# Arithmetic of Hyperbolic Manifolds

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## 1. Introduction

By a hyperbolic 3-manifold we mean a complete orientable hyperbolic 3-manifold of finite volume, that is a quotient $\mathbb{H}^3/\Gamma$ with $\Gamma \subset \mathrm{PSL}_2\mathbb{C}$ a discrete subgroup of finite covolume (here briefly "a Kleinian group").

Among hyperbolic 3-manifolds, the arithmetic ones form an interesting, and in many ways more tractable, subclass. The tractability comes from the availability of arithmetic tools and invariants. For example, an arithmetic manifold $M = \mathbb{H}^3/\Gamma$ is determined up to commensurability by its defining field $k$ (a number field with exactly one complex place) and quaternion algebra $A$ (which is ramified at all real places of $k$). Any such pair $(k, A)$ determines a unique commensurability class of arithmetic hyperbolic 3-manifolds.

One aim of this paper is to try to extend arithmetic considerations to more general hyperbolic 3-manifolds. For example, a commensurability invariant pair $(k(M), A(M))$ consisting of a non-totally-real number field and a quaternion algebra over it is defined for any $M = \mathbb{H}^3/\Gamma$ (Sects. 2 and 3; we also write $(k(\Gamma), A(\Gamma))$), but it fails to be a complete commensurability invariant of $M$—non-commensurable $M$ can have the same $A(M)$ (see Sect. 10 for examples). Nevertheless, $k(M)$ and $A(M)$ do contain useful information—for instance (Theorem 3.2 and Proposition 3.3) ramification of $A(M)$ at a finite prime forces subgroups of $\Gamma$ to have non-trivial abelianizations, $A(M)$ gives a good amphicheirality test (Proposition 3.4), and $k(\Gamma)$ composes under amalgamation of Kleinian groups along a non-elementary Kleinian group (Theorem 2.8 and **[NR1]**) and is therefore a mutation invariant.

The trace field $\mathbb{Q}(\operatorname{tr}\Gamma)$ of $\Gamma$ is not a commensurability invariant and the field $k(\Gamma)$ is in fact the smallest field among the trace fields of finite index subgroups of $\Gamma$. We call it the ***invariant trace field*** of $\Gamma$. We show that the trace field $\mathbb{Q}(\operatorname{tr}\Gamma)$ is a Galois $(\mathbb{Z}/2)^m$-extension of the invariant trace field $k(\Gamma)$ and there is a precise Galois relationship between subgroups of $\Gamma$ and their trace fields, the largest subgroup of $\Gamma$ with trace field $k(\Gamma)$ being normal with quotient $(\mathbb{Z}/2)^m$ (Theorem 2.2).

It turns out that the existence of parabolic elements in $\Gamma$ facilitates many arithmetic questions. For example, in this case $A(\Gamma)$ is just the matrix algebra $\mathrm{M}_2(k(\Gamma))$. Moreover, $k(\Gamma)$ equals the field generated by the tetrahedral parameters of the ideal tetrahedra of any ideal triangulation of $\mathbb{H}^3/\Gamma$ (Theorem 2.4). If one conjugates $\Gamma$ so that three parabolic fixed points are at $0$, $1$, and $\infty$ in $\mathbb{C} \cup \{\infty\} = \partial\overline{\mathbb{H}}^3$, then $k(\Gamma)$ is also the field generated by all parabolic fixed points (Lemma 2.5).

As another example, an arithmetic orbifold with cusps cannot have geodesics shorter than $0.431277313$ (cf. Theorem 4.6 and Corollary 4.7); a corresponding result for com-

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pact orbifolds is conjectured (with a bound of $0.09174218$) but depends on Lehmer's conjecture of number theory.

We illustrate the greater tractability of non-compact manifolds also by doing some explicit computations for a family of manifolds related to Dehn surgeries on one component of the Whitehead link. Before describing this we need some terminology.

