Contemporary Mathematics

Algorithms for polynomials in two variables

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Abstract. Vladimir Shpilrain and Jie-Tai Yu have asked for an effective algorithm to decide if two elements of $\mathbb{C}[x, y]$ are related by an automorphism of $\mathbb{C}[x, y]$. We describe here an efficient algorithm, due to the second author, that decides this question and finds the automorphism if it exists. We also discuss some examples related to work of Shpilrain and Yu. Part of the purpose of this paper is to advertise the use of splice diagrams in studying $\mathbb{C}[x, y]$.

1. Introduction

We give an efficient algorithmic answer to the following problem, which was posed by Shpilrain and Yu [**15**] and resolved by them in special cases. The algorithm is due to the second author in [**19**], where it is described in terms of Newton polygons. Here we describe it in terms of splice diagrams.

Problem 1.1. Given polynomials $f, g \in \mathbb{C}[x, y]$, decide if there exists a polynomial automorphism $\phi \in \operatorname{Aut}\mathbb{C}[x, y]$ with $f\phi = g$ and find ϕ if it exists.

As we discuss in the final section, $\mathbb{C}$ could be replaced by any field, although in finite characteristic the description in terms of splice diagrams given here would need some modification.

Any $\phi \in \operatorname{Aut}\mathbb{C}[x, y]$ induces a polynomial bijection $\mathbb{C}^2 \to \mathbb{C}^2$ (which we also call ϕ) and vice versa. From this point of view, the problem asks for a polynomial bijection $\phi\colon \mathbb{C}^2 \to \mathbb{C}^2$ which makes the diagram

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "\phi"] \arrow[d, "g"] & \mathbb{C}^2 \arrow[d, "f"] \\ \mathbb{C} \arrow[r, equal] & \mathbb{C} \end{tikzcd}$$

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commutative. We shall use the terminology "f is right-equivalent to g" for this. From a topological point of view it is also natural to ask about "right-left equivalence", which asks for polynomial bijections $\phi\colon \mathbb{C}^2 \to \mathbb{C}^2$ and $\psi\colon \mathbb{C} \to \mathbb{C}$ making

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "\phi"] \arrow[d, "g"] & \mathbb{C}^2 \arrow[d, "f"] \\ \mathbb{C} \arrow[r, "\psi"] & \mathbb{C} \end{tikzcd}$$

commutative (ψ is an affine map $\psi(z) = az + b$ with $a \neq 0$). Our results apply also to this. We shall use the abbreviation "equivalence" to mean right-equivalence.

Shpilrain and Yu say f and g in $\mathbb{C}[x,y]$ are "isomorphic" if the curves $f(x,y) = 0$ and $g(x,y) = 0$ are isomorphic as affine schemes; that is, their rings of functions $\mathbb{C}[x,y]/(f)$ and $\mathbb{C}[x,y]/(g)$ are isomorphic. In the final section we discuss some of their examples, and additional examples motivated from their work, of isomorphic but non-equivalent polynomials.

We first recall the structure of the automorphism group $\operatorname{Aut}\mathbb{C}[x,y]$. It has two subgroups

$$A := \operatorname{Aff}\mathbb{C}^2 = \{(x,y) \mapsto (ax + by + s, cx + dy + t) : a,b,c,d,s,t \in \mathbb{C}\}$$
$$B := \{(x,y) \mapsto (ax + g(y), dy + t) : g(y) \in \mathbb{C}[y], a,d,t \in \mathbb{C}\}$$

B is called the "Jonquière subgroup".

Theorem 1.2 (Jung [**7**], 1942). *$\operatorname{Aut}\mathbb{C}[x,y]$ is generated by its subgroups A and B.*

This theorem implies that any $\phi \in \operatorname{Aut}\mathbb{C}[x,y]$ has an expression

$$\phi = \phi_1\phi_2\dots\phi_n$$

with ϕ_i in A for i even and in B for i odd or vice versa. Moreover, we may assume that if $n > 1$ then no ϕ_i is in $A \cap B$. We call such an expression *normal form.*

If each ϕ_i in B is actually in the subgroup of *triangular automorphisms*

$$T := \{(x,y) \mapsto (x + g(y), y) : g(y) \in \mathbb{C}[y]\}.$$

we shall speak of *strict normal form.* We can always change a normal form representation with $n > 1$ to strict normal form by applying changes of the form: replace ϕ_i and ϕ_{i+1} by $\phi_i\chi$ and $\chi^{-1}\phi_{i+1}$ with $\chi \in A \cap B$.

The following theorem is stated in slightly different form in [**19**].

Theorem 1.3 (Wightwick [**19**]). *Suppose $f \in \mathbb{C}[x,y]$ is a non-constant polynomial and $\phi \in \operatorname{Aut}\mathbb{C}[x,y]$. Suppose $\phi = \phi_1\phi_2\dots\phi_n$ is a normal form expression for ϕ, and put $f_i = f\phi_1\phi_2\dots\phi_i$ for $i = 1,\dots,n$, and $f_0 = f$. Then the sequence of degrees $\deg(f_i)$ satisfies*

$$\deg(f_0) \geq \cdots \geq \deg(f_k) \leq \cdots \leq \deg(f_n), \quad \textit{for some } 0 \leq k \leq n.$$

Moreover, there is at most one $\phi_i \in B$ for which $\deg(f_{i-1}) = \deg(f_i)$, and if this happens then $\deg(f_i)$ is the minimal degree $\deg(f_k)$.

This theorem implies the result (van der Kulk [**9**], 1953) that $\operatorname{Aut}\mathbb{C}[x,y]$ is the amalgamated free product of A and B, amalgamated along $A \cap B$. For if not, there would be a non-trivial normal form representation of $\phi = 1$ with $\phi_1 \in B$, and applying the theorem with this ϕ to $f = x$ gives a contradiction.

