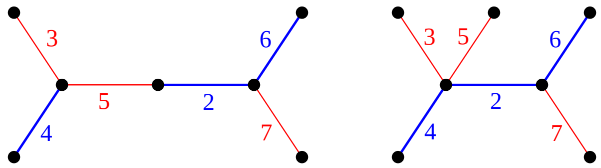

FIGURE 8. A tree diagram Γ and its twisted counterpart.

Combining the ideas used in Theorems 3.9 and 5.4, we can now show that all tree are reflection rigid up to diagram twisting.

Theorem 5.7. *If Γ is a finite diagram which is a tree, then the corresponding Coxeter group is reflection rigid up to diagram twisting.*

Proof. Let (W, S) be the Coxeter system corresponding to Γ, let $S' \subseteq R_S$ be another Coxeter generating set for W, and let Γ' be its Coxeter diagram. By Theorem 3.8 $|S| = |S'|$. By Lemma 3.6 Γ_{odd} and Γ'_{odd} and have the same number of connected components. Moreover, there is a bijection between these components and the conjugacy classes in $R_S = R_{S'}$ so that we have an explicit bijection between the components of Γ_{odd} and components of Γ'_{odd}.

Since the edges in Γ are in bijection with the conjugacy classes of maximal finite subgroups (Theorem 1.8), and since each of these subgroups is reflection rigid (Theorem 3.10), Γ and Γ' has the same number of edges and the same multiset of edge labels.

Moreover, such a dihedral subgroup will be generated by two reflections in particular conjugacy classes of $R_S = R_{S'}$, and all of the conjugates of this subgroup will be generated by two reflections in exactly the same conjugacy classes. This shows that we can also determine which connected component of Γ_{odd} contains a particular odd labeled edge and which two connected components are joined by a particular even labeled edge. Finally note that Γ' must be connected. The connectivity of Γ', the number of vertices and the number of edges combine to show that Γ' is a tree. In addition, we have been able to reconstruct enough of the tree to apply Example 5.6 to see that Γ and Γ' must be twist equivalent. ☐

6. Additional examples

In this section we present an additional example of diagram twisting which suggests that it may be computational difficulty to determine whether two finite diagrams are equivalent up to diagram twisting.

Example 6.1. Let Γ' be a complete graph on n vertices where the subdiagram Γ'_{odd} is connected and all of the edge labels of Γ' are either 2 or 3. It is easy to check that Γ' admits no nontrivial diagram twists.

Next, let Γ be the disjoint union of Γ' and a new vertex v with a few additional edges connecting v to vertices in Γ'. For example, suppose that there is only one additional edge e connecting v with Γ' and that e has an edge label of 4. Using diagram twisting, we can move the endpoint of e to any vertex in Γ'. To see this, consider a geodesic in Γ'_{odd} from the current endpoint of e to the desired endpoint. Since this is geodesic, the subdiagram of Γ' determined by the k vertices in the geodesic will be a subdiagram of type A_k. Twisting v around this type A_k subdiagram will move the endpoint as desired.

On the other hand, if there are several edges connecting v to Γ', each with distinct edge labels, then not all permutations of their endpoints are possible since the additional edges may prevent certain types of twisting. For example, if Γ'_{odd} is the tree shown in Figure 9 and there are edges connecting v to each of the labeled vertices, then the connections to vertices a and b

are trapped where they are, but the remaining connections to vertices c, d, e, and f can be permuted freely.

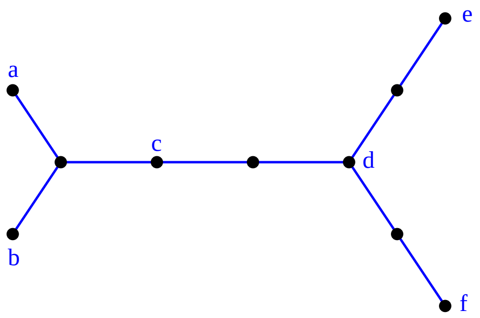

FIGURE 9. The diagram Γ_{odd} described in Example 6.1.