The most striking dichotomy in the arithmetic of hyperbolic groups is a result of Margulis (cf. **[Z**, Ch. 6**]**). Define the "commensurator of $\Gamma$" to be

$$\mathrm{Comm}(\Gamma) = \{g \in \mathrm{Isom}(\mathbb{H}^3) : [\Gamma{:}\Gamma \cap g^{-1}\Gamma g] < \infty\},$$

and let $\mathrm{Comm}^+(\Gamma)$ be its orientation preserving subgroup. Margulis shows that $\mathrm{Comm}^+(\Gamma)$ is discrete (and hence contains $\Gamma$ with finite index) if and only if $\Gamma$ is non-arithmetic. Thus in this case $\mathrm{Comm}^+(\Gamma)$ is the unique maximal element of the commensurability class of $\Gamma$. In the arithmetic case Borel showed that there are infinitely many maximal elements in the commensurability class (**[Bo]**).

As several people have observed, $\mathrm{Comm}(\Gamma)$ can be defined purely group-theoretically. Define a "virtual automorphism" of $\Gamma$ to be an isomorphism $\phi{:}\,\Gamma_1 \to \Gamma_2$ between subgroups of finite index in $\Gamma$ and define two virtual automorphisms to be "equivalent" if they agree on some subgroup of $\Gamma$ of finite index. Then Mostow rigidity easily implies that $\mathrm{Comm}(\Gamma)$ is the group of equivalence classes of virtual automorphisms under composition. (We like to call this version of $\mathrm{Comm}(\Gamma)$ the "abstract commensurator" of $\Gamma$, since in geometric situations without rigidity, for instance Fuchsian groups, it is much larger than the geometric commensurator. It has also been invented by group theorists, since it is interesting for other groups too; a simple example is $\mathrm{Comm}(\mathbb{Z}^n) = \mathrm{GL}_n\mathbb{Q}$. Bass and Kulkarni (**[BK]**) have investigated it for lattices on trees.)

From a geometric point of view, a virtual automorphism $\phi{:}\,\Gamma_1 \to \Gamma_2$ for $\Gamma$ represents an isometry (or just an isotopy class of homeomorphisms) between the two finite covers $\mathbb{H}^3/\Gamma_1$ and $\mathbb{H}^3/\Gamma_2$ of $M = \mathbb{H}^3/\Gamma$. We call such an isometry a "virtual symmetry" of $M$. We call it a "hidden symmetry" if it does not lift from a symmetry of $M$, i.e., $\phi$ lies in $\mathrm{Comm}(\Gamma) - N(\Gamma)$, where $N(\Gamma)$ is the normalizer of $\Gamma$ in $\mathrm{Isom}(\mathbb{H}^3)$. Define virtual symmetries to be equivalent if they have a common lift to some mutual finite cover of $M$ (caution: the set of equivalence classes does not form a group). Thus Margulis' theorem implies that arithmeticity of $M$ is equivalent to the existence of infinitely many non-equivalent hidden symmetries, while for non-arithmetic $M$ there is a finite cover of $M$ whose symmetries give all virtual symmetries of $M$.

As an example, if $\mathcal{O}_d$ is the ring of integers of $\mathbb{Q}(\sqrt{-d})$, then $\mathrm{Comm}\,\mathrm{PSL}_2\mathcal{O}_d$ is $\mathrm{PGL}_2\mathbb{Q}(\sqrt{-d})$ extended by an orientation reversing involution (complex conjugation). The existence of hidden symmetries for the orbifold $M = \mathbb{H}^3/\mathrm{PSL}_2\mathcal{O}_1$ is nicely illustrated by examples occurring in the literature: the complement $B$ of the Borromean rings in $S^3$ covers $M$ in at least two inequivalent ways—an embedding of $\pi_1 B$ as a normal subgroup of $\mathrm{PSL}_2\mathcal{O}_1$ is given in **[FN]** and a non-normal embedding is given in **[Ri]**; it is not hard to check that an element of $\mathrm{Comm}\,\mathrm{PSL}_2\mathcal{O}_1$ which conjugates one of these embeddings to the other cannot generate a discrete group with $\mathrm{PSL}_2\mathcal{O}_1$.