The theorem clearly implies that n is at most $2(\deg(f_0)+\deg(f_n))+1$. It is also easy to see that that each ϕ_i has degree at most $\max(\deg(f_{i-1}), \deg(f_i))$ (this also follows from the proof in [**19**]). This already implies an algorithm for Problem 1.1, since it implies a bound of N^{4N+1} on the degree of the automorphism ϕ with $N = \max(\deg(f), \deg(g))$, so finding ϕ amounts to solving a system of algebraic equations in the coefficients of this automorphism, for which computational techniques are known. However, we promised an efficient algorithm, and this algorithm would be inefficient even if this bound on $\deg(\phi)$ were much improved (as it can be).

We will describe the efficient algorithm in terms of the splice diagrams introduced in the book [**6**]. They were used there to describe the local topology of plane curve singularities. Later, in [**11**], they were used to discuss the global topology of polynomial maps $f\colon \mathbb{C}^2 \to \mathbb{C}$, which is what interests us here.

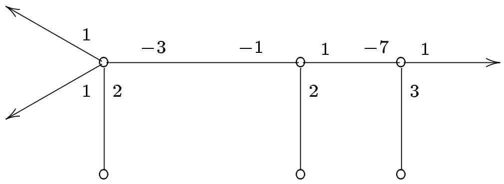
2. Splice diagrams

Associated to a fiber $f^{-1}(c) \subset \mathbb{C}^2$ of a non-constant polynomial $f \in \mathbb{C}[x,y]$ is a combinatorial invariant: the *splice diagram* $\Gamma(f^{-1}(c))$ for $f^{-1}(c)$. It is a tree with certain numerical and other decorations on it. There are finitely many values of c for which this diagram differs from its generic value. The generic diagram is called the *regular splice diagram* for f, denoted $\Gamma(f)$. As we shall describe, $\Gamma(f^{-1}(c))$ encodes behaviour at infinity of $f^{-1}(c)$. Therefore, for the finitely many c for which $\Gamma(f^{-1}(c))$ is not regular we speak of $f^{-1}(c)$ being *irregular at infinity* or having "singularities at infinity."

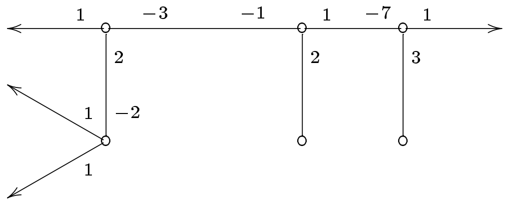
For example, the so called Briançon polynomial,

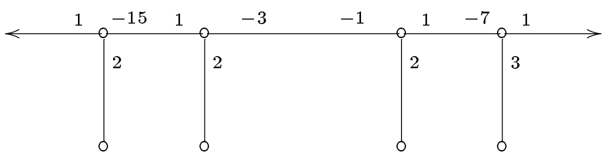
$$f(x,y) = x^2(1+xy)^4 + 3x(1+xy)^3 + (3 - \frac{8}{3}x)(1+xy)^2 - 4(1+xy) + y,$$

discussed in detail in [**3**], has regular splice diagram:

$\Gamma(f) =$ 

and two fibers which are irregular at infinity:

$\Gamma(f^{-1}(0)) =$ 

$\Gamma(f^{-1}(\frac{-16}{9})) =$ 

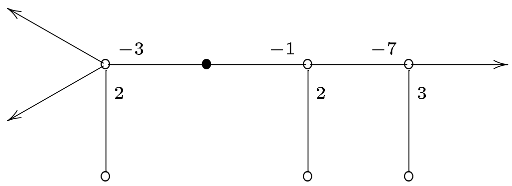
We explain such diagrams in greater detail below, but first we explain why they are useful invariants.

First, $\Gamma(f^{-1}(c))$ is easily computable (by hand; or explicit computer code is also available in Magma [**4**]). In fact, $\Gamma(f^{-1}(c))$ is simply a graphical representation of the exponents of the topologically relevant terms in the Puiseux expansions at infinity of $f(x,y) = c$, and Newton already knew how to compute Puiseux expansions. The regular splice diagram $\Gamma(f)$ is easily derived purely combinatorially from $\Gamma(f^{-1}(c))$ (see [**13**]), so one does not need to know a regular value of f to find $\Gamma(f)$.

Second, each $\Gamma(f^{-1}(c))$ individually is an invariant under equivalence of f, while the collection of them is an invariant under right-left equivalence (in particular, the regular splice diagram $\Gamma(f)$ is invariant under right-left equivalence of f).

Actually, splice diagrams exist in two versions, the rooted and unrooted splice diagrams, and the above are the unrooted versions. The rooted splice diagrams will be denoted $\Gamma_\bullet(f^{-1}(c))$ and $\Gamma_\bullet(f)$ respectively. For the above examples the rooted diagrams differ from the unrooted ones by the addition of a root vertex (drawn as a filled dot "•" with adjacent numeric weights 1) in the center of the edge with weights $-3, -1$.

We usually omit edge weights 1. For example, the rooted splice diagram for Briançon's polynomial is therefore

$\Gamma_\bullet(f) =$ 

The unrooted splice diagrams encode topology of $f\colon \mathbb{C}^2 \to \mathbb{C}$, while the rooted splice diagrams encode the same topology together with its relationship to a generic line in $\mathbb{C}^2$. Thus, the rooted splice diagram is *not* an invariant of equivalence, since the concept of generic line is not invariant under polynomial automorphisms. The topological meaning of these diagrams is detailed in [**11**] and surveyed in [**12**]. We will here give a brief description of the algebraic meaning. See also [**5**] for an extended survey and [**18**] for details of how to compute splice diagrams using Newton polygons.

We first describe the general structure of the splice diagrams that arise in our situation (the splice diagrams used to describe plane curve singularities are similar except that edge determinants are positive in Item 2.1.5).

2.1. Properties of splice diagrams.

2.1.1. Each *arrowhead* in the splice diagram corresponds to a branch at infinity of $f^{-1}(c)$ (a *branch at infinity* can be taken to be a component of the intersection of $f^{-1}(c)$ with the complement of a large ball in $\mathbb{C}^2$).