There are a number of other problems which can arise making it difficult to decide at a glance whether two diagrams are twist equivalent. There is even a possibility that deciding whether two diagrams are twist equivalent is an NP complete problem.

7. Artin groups

Although all of the recent rigidity results were proved for Coxeter groups exclusively, these results can often be quickly extended to Artin groups. The main tool for such an extension is the following technical lemma.

Lemma 7.1. *Let (A, S) be an Artin system and let Γ_S be the corresponding diagram. If S' is another Artin generating set for A and $R_S = R_{S'}$, then Γ_S and $\Gamma_{S'}$ define isomorphic Coxeter groups. Moreover, this Coxeter group W contains copies of S and S' and $R_S = R_{S'}$ in W.*

Proof. Let W and W' denote the Coxeter groups defined by Γ_S and $\Gamma_{S'}$, respectively. Recall that W is obtained from the group A by adding a relation of the form $s^2 = 1$ for each $s \in S$, but since the elements of $R_S \subseteq A$ are conjugates of the elements of S, it is equivalent to add a relation of the form $s^2 = 1$ for each $s \in R_S$. Similarly, the group W' is the quotient of A by adding a relation of the form $s^2 = 1$ for each $s \in R_{S'}$. Since $R_S = R_{S'}$, these quotients are identical and W is isomorphic to W'. The final assertion now follows immediately. □

Theorem 7.2. *Let Γ be a diagram, let (W, S) be the corresponding Coxeter system and let (A, S) be the corresponding Artin system. If (W, S) is reflection rigid up to diagram twisting, then (A, S) will be reflection rigid up to diagram twisting.*

Proof. Let $S' \subseteq R_S$ be another Artin generating set for A. By Lemma 7.1, the Coxeter group W defined by Γ is isomorphic to the one defined by $\Gamma_{S'}$ and $R_S = R_{S'}$ in W. Since (W, S) is reflection rigid up to diagram twisting by assumption and since S' is another Coxeter generating set for W with $S' \subseteq R_{S'} = R_S$, we can conclude that the diagrams Γ_S and $\Gamma_{S'}$ are equivalent up to diagram twisting. Since this is true for every Artin generating set $S' \subseteq R_S$ in A, the Artin system (A, S) is reflection rigid up to diagram twisting. $\square$

As corollaries, we immediate produce Artin versions of Theorem 3.10, Theorem 3.9 and Theorem 5.7. Note that an Artin group of finite type is simply one defined by a diagram whose Coxeter group is finite.

Corollary 7.3. *If Γ is the diagram of a finite type Artin group, then Γ is reflection rigid.*

Corollary 7.4. *If Γ is a finite diagram and all of its edge labels are even, then its Artin system is reflection rigid.*

Corollary 7.5. *If Γ is a finite diagram whose graph is a tree, then its Artin system is reflection rigid up to diagram twisting.*

8. Conjectures

The natural conjecture at this point is the following:

Conjecture 8.1. *Let W be a Coxeter group with Coxeter generating sets S and S'. If $R_{S'} = R_S$, then $\Gamma_S \sim \Gamma_{S'}$. In other words, Coxeter systems are reflection rigid up to diagram twisting.*

As evidence for this conjecture, we note that Theorem 3.10, Theorem 3.9 and Theorem 5.7 are special cases in which it is known to be true. A similar conjecture can be made for Artin systems.

Conjecture 8.2. *Let A be an Artin group with Artin generating sets S and S'. If $R_{S'} = R_S$, then $\Gamma_S \sim \Gamma_{S'}$. In other words, Artin systems are reflection rigid up to diagram twisting.*

This time, the relevant special cases are Corollary 7.3, Corollary 7.4 and Corollary 7.5. Since Example 2.5 fails to extend to the Artin case, it is quite possible that Artin groups are actually *rigid* up to diagram twisting. This would however be quite a strong result, and it is therefore probably false.

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