Note that the question about whether hidden symmetries exist for $M = \mathbb{H}^3/\Gamma$ is just the question about whether $\mathrm{Comm}(\Gamma)$ (hidden symmetries) equals the normalizer of $\Gamma$ (symmetries), i.e., whether $\Gamma$ is normal in $\mathrm{Comm}(\Gamma)$.

In Sections 5–8 of this paper we consider the manifold $C(p,s)$ which is the complement of a $p$ link chain in $S^3$ with $s$ left half twists (or $-s$ right half twists) pictured in Fig. 1 (we also make sense of this manifold—up to commensurability—for $p = 0$). We show it has a hyperbolic structure if and only if $\{|p+s|, |s|\} \not\subseteq \{0,1,2\}$ and determine its commensurator (i.e., the commensurator of its fundamental group) and some arithmetic invariants in this case. These computations use that $C(p,s)$ is commensurable with the result of $(p, s/2)$ Dehn surgery on one component of the Whitehead link (one can make sense of $(p,q)$ surgery for $q$ a half-integer; see Sect. 5). These Dehn surgeries on the Whitehead link are also discussed in **[HMW]**.

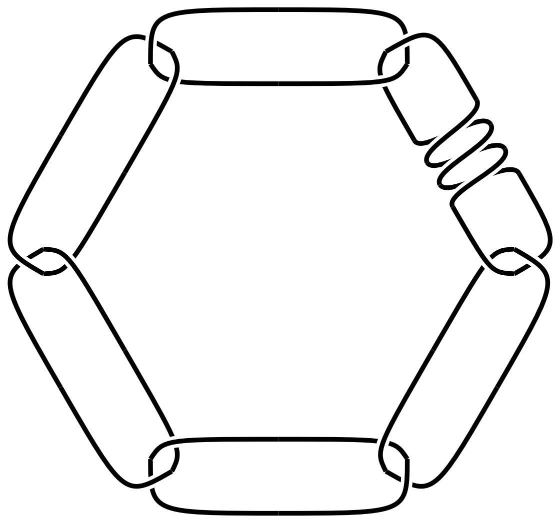

**Figure 1.** $C(p,s)$ for $p = 6,\ s = -3$

We give the finite list of $(p,s)$ for which $C(p,s)$ is arithmetic—it is then commensurable with $\mathrm{PSL}_2\mathcal{O}_d$ with $d \in \{1,2,3,7,15\}$ so the commensurator is $\mathrm{PGL}_2\mathbb{Q}(\sqrt{-d})$ extended by complex conjugation. In all other cases we show that $\pi_1 C(p,s)$ is normal in its orientation preserving commensurator, with quotient the "obvious" group of orientation preserving symmetries of $C(p,s)$, and an orientation reversing commensuration exists if and only if $p+s = \pm s$ or $p+s = 0$ or $s = 0$.

We say $\Gamma$ (or $\mathbb{H}^3/\Gamma$) has ***integral traces*** if $\{\mathrm{tr}\,\gamma : \gamma \in \Gamma\}$ consists of algebraic integers; it then follows that $\Gamma$ can be conjugated to a subgroup of $\mathrm{PSL}_2\mathbb{A}$, where $\mathbb{A}$ is the ring of algebraic integers. In **[Ba2]** Bass shows that if a hyperbolic manifold $M = \mathbb{H}^3/\Gamma$ has no closed incompressible surface of genus $> 1$ then $\Gamma$ has integral traces. Note that the property of having integral traces is a commensurability invariant (since an element $\gamma \in \mathrm{PSL}_2\mathbb{C}$ has algebraic integer trace if and only if it has algebraic integer eigenvalues, which is true if it is true for some power of $\gamma$). In Theorem 6.3 we determine which $C(p,s)$ have non-integral traces, namely those with $p+s = \pm s$ and

$s$ equal to $\pm 2$ times an odd prime power. These appear to be the first examples in the literature of hyperbolic manifolds with non-integer traces.