2.1.2. Each edge emanating from the *root vertex* corresponds to a point at infinity of $f^{-1}(c)$ (a *point at infinity* is a point of intersection of the closure of $f^{-1}(c)$ in $\mathbb{C}P^2$ with the line at infinity $\mathbb{C}P^1$). All edge weights adjacent to the root vertex are 1.

2.1.3. Any other vertex of the splice diagram is either

- a *leaf*: a non-arrowhead vertex of valency 1. It has no adjacent weights.
- a *node*: a non-arrowhead vertex of valency ≥ 2. It has pairwise coprime integer weights associated to its adjacent edges.

Each edge will thus have a weight at its near end, seen from the node, and may have a weight at its far end. We call weights "near" or "far" correspondingly.

2.1.4. All near weights are positive and at most one of the near weights at any node differs from 1 (edge weights 1 are usually omitted in the diagram). A far weight can be any integer.

2.1.5. The *edge determinant* of an edge connecting two nodes (one of which may be the root vertex) is the product of the two weights on the edge minus the product of the weights adjacent to the edge. All edge determinants are negative. For example, in the rooted diagram above the three edge determinants are, from left to right, $-3-2=-5$, $-1-2=-3$, and $-7-(-1\times 2\times 3)=-1$.

2.1.6. (**Reduction 1**) Any edge with edge weight 1 leading to a leaf may be deleted (with the leaf).

2.1.7. (**Reduction 2**) Any node (other than the root vertex) of valency 2 may be removed, and the adjacent edges coalesced into a single edge.

DEFINITION 2.1. A diagram is *reduced* if neither Reduction 1 nor Reduction 2 can be applied to it. In particular, all non-root nodes have valency ≥ 3.

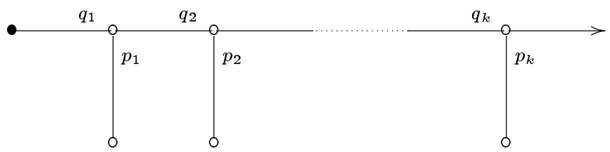
We will assume diagrams are reduced unless otherwise stated, and the notations $\Gamma_\bullet(f^{-1}(c))$, $\Gamma_\bullet(f^{-1}(c))$, $\Gamma(f^{-1}(c))$, and $\Gamma(f)$ will always mean reduced diagrams.

Here, the *unrooted splice diagram* $\Gamma(f^{-1}(c))$ or $\Gamma(f)$ is the result of changing the root vertex of the corresponding rooted diagram to an ordinary vertex and then reducing by Reductions 1 and/or 2 as necessary. However, there is an exceptional case described at the end of this section, where a further reduction must be done to eliminate a vanishing edge determinant.

2.2. Splice diagrams via Puiseux expansion. The rooted diagram relates to the algebra as follows. Pick an arrowhead and consider the corresponding branch at infinity of $f(x)=c$. Change coordinates linearly so this branch occurs at the point at infinity $[x{:}y]=[1{:}0]$ and consider the Puiseux expansion at this branch in the "Newton form"

$$y = x^{q_1/p_1}(a_1 + x^{d_1/p_1p_2}(a_2 + x^{d_2/p_1p_2p_3} + \dots)\dots)).$$

Here $q_1 < p_1$ and $d_i < 0$ for each i. One can show that there is a k such that $p_i = 1$ for $i > k$. We only need the terms up to this point. We then form the (maybe unreduced) splice diagram

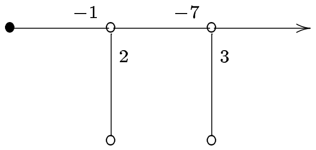


where the q_i for $i \geq 2$ are chosen inductively so that d_i is the edge determinant $q_{i+1} - q_ip_{i+1}$ for each i. Finally, we assemble these diagrams for the different branches at infinity of $f(x)=c$ by merging the parts that correspond to identical initial segments of the Puiseux expansions and then reduce the diagram as necessary.

For example, the right-hand arrowhead in the above diagrams for the Briançon polynomial corresponds to a branch of $f(x,y)=c$ where the Puiseux expansion at infinity is (writing the expansion as x in terms of y since the branch is at [0:1] rather than [1:0])

$$x = y^{-1/2}(i + y^{-1/6}(\frac{i}{2} + \dots)\dots).$$

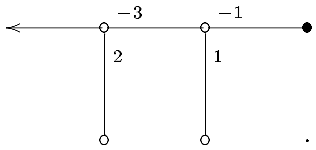
This gives the diagram



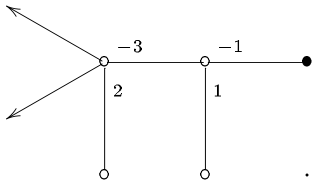
There are generically two branches of $f(x,y)=c$ at the point [1:0] at infinity and they have Puiseux expansions

$$y = x^{-1}(-1 + x^{-1/2}(\alpha + \dots)\dots)$$

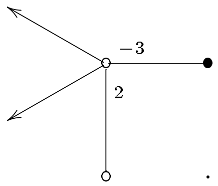
with $\alpha = \sqrt{4/3 \pm \sqrt{16/9 + c}}$. For $c \neq 0, -16/9$ this gives two branches each with (unreduced) diagram



Combining these two branches gives the (unreduced) diagram



which reduces by reduction moves of Items 2.1.6. and 2.1.7. above to



Combining this with the diagram for the branch at [0:1] gives the regular rooted splice diagram shown earlier.

The two values $c = 0, -16/9$ are clearly special for the above Puiseux expansion. When $c = 0$, in addition to the branch at [1:0] described above with $\alpha = \sqrt{8/3}$, there are two branches corresponding to $\alpha = 0$:

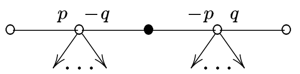
$$y = x^{-1}(-1 + x^{-1}(\beta + \dots)\dots), \quad \beta = -\frac{3}{4} \pm \sqrt{3}.$$

When $c = -16/9$, so $\alpha = \sqrt{4/3}$, there is just one branch at [1:0] and it has an additional relevant term in its Puiseux expansion:

$$y = x^{-1}(-1 + x^{-1/2}(\sqrt{4/3} + x^{-1/2}(-3/4 + \dots)\dots)).$$

These give the two irregular diagrams shown earlier.