In Sections 9 and 10 we collect some questions and comments.

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## 2. Trace-fields

**2.1.** Let $\Gamma$ be a Kleinian group of finite covolume. It is well known, and follows from rigidity, that the trace field $\mathbb{Q}(\operatorname{tr}\Gamma)$ has finite degree over $\mathbb{Q}$. However, $\mathbb{Q}(\operatorname{tr}\Gamma)$ is not an invariant of the commensurability class of $\Gamma$ (see **[Bo]**, **[R3]**, and Sect. 6 of this paper for examples), although it is not far removed. For if we denote $\Gamma^{(2)} = gp\{\gamma^2 : \gamma \in \Gamma\}$, then $\Gamma^{(2)}$ is normal in $\Gamma$ with $\Gamma/\Gamma^{(2)}$ finite abelian of exponent 2, and the following was proved in **[R3]**.

**Theorem 2.1.** *With $\Gamma$ as above, $\mathbb{Q}(\operatorname{tr}\Gamma^{(2)})$ is an invariant of the commensurability class of $\Gamma$. Moreover*

$$\mathbb{Q}(\operatorname{tr}\Gamma^{(2)}) = \mathbb{Q}\left((\operatorname{tr}\gamma)^2 : \gamma \in \Gamma\right). \qquad \square$$

**Notation.** We shall denote $\mathbb{Q}(\operatorname{tr}\Gamma^{(2)})$ by $k(\Gamma)$ throughout the rest of this paper. We call it ***the invariant trace field***.

Theorem 2.1 implies that $\mathbb{Q}(\operatorname{tr}\Gamma) = k(\Gamma)(\sqrt{r_1}, \dots, \sqrt{r_m})$ for some $r_1, \dots, r_m \in k(\Gamma)$, and hence $\operatorname{Gal}(\mathbb{Q}(\operatorname{tr}\Gamma)/k(\Gamma)) = (\mathbb{Z}/2)^m$. The following theorem describes the relationship between subgroups of $\Gamma$ and their trace fields.

**Theorem 2.2.** *Assume $\Gamma$ has no 2-torsion. Let $K$ be a field satisfying $k(\Gamma) \subseteq K \subseteq \mathbb{Q}(\operatorname{tr}\Gamma)$.*

(1) $\Gamma_K := \{\gamma \in \Gamma : \operatorname{tr}\gamma \in K\}$ *is a normal subgroup of $\Gamma$ with $\Gamma/\Gamma_K$ finite abelian of exponent 2.*

(2) *There is a non-singular bilinear pairing*

$$\begin{array}{ccc} \Gamma/\Gamma_K \otimes \operatorname{Gal}(\mathbb{Q}(\operatorname{tr}\Gamma)/K) & \to & \mathbb{Z}/2 = \{\pm 1\}; \\ \gamma \otimes g & \mapsto & \frac{(\operatorname{tr}\gamma)^g}{\operatorname{tr}\gamma} \end{array}$$

*in particular, $\Gamma/\Gamma_{k(\Gamma)} = (\mathbb{Z}/2)^m$.*

(3) *The above results hold in the presence of 2-torsion if, when $\operatorname{tr}\gamma = 0$ (i.e., $\gamma^2 = 1$), one replaces $\operatorname{tr}\gamma$ in the above definitions by $\operatorname{tr}\gamma'$ for an element $\gamma' \in \gamma\Gamma^{(2)}$ with non-zero trace.*

Recall that any cusped hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ is topologically the interior of a compact manifold-with-boundary $\overline{M}$, whose boundary consists of tori. The following result was proved in **[R3]** for knot complements.