2.3. Exceptional splice diagrams. As mentioned earlier in Definition 2.1, there is an exceptional case for the relationship between reduced rooted and unrooted diagrams. The following rooted splice diagram will be called *exceptional* if $p \geq q \geq 1$, $gcd(p, q) = 1$, and $pd > 1$:



(d arrows in each bunch, right edge omitted if $q = 1$).

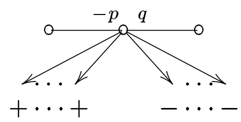
This is the regular reduced rooted splice diagram for $f(x, y) = x^{dp}y^{dq}$, for example.

(This $f(x, y)$ has one irregular fiber $f^{-1}(0)$ with splice diagram

$$(dp) \longleftarrow\!\!\bullet\!\!\longrightarrow (dq).$$

The weights in parentheses refer to multiplicity of nonreduced branches at infinity. There are other polynomials with the same regular rooted splice diagram which have different irregular ones, see [**11**].)

The reduced unrooted splice diagram for this rooted diagram is



(right edge omitted if $q = 1$.)

There are d arrows with $+$ signs and d with $-$ signs. The signs refer to orientations. See [**11**] for an explanation.

2.4. Non-reduced fibers. As above, if a polynomial f has non-reduced fibers then weights at arrowheads of the splice diagram are used to indicate multiplicities of components of such a fiber. Such a diagram is of course an irregular diagram.

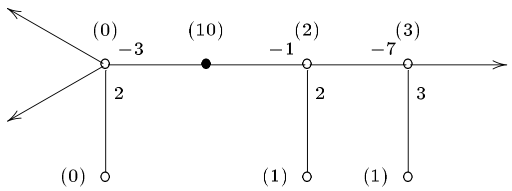
3. Invariants from splice diagrams

Much of this section is not essential for what follows, but the invariants encoded in splice diagrams are often helpful in computing them, since they give strong numerical constraints.

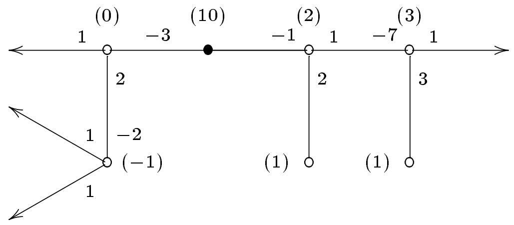
Given a vertex v of a splice diagram Γ, the *linking coefficient at* v (also called *multiplicity at* v) is the sum over arrowheads w of Γ:

$$\ell_v = \sum_w \ell(v, w),$$

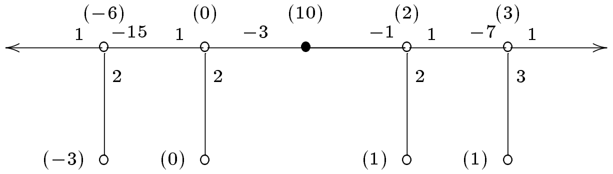
where $\ell(v, w)$ is the product of edge weights directly adjacent to but not on the path from v to w in Γ. For example, in the regular and irregular rooted diagrams for the Briançon polynomial these linking coefficients are as indicated in parentheses in the diagrams:

$\Gamma_\bullet(f) =$


$\Gamma_\bullet(f^{-1}(0)) =$



$\Gamma_\bullet(f^{-1}(\frac{-16}{9})) =$



3.1. Properties relating to linking coefficients.

3.1.1. The degree of a polynomial is always the linking coefficient $\ell_\bullet$ at the root vertex.

3.1.2. A splice diagram is regular if and only if it has no negative linking coefficients and the fiber in question is reduced (i.e., there are no multiplicity weights at arrowheads).

3.1.3. A polynomial has fibers with irregular splice diagrams if and only if the regular splice diagram has at least one zero linking coefficient. Each irregular diagram determines the regular one and the regular one strongly constrains the number and form of the irregular ones ([**13**]).

3.1.4. The Euler characteristic of the generic fiber of f is

$$\chi_{reg} := \sum_{v \in \operatorname{vert} \Gamma(f)} (2 - \delta_v)\ell_v,$$

where $\operatorname{vert}\Gamma(f)$ is the set of non-arrowhead vertices of $\Gamma(f)$ and δ_v is valency of vertex v (number of edges at v).

3.1.5. For any $c \in \mathbb{C}$ define the *Milnor number at infinity* of $f^{-1}(c)$ as

$$\lambda_c := \sum_{\substack{v \in \operatorname{vert} \Gamma(f^{-1}(c)) \\ \ell_v < 0}} (2 - \delta_v)\ell_v,$$

so $\lambda_c = 0$ unless $\Gamma(f^{-1}(c))$ is irregular. Moreover, if $f^{-1}(c)$ is a reduced fiber, define its *total Milnor number* as the sum of Milnor numbers:

$$\mu_c := \sum_{p \in f^{-1}(p)} \mu_p$$

(since $\mu_p = 0$ at a non-singular point, this is a finite sum). Then the Euler characteristic of a reduced fiber $f^{-1}(c)$ is given by

$$\chi(f^{-1}(c)) = \mu_c + \sum_{v \in \operatorname{vert} \Gamma(f^{-1}(c))} (2 - \delta_v)\ell_v = \mu_c + \lambda_c + \chi_{reg}.$$

3.1.6. (Suzuki [**17**]) For any f one has $1 - \chi_{reg} = \sum_{c\in\mathbb{C}} \big(\chi(f^{-1}(c) - \chi_{reg}\big)$, so if all fibers of f are reduced (i.e., f has only isolated singularities) then

$$1 - \chi_{reg} = \sum_{c \in \mathbb{C}} (\mu_c + \lambda_c).$$

In the formulae of Items 3.1.4. and 3.1.5. one can use either the rooted or unrooted diagram (in fact the the formulae do not need reduced diagrams).

For any polynomial, as soon as singularities and "singularities at infinity" are found contributing sufficient μ_c and λ_c to satisfy Suzuki's formula, one knows that all non-generic fibers have been found. For example, for Briançon's polynomial, Items 3.1.4. and 3.1.5. give:

$$\chi_{reg} = -3; \quad \lambda_0 = 1, \quad \lambda_{-16/3} = 3.$$

These already satisfy Suzuki's formula, so the fibers $f^{-1}(0)$ and $f^{-1}(-16/3)$ are indeed the only fibers with singularities at infinity and there are no "finite" singularities. Briançon's polynomial was the first non-singular polynomial discovered that also has all fibers connected [**3**], although Briançon constructed it for a different purpose.

4. Newton polygons and splice diagrams

Given a non-constant polynomial $f(x, y)$, our first step will be to put it into as simple form as possible. We first describe how the part of $\Gamma_\bullet(f)$ nearest the root can be read off from the Newton polygon of f. For more details see [**11**] or [**18**].

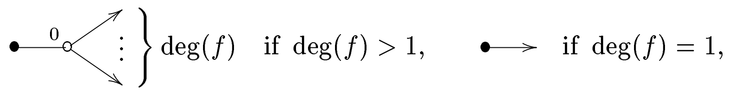
The *Newton polygon* of f is the convex hull of the set of points

$$\{(a, b) \in \mathbb{R}^2 : x^a y^b \text{ is a monomial of } f\} \cup \{(0, 0)\}.$$

(This would better be called the "regular Newton polygon" and the adjective "regular" dropped if $(0, 0)$ is not explicitly added, but we shall only use this version.)

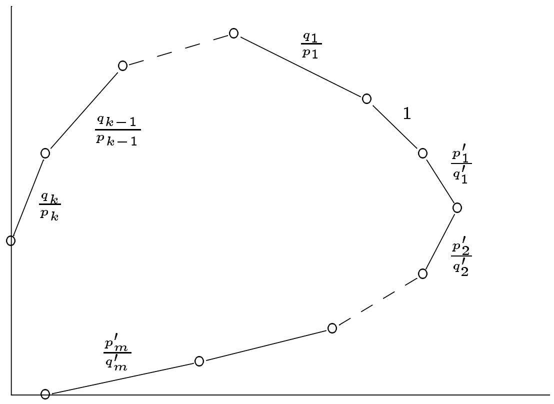
If the Newton polygon consists of a single line segment then either:

- f is a polynomial in just one of its variables, in which case the regular rooted splice diagram is

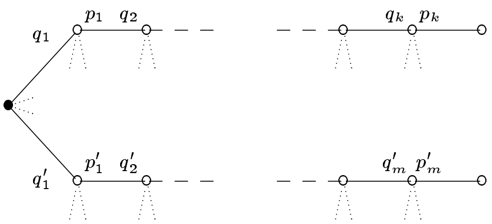


- $f(x, y) = g(x^p y^q)$ for some polynomial g of degree d, say, in which case the regular rooted splice diagram is $\longleftarrow\!\bullet\!\longrightarrow$ if $p = q = d = 1$, and of the exceptional type of Subsection 2.3 otherwise.

We assume from now on that the Newton polygon has non-empty interior. Suppose it is as follows, where the labels on segments are the negative of their slopes:



Then part of (a possibly unreduced version of) the rooted splice diagram is as follows. This diagram can be reduced if p_k or p'_m is 1.

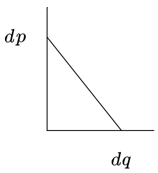


The number of additional edges (indicated by dotted lines) at each vertex is computed as follows. Consider a vertex with adjacent weights p, q, say. It corresponds to an edge of the Newton polygon of slope $-q/p$ or $-p/q$. The monomials of $f(x, y)$ for this edge sum to an expression of the form

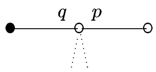
$$cx^a y^b \prod_{k=1}^{s} (x^p + c_k y^q)^{r_k} \quad \text{or} \quad cx^a y^b \prod_{k=1}^{s} (x^q + c_k y^p)^{r_k}.$$

Then s is the number of edges in question.

The segment of slope -1 may have length 0. And it will occur at an end of the chain of edges shown (rather than in the middle) if all the slopes shown are greater than -1 or all are less than -1, in which case the root vertex has valency 1. For example, if p, q are coprime integers with $q < p$ and $f(x, y)$ has Newton polygon



then

$\Gamma_\bullet(f) =$  .

This is the type of polynomial studied in [**15**].

5. Algorithms

5.1. Linear positioning. We first perform a linear automorphism to put points at infinity of f in "standard position".

Write $f(x, y) = f_0(x, y) + f_1(x, y) + \cdots + f_N(x, y)$ where $f_i(x, y)$ is the homogeneous part of degree i. The points at infinity for any fiber $f^{-1}(c)$ are the points of vanishing of the linear factors of $f_N(x, y)$. By a linear change of coordinates one of them can be put at $[x{:}y] = [1{:}0]$ and, if there is more than one, another at $[x{:}y] = [0{:}1]$. For the Newton polygon this means that the part lying on the line $x + y = N$ either does not include the points $(0, N)$ and $(N, 0)$ or consists only of the former point.

We shall assume from now on that f has been positioned in this way.

5.2. Reduction of degree. A reduced rooted splice diagram is *minimal* if it results from the reduced unrooted splice diagram by placing a root vertex either at an existing vertex or on an existing edge. We also call the exceptional diagrams discussed in Subsection 2.3 minimal.

PROPOSITION 5.1. *Assume we have applied a linear automorphism as above so* [1:0] *is a point at infinity for* $f(x,y)$. *Then, if the reduced rooted splice diagram for* $f(x,y)$ *is not minimal, there is an automorphism of the form* $(x,y) \mapsto (x - cy^q, y)$ *which reduces the degree of* f.