**Corollary 2.3.** *Let* $M = \mathbb{H}^3/\Gamma$ *be a manifold* (*that is,* $\Gamma$ *has no torsion*). *If* $\mathrm{Cok}(H_1(\partial\overline{M};\mathbb{Z}) \to H_1(\overline{M};\mathbb{Z}))$ *is finite of odd order then* $k(\Gamma) = \mathbb{Q}(\mathrm{tr}\,\Gamma)$. *In particular, this holds if* $M$ *is the complement of a link in a* $\mathbb{Z}/2$*-homology sphere.*

*Proof of Corollary.* Note that if $P$ is the subgroup of $\Gamma$ generated by parabolic elements, then $\Gamma/P = \mathrm{Cok}(H_1(\partial\overline{M};\mathbb{Z}) \to H_1(\overline{M};\mathbb{Z}))$, so $\Gamma/\Gamma^{(2)}P$ is the largest quotient of $\mathrm{Cok}(H_1(\partial\overline{M};\mathbb{Z}) \to H_1(\overline{M};\mathbb{Z}))$ of exponent 2. Thus the condition of the corollary is equivalent to: $\Gamma = \Gamma^{(2)}P$. But certainly $P \subseteq \Gamma_{k(\Gamma)}$, so $\Gamma = \Gamma_{k(\Gamma)}$, so $k(\Gamma) = \mathbb{Q}(\mathrm{tr}\,\Gamma)$. □

*Proof of Theorem 2.2.* (1) For any $B \in \mathrm{SL}_2\mathbb{C}$ one has the matrix equation $B^2 - \mathrm{tr}\,(B)B + I = 0$. Left-multiplying this by $A$ and taking trace gives

$$\mathrm{tr}\,(B)\,\mathrm{tr}\,(AB) = \mathrm{tr}\,(AB^2) + \mathrm{tr}\,A.$$

If $A \in \Gamma^{(2)}$ then this equation implies $\mathrm{tr}\,(B)\,\mathrm{tr}\,(AB) \in k(\Gamma)$, so if $B \in \Gamma_K$ then $AB \in \Gamma_K$ (no element of $\Gamma$ has trace $0$ by our exclusion of 2-torsion). Thus $\Gamma^{(2)}\Gamma_K \subseteq \Gamma_K$. Now suppose both $A$ and $B$ are in $\Gamma_K$. Then $B^2A \in \Gamma^{(2)}\Gamma_K \subseteq \Gamma_K$, so $AB^2 \in \Gamma_K$ (since $B^2A$ and $AB^2$ have the same trace). The above equation thus implies $\mathrm{tr}\,(AB) \in K$, so $AB \in \Gamma_K$. Since trivially $A \in \Gamma_K$ implies $A^{-1} \in \Gamma_K$, $\Gamma_K$ is a subgroup. It is normal with $\Gamma/\Gamma_K$ finite abelian of exponent 2 because this is so for any supergroup of $\Gamma^{(2)}$ in $\Gamma$.

(2) It is trivial that the pairing of (2) is linear in $g$ if one fixes $\gamma$. For the linearity in $\gamma$, let $\alpha$ and $\beta$ be elements of $\Gamma$ represented by matrices $A$ and $B$. By Theorem 2.1, $\mathrm{tr}\,A = \sqrt{a}$ and $\mathrm{tr}\,B = \sqrt{b}$ for some $a, b \in k(\Gamma)$. Moreover, $AB^2 \in \Gamma_{\mathbb{Q}(\sqrt{a})}$, so $\mathrm{tr}\,AB^2 = c\sqrt{a}$ for some $c \in k(\Gamma)$. The above trace equation gives $\mathrm{tr}\,AB = ((c{+}1)/b)\sqrt{ab}$, from which the desired linearity follows. The pairing of (2) is nonsingular by definition of $\Gamma_K$, and part (3) of the theorem is also clear. □

**2.2.** In this subsection we give a geometric description of the invariant trace field $k(\Gamma)$ in case $M = \mathbb{H}^3/\Gamma$ is a cusped hyperbolic manifold.

By **[EP]**, $M$ has at least one triangulation by ideal tetrahedra:

$$M = S_1 \cup S_2 \cup \ldots \cup S_n,$$

where each $S_j$ is an ideal tetrahedron in $\mathbb{H}^3$. As discussed in **[T**, chapt. 4**]**, the tetrahedron $S_j$ is described up to isometry by a single complex number $z_j$ with positive imaginary part (the ***tetrahedral parameter*** of $S_j$) such that the Euclidean triangle cut off any vertex of $S_j$ by a horosphere section is similar to the triangle in $\mathbb{C}$ with vertices $0$, $1$, $z_j$. Alternatively, $z_j$ is the cross-ratio of the vertices of $S_j$ (considered as points of $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$). This tetrahedral parameter depends on a choice (an edge of $S_j$ or an oriented ordering of its vertices); changing the choice replaces $z_j$ by $1/(1-z_j)$ or $1-1/z_j$. Denote the field $\mathbb{Q}(z_j : j = 1, \ldots, n)$ by $k_\Delta M$ or $k_\Delta\Gamma$. *A priori* $k_\Delta\Gamma$ might depend on the choice of triangulation, but we have:

**Theorem 2.4.** $k_\Delta\Gamma = k(\Gamma)$.

*Proof.* Denote $k_\Delta\Gamma$ by $k_\Delta$ for short. If we lift the triangulation of $M$ to $\mathbb{H}^3$ we get a tesselation of $\mathbb{H}^3$ by ideal tetrahedra. Let $V$ be the set of vertices of these tetrahedra in the sphere at infinity. Let $k_1$ be the field generated by all cross ratios of 4-tuples of points of $V$. Position $V$ by an isometry of $\mathbb{H}^3$ (upper half-space model) so that three of its points are at $0,\ 1,$ and $\infty$, and let $k_2$ be the field generated by the remaining points of $V$. This $k_2$ does not depend on which three points we put at $0,\ 1,\ \infty$; in fact:

**Lemma 2.5.** $k_1 = k_2 = k_\Delta$.

*Proof.* $k_1 \subseteq k_2$ since $k_1$ is generated by cross-ratios of elements of $k_2$ while $k_2 \subseteq k_1$ because the cross-ratio of $0,\ 1,\ \infty,$ and $z$ is just $z$. $k_\Delta \subseteq k_1$ is trivial. Finally, put three vertices of one tetrahedron of our tesselation at $0,\ 1,$ and $\infty$, and then $k_2 \subseteq k_\Delta$ is a simple induction on noting that, for any field $l$, if three vertices and the tetrahedral parameter of an ideal tetrahedron $S \subseteq \mathbb{H}^3$ are in $l \cup \{\infty\}$, then so is the fourth vertex. $\square$

Now suppose we have positioned $V$ as above. Any element $\gamma \in \Gamma$ maps $0,\ 1,$ and $\infty$ to points $w_1,\ w_2,$ and $w_3$ of $V \subseteq k_\Delta \cup \{\infty\}$. Thus $\gamma$ is given by a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ whose entries satisfy

$$\begin{aligned} b \qquad\qquad - dw_1 &= 0, \\ a + b - cw_2 - dw_2 &= 0, \\ a \qquad - cw_3 \qquad\quad &= 0. \end{aligned}$$

We can solve this for $a,\ b,\ c,\ d$ in $k_\Delta$ and then $\gamma^2$ is represented by the element

$$\frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 \in \mathrm{PSL}_2 k_\Delta.$$

Thus $k(\Gamma) = \mathbb{Q}(\operatorname{tr}\Gamma^{(2)}) \subseteq k_\Delta$.