PROOF. If the reduced rooted splice diagram is not minimal, the root vertex has just one edge emanating and that edge has far weight 1. In terms of the Newton polygon this means that for some $q > 1$ there is a segment of slope $-q$ touching the y-axis (q is the near weight at the vertex adjacent to the root).

Then $f(x,y)$ has the form

$$f(x,y) = by^a \prod_{k=1}^{s} (x + c_k y^q)^{r_k} + \sum_{qi+j<qS+a} c_{ij} x^i y^j, \quad \text{where } S = \sum_{k=1}^{s} r_k.$$

Composing f with any of the automorphisms $(x,y) \mapsto (x - c_k y^q, y)$ will reduce the degree of f. □

By iterating the procedure of the above proposition we can apply automorphisms to $f(x,y)$ until its reduced rooted splice diagram is minimal. This procedure reduces the degree of f. Thus some minimal rooted splice diagram corresponds to an automorphic image of f of least possible degree. However, there may be several minimal rooted diagrams, and not all of them will give the absolute minimal degree among automorphic images of f.

In any case, we assume from now on that $\Gamma_\bullet(f)$ is minimal.

5.3. Changing reduced splice diagram. A reduced unrooted splice diagram $\Gamma = \Gamma(f^{-1}(c))$ may have several places where one can put a root vertex to make it into a rooted splice diagram. In [**11**] it is shown how to realise each resulting rooted splice diagram as $\Gamma_\bullet\big((f\phi)^{-1}(c)\big)$ for a suitable polynomial automorphism ϕ. Since this is an essential part of our algorithm we describe it here.

We will assume Γ is non-exceptional since each exceptional Γ has a unique minimal rooted splice diagram (see Subsection 2.3).

We first describe how to find all potential root vertices for Γ. Choose one possible root vertex. Consider a simple path from this vertex to another vertex with the properties:

- The edge weights *on* the path are all positive except maybe at the very end of the path;
- If an edge weight *adjacent* to the path is > 1 then it is the only edge weight > 1 at its node. A node where this happens is called a *contributing node* (following [**11**]) and we say the path "passes" this contributing node. (If a path through a contributing node has the weight > 1 on it rather than adjacent to it, we say it "goes through" the contributing node.)

Take the subgraph Γ_0 of Γ that is the union of all these paths. Each possible position for a root node lies at a vertex or on an edge of this subgraph. Their positions are found as follows. A *directed path* in Γ_0 is a simple path starting at a vertex or edge midpoint and departing each node along it by an edge with maximal weight among the edge-weights of Γ at that node. The condition of negative edge determinants implies that a maximal directed path will either start at a leaf, in the middle of an edge, or at a node with all adjacent weights 1 and it will end at a vertex with non-positive edge weight, an arrowhead, or at a contributing vertex. The starting points of maximal directed paths in Γ_0 are the possible positions for root vertices. In fact every half-edge of Γ_0 is on a unique maximal directed path, so these maximal directed paths give a flow on Γ_0 and the possible root vertices are the sources of this flow (see [**11**]).

Any simple path in Γ_0 will be a union of directed paths laid end-to-end with directions alternating. The direction can only change from forward to backward as we pass a contributing vertex and can only change from backward to forward as we go through a potential root vertex. We shall call two potential root vertices "adjacent" if the path between them goes through no other potential root vertex, so it changes direction exactly once, and this happens as it passes some contributing node. (The path may, however go through other contributing nodes without "passing" them—see Item 2. above.)

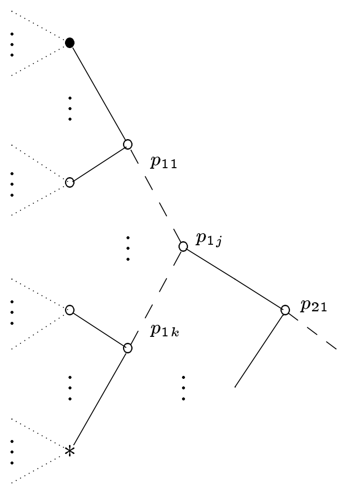


FIGURE 1. Adjacent potential root vertices

A typical path between adjacent potential root vertices is the path from $\bullet$ to $*$ in Figure 1 (the p_{ij} are greater than 1 and increase to the right in the graph). We are omitting vertices that are neither contributing nodes nor potential root vertices in this picture (they are all of valency at most 2 in Γ_0). The vertices at the left of the picture are pairwise adjacent potential root vertices.

If f does have several minimal rooted splice diagrams we need to find automorphisms that move us from one to another. It suffices to move between adjacent potential root vertices.

Suppose the root of $\Gamma_\bullet(f)$ is at the top left in Figure 1. Let the rightward edge at that node correspond to the point [1:0] at infinity. Then the Newton polygon will have segments on its boundary with slopes $-p_{11}, \dots, -p_{1j}, \dots, -p_{21}, \dots$. There may be segments with other slopes corresponding to vertices of $\Gamma_\bullet(f)$ that are valency two in Γ_0 and therefore not drawn in the figure. Suppose we want to move the root vertex to the position $*$. This moves it past the contributing node v with adjacent weight p_{1j}. The terms of $f(x,y)$ corresponding to the Newton polygon segment of slope $-p_{1j}$ give a polynomial $x^a y^b \prod_{i=1}^{s}(x+c_i y^{p_{1j}})^{r_i}$ with each factor $(x+c_i y^{p_{1j}})^{r_i}$ corresponding to one of the relevant edges emerging to the left from v. If the edge at v towards $*$ is the one with index i then the automorphism $(x,y)\mapsto(x-c_i y^{p_{1j}},y)$ moves the root vertex as desired. However, the resulting diagram may need further automorphisms as in Subsection 5.2 to reduce it to a minimal diagram again (an automorphism that gives a minimal diagram will be of the form $(x,y)\mapsto(x-g(y),y)$ with $g(y)$ a polynomial with highest order term $c_i y^{p_{1j}}$). Moreover, the result may not be to move the root vertex to $*$, but rather to some other of the potential roots to the left of v via the i-th edge out of vertex v, for instance, the vertex above $*$ in the figure. But we can iterate the procedure to try to move from the new root to $*$. Since each iteration involves a contributing vertex closer to $*$, this iterative procedure eventually ends with our node at $*$.

5.4. Comparing polynomials. Given two polynomials f and g in $\mathbb{C}[x,y]$, their (reduced unrooted) splice diagrams are likely to distinguish them if they are not equivalent (or even not right-left-equivalent), so the first thing to do is compute these and reduce them as above. Often f and g can be distinguished without computing the whole splice diagram, or they may be distinguished by an irregular splice diagram or by positions of singularities or of irregular fibers. We will just consider the regular splice diagram here.

We may thus assume that we have applied automorphisms to f and g so they have isomorphic rooted regular splice diagrams and these diagrams are minimal. After fixing g, there may be several ways of introducing a root in the unrooted splice diagram for f to make the diagrams isomorphic, and several isomorphisms, so there may be several cases to try. For each case we proceed as follows.

First, apply a linear automorphism as in subsection 5.1, if it exists, so that the points at infinity of f and g are the same for corresponding edges at the root vertices of $\Gamma_\bullet(f)$ and $\Gamma_\bullet(g)$. This automorphism will exist if there are three or less points at infinity, but it is uniquely determined up to scaling by three points, so if there are four or more points at infinity it may not exist, in which case there is no automorphism taking f to g (or even f to $\alpha g+\beta$ with $\alpha,\beta\in\mathbb{C}$) for the given isomorphism of splice diagrams.

Once points at infinity are matched, the problem of finding an automorphism, if it exists, taking f to g (or f to some $\alpha g+\beta$) for this particular isomorphism of splice diagrams reduces, by the following theorem, to solving simple equations in a very limited number of variables (α,β, and the coefficients of the ϕ described in the theorem).

THEOREM 5.2. *Suppose $f=f(x,y)$ has minimal rooted splice diagram. Assume also that a point at infinity occurs at* [1:0], *and a second, if it exists, at* [0:1], *and that f is not a polynomial in y alone. Suppose ϕ is an automorphism that does not move the root vertex of f and does not move any point at infinity*[1]*. Then*

- *If f has one point at infinity then ϕ is in the Jonquière subgroup B, that is, it is of the form $\phi(x,y)=(bx+h(y),dy+t)$. The degree of h is bounded by the*

[1]From an algebraic point of view, to say an automorphism moves neither the root vertex nor any point at infinity means that each branch at infinity of a generic fiber $f^{-1}(c)$ should be at the same point at infinity before and after applying the automorphism.

absolute value of slope of the top segment of the Newton polygon boundary (in particular, $\deg(h) \le \deg(f)$*).*

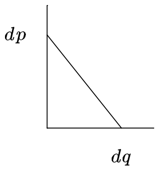
- *If f has two points at infinity then $\phi(x,y) = (ax+s, dy+t)$.*
- *If f has three or more points at infinity then $\phi(x,y) = (ax+s, ay+t)$.*

PROOF. The specific form of ϕ described in the theorem is easily seen if ϕ is affine, so assume it is not. Since the automorphism ϕ does not change the splice diagram, it does not change the degree of f (see Item 3.1.1 in Section 3). By Theorem 1.3, a normal form representation for the automorphism ϕ involves exactly one factor from B. We can thus write it as $\phi_1 \circ \phi_2 \circ \phi_3$, with $\phi_2 \in B - A$ and $\phi_1, \phi_3 \in A$, but with no restriction on non-triviality of ϕ_1, ϕ_3. We can modify ϕ_2 and ϕ_3 to make ϕ_2 triangular, say $(x,y) \mapsto (x+h(y), y)$ with $\deg(h) > 1$. Denote $f_i = f\phi_1 \dots \phi_i$. By Theorem 1.3, f, f_1, f_2, f_3 all have the same degree N say. Note that f_1 has the same number of points at infinity as f. If $f_1(x,y)$ had a point at infinity other than [1:0] then it would have a monomial $x^a y^{N-a}$ with $a > 1$. Taking the largest such a, it would follow that $f_2 = f_1\phi_2$ has a monomial $y^{a\deg(h)+N-a}$ and thus has degree larger than N. Thus f_1 has just one point at infinity at [1:0], so ϕ_1 fixes this point and is therefore in $A \cap B$. The same argument applied to $f = f_3\psi_3\psi_2\psi_1$ with $\psi_i = \phi_i^{-1}$ shows $\psi_3 \in A \cap B$ so $\phi_3 \in A \cap B$. Thus $\phi \in B$, as claimed.

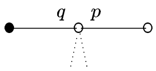
The statement about the degree of $h(y)$ follows by noting that, if the slope of the top segment in the Newton polygon is $-p/q$ and if $\deg(h) > p/q$ then a monomial $x^{N-rq}y^{rp}$ corresponding to the lowest point on this top segment of the Newton polygon for f leads to a monomial $y^{(N-rq)\deg(h)+rp}$ for $f\phi$. Since $(N-rq)\deg(h)+rp > (N-rq)p/q+rp = Np/q > N$, this completes the proof. $\square$

6. Examples and discussion

6.1. Uniquely one point at infinity. If p, q are coprime integers with $1 < q < p$ and $f(x,y)$ has Newton polygon



then

$\Gamma_\bullet(f) =$ 

is the unique minimal splice diagram, so f cannot be equivalent (or even right-left-equivalent) to another polynomial of the same type with different p, q, or d. This is the case proved in [**15**].

More generally, if the top segment of the Newton polygon boundary has slope $-p/q$ with p, q as above then there is one point at infinity and the minimal splice diagram is unique. Thus the invariants extractable from this diagram are invariants of the equivalence class of f. In particular, p, q, $\deg(f)$, and the lines of negative slope on the Newton polygon boundary are invariants.