For the reverse inclusion we shall use the following lemma which was observed in **[R4]**. Indeed, given the lemma, which says that $\Gamma$ can be conjugated to lie in $\mathrm{PSL}_2\mathbb{Q}(\operatorname{tr}\Gamma)$, the points of $V$, which are the fixed points of parabolic elements of $\Gamma$, lie in $\mathbb{Q}(\operatorname{tr}\Gamma)$, since the fixed point of a parabolic element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is $b/(1-a) = (1-d)/c$. Thus, by Lemma 2.5, $k_\Delta \subseteq \mathbb{Q}(\operatorname{tr}\Gamma)$. On the other hand, $k_\Delta$ is clearly an invariant of the commensurability class of $\Gamma$, so we can apply this to $\Gamma^{(2)}$ to see $k_\Delta \subseteq \mathbb{Q}(\operatorname{tr}\Gamma^{(2)}) = k(\Gamma)$. $\square$

**Lemma 2.6.** *A non-cocompact finite volume Kleinian group* $\Gamma$ *has a faithful discrete representation in* $\mathrm{PSL}_2\mathbb{Q}(\operatorname{tr}\Gamma)$.

*Proof.* By putting a lift of a cusp at $\infty$, another lift of a cusp at $0$, and the image of $0$ under a parabolic element $T_1$ which fixes $\infty$ at $1$, we arrange that $\Gamma$ contains the elements

$$T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

($T_2$ is any parabolic which fixes $0$). Since $T_1T_2$ has trace $2+\alpha$ we see that $\alpha \in \mathbb{Q}(\operatorname{tr}\Gamma)$. The lemma now follows from **[Mac**, Prop. 3.1**]** which says that the coefficient field of a non-abelian subgroup $\Gamma$ of $\mathrm{PSL}_2\mathbb{C}$ is generated by $\mathbb{Q}(\operatorname{tr}\Gamma)$ and the coefficients of any two non-commuting elements of $\Gamma$. $\square$

**2.3.** A horospherical section of a cusp of a hyperbolic manifold $M$ is a flat torus. This torus is isomorphic to $\mathbb{C}/\Lambda$, for some lattice $\Lambda \subset \mathbb{C}$, and the ratio of two generators of $\Lambda$ is the ***conformal parameter*** of the flat torus, which we call the ***cusp parameter*** of the cusp of $M$. It depends on the choice of generators of $\Lambda$, but a different choice changes it by an integral Möbius transformation, so the field it generates is independent of choices. The field generated by the cusp parameters of all the cusps of $M = \mathbb{H}^3/\Gamma$ is called the ***cusp field*** of $M$ (or of $\Gamma$). The cusp field is clearly a commensurability invariant, so we define the cusp field of an orbifold to be the cusp field of some manifold cover.

Since an ideal triangulation of $M$ induces triangulations of the cusp tori, from which the cusp parameters may be computed, Theorem 2.4 has the corollary:

**Proposition 2.7.** *The cusp field of a hyperbolic orbifold is contained in the invariant trace field.* □

As we point out in Sect. 10, the cusp field can be smaller than the invariant trace field, even for hyperbolic knot complements.

**2.4.** The following theorem is from **[NR1]**. Our formulation here is stronger, but the proof in **[NR1]** applies verbatim.

**Theorem 2.8.** *If the Kleinian group $\Gamma$ is generated by subgroups $\Gamma_1$ and $\Gamma_2$ and $\Gamma_1 \cap \Gamma_2$ is non-elementary then $k(\Gamma) = k(\Gamma_1)k(\Gamma_2)$ (join of fields).* □

## 3. Quaternion algebras

**3.1.** In this section we associate to a Kleinian group $\Gamma$ a quaternion algebra over the trace field $\mathbb{Q}(\operatorname{tr}\Gamma)$. Namely, let $\bar{\Gamma}$ be the inverse image of $\Gamma \subset \mathrm{PSL}_2\mathbb{C}$ in $\mathrm{SL}_2\mathbb{C}$ and denote by $\mathbb{Q}\Gamma$ the algebra of all $\mathbb{Q}$-linear combinations of elements of $\bar{\Gamma}$. It follows from the characteristic equation