6.2. Recognising a coordinate. If $g(x,y)=x$ then $\Gamma_\bullet(g)=\bullet\!\!\longrightarrow$, so if f is equivalent to g and $\deg(f)>1$ then, after applying a linear automorphism as in Subsection 5.1, there must be an automorphism that reduces degree derivable from the Newton polygon as in Subsection 5.2.

The Abhyankar-Moh-Suzuki theorem ([**1**], [**17**]) says that if $f\in\mathbb{C}[x,y]$ has a non-singular contractible fibre then f is a coordinate (meaning equivalent to x). There are now many proofs of this, and they all proceed by (implicitly or explicitly) constructing the linear and triangular automorphism which together reduce the degree of f. A rather different algorithm to recognise a coordinate is given in [**14**].

6.3. Isomorphic non-equivalent curves with one place at infinity. In [**16**] Shpilrain and Yu give the following interesting family of examples. Let

$$f(x,y)=x-h(x^q,y)\quad\text{and}\quad g(x,y)=x-h^q(x,y).$$

Then f and g are isomorphic. Recall this means the algebraic sets $f^{-1}(0)$ and $g^{-1}(0)$ are isomorphic as affine schemes, that is, there is a ring isomorphism

$$\mathbb{C}[x,y]/\big(x-h(x^q,y)\big)\ \to\ \mathbb{C}[x,y]/\big(x-h^q(x,y)\big).$$

It is given by

$$\begin{matrix} x\mapsto h(x,y) \\ y\mapsto y\end{matrix}\qquad\text{with inverse}\qquad\begin{matrix} x\mapsto x^q \\ y\mapsto y\end{matrix}$$

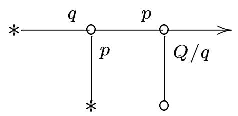
In the earlier paper [**15**] they applied this in the special case

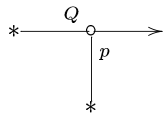
$$f(x,y)=x-x^{q_1q_2\dots q_n}-y^p,\qquad gcd(q_1q_2\dots q_n,p)=1$$

and $q=q_1q_2\dots q_i$ for $i=1,2,\dots,n-1$, to give examples of arbitrarily many affine plane curves with one place at infinity that are isomorphic but differently embedded in $\mathbb{C}^2$ (by Abhyankar and Singh [**2**] the number of embeddings of a one-place curve is always finite).

The rooted splice diagrams for this example are, with $Q=q_1\dots q_n$, $q=q_1\dots q_i$:

$$\Gamma_\bullet\left(x-(x^q+y^p)^{Q/q}\right)=$$



$=$  if $i=n$,

with root vertex at whichever of the two vertices marked $*$ has the smaller adjacent weight.

6.4. Families of isomorphic non-equivalent curves. In contrast to the situation for one place at infinity, a curve with several places at infinity may have a family of inequivalent embeddings in $\mathbb{C}^2$. An example is given by $\mathbb{C}-\{0,1\}$ embedded in $\mathbb{C}^2$ as the fiber $f^{-1}(c)$ of $f(x,y)=(xy+1)(x(xy+1)+1)$ for any $c\neq 0,1$.

Indeed, one can check that an explicit isomorphism $f^{-1}(c)\to\mathbb{C}-\{0,1\}$ is $(x,y)\mapsto z:=(xy+1)/c$. The generators z,z^{-1}, and $(1-z)^{-1}$ of the ring of functions on $\mathbb{C} - \{0,1\}$ are given in terms of the functions x, y on $f^{-1}(c)$, and conversely, by

$$\begin{aligned}
z &= \frac{xy+1}{c} \\
z^{-1} &= x(xy+1)+1 \\
(1-z)^{-1} &= \frac{c^2+cxy+y+xy(xy+1)}{c(c-1)}\ ,
\end{aligned}$$

$$\begin{aligned}
x &= \frac{1}{c}(1-z)z^{-2} \\
y &= cz^2(cz-1)(1-z)^{-1}.
\end{aligned}$$

Composing and simplifying, one finds an isomorphism

$$\mathbb{C}[x,y]/(f(x,y)-c) \cong \mathbb{C}[X,Y]/(f(X,Y)-d)$$

given by

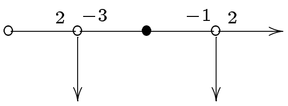
$$X = \frac{cx}{d}, \qquad Y = \frac{cd(d-1)y + d(d-c)(xy+1)^2}{c^2(c-1)}.$$

Different fibers $f^{-1}(c)$ and $f^{-1}(d)$ are not even right-left equivalent, since the map ψ of a right-left equivalence

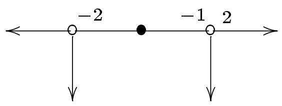
$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, "\phi"] \arrow[d, "f"] & \mathbb{C}^2 \arrow[d, "f"] \\
\mathbb{C} \arrow[r, "\psi"] & \mathbb{C}
\end{tikzcd}$$

would have to fix the two irregular values 0 and 1 of f and would thus be the identity, so it cannot map c to d if $c \neq d$.

This polynomial $f(x,y)$ has regular splice diagram

$\Gamma_\bullet(f) =$ 

and a single irregular diagram at the smooth reducible fiber $f^{-1}(0)$:

$\Gamma_\bullet(f^{-1}(0)) =$ 

The fiber $f^{-1}(1)$ is also reducible, with a normal crossing singularity at $(0,-1)$, but it is regular at infinity. The contributions from this singularity and the singularity at infinity of the fiber over 0 fulfil Suzuki's formula (end of Section 3), so we see again that there are no other non-generic fibers.

This is just one example of a polynomial with all generic fibers isomorphic. Such polynomials are called *isotrivial* and have been classified (see Kaliman [**8**]).

6.5. Other fields. Our base field has been $\mathbb{C}$ for convenience and to emphasize the geometric underpinnings for the approach. However, algebraic closure is certainly not essential since one can always perform computation in extension fields as necessary. The assumption of characteristic zero is also inessential for the algorithm: the proof of Theorem 1.3 in [**19**] goes through with no change and the algorithm, as described there without splice diagrams, also works. However, the theory of splice diagrams would need some modification in finite characteristic.

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