$$A^2 - (\operatorname{tr} A)A + I = 0$$

that $\mathbb{Q}\Gamma$ contains $\tau I$ for any $\tau$ in the trace field $\mathbb{Q}(\operatorname{tr}\Gamma)$. Hence $\mathbb{Q}\Gamma$ is an algebra over $\mathbb{Q}(\operatorname{tr}\Gamma)$ (and equals $k\Gamma$ for any subfield $k$ of $\mathbb{Q}(\operatorname{tr}\Gamma)$ ). It is in fact a quaternion algebra over this field (cf. **[Ba1**, Prop. 2.2**]** or **[Ba2]**).

Note that the result of Macbeath quoted in the proof of Lemma 2.6 follows from the observation that a quaternion algebra is generated over its base field by any two non-commuting elements. For the same reason, although $\mathbb{Q}\Gamma$ is not a commensurability invariant, $\mathbb{Q}\Gamma^{(2)}$ is. Together with the trace field $\mathbb{Q}(\operatorname{tr}\Gamma)$, this commensurability invariant determines $\mathbb{Q}\Gamma$ by the equation

$$\mathbb{Q}\Gamma = \mathbb{Q}\Gamma^{(2)} \otimes_{k(\Gamma)} \mathbb{Q}(\operatorname{tr}\Gamma).$$

**Notation.** We shall denote $\mathbb{Q}\Gamma^{(2)}$ by $A(\Gamma)$ and call it the ***invariant quaternion algebra*** of $\Gamma$.

Lemma 2.6 tells us that the invariant quaternion algebra $A(\Gamma)$ gives no more information than the invariant trace field $k(\Gamma)$ when $\Gamma$ has cusps— $A(\Gamma)$ then equals the matrix algebra $\mathrm{M}_2(k(\Gamma))$. However, in general $A(\Gamma)$ is more interesting, for instance $A(\Gamma)$ is a complete commensurability invariant for arithmetic $\Gamma$ and equals $\mathrm{M}_2(k(\Gamma))$ in this case if and only if $\Gamma$ has cusps, as we discuss in the next section.

Recall (**[Vig]**) that a quaternion algebra $A$ over a number field $k$ is determined up to isomorphism by the set $S(A)$ of primes of $k$ (finite or infinite) at which $A$ is ramified; $S(A)$ has a finite even number of elements and any finite even set of primes of $k$ is the ramification set of some quaternion algebra over $k$.

Little seems to be known about what the invariant $A(\Gamma)$ of $\Gamma$ says about the geometry of $\mathbb{H}^3/\Gamma$. If $\Gamma$ is arithmetic then Clozel **[Cl]** has shown

**Theorem 3.1.** *If $\Gamma$ is arithmetic and $S(A(\Gamma))$ contains no finite primes of $k = k(\Gamma)$ (or, more generally, if for each finite $\nu \in S(A(\Gamma))$, the local field $k_\nu$ contains no quadratic extension of $\mathbb{Q}_p$, where $p$ is the rational prime that $\nu$ divides), then $\Gamma$ has a subgroup of finite index with infinite abelianization—in particular, $\mathbb{H}^3/\Gamma$ is almost sufficiently large.* □

On the other hand, using an observation of **[KLS]**, the existence of finite ramification for $A(\Gamma)$ also implies strong results about abelian quotients of finite index subgroups of $\Gamma$, as we now describe. We formulate the result in terms of $\mathbb{Q}\Gamma$, so let $\Gamma$ be a Kleinian group and denote $\mathbb{Q}(\operatorname{tr}\Gamma) = k$ and $\mathbb{Q}\Gamma = A$.

**Theorem 3.2.** *Suppose $A$ is ramified at the finite prime $\nu$ and let $p$ be the rational prime that $\nu$ divides. Then $\Gamma$ has a normal subgroup $\Delta$ with finite cyclic quotient (of order dividing $p^t - 1$ for some $t \le 2[k{:}Q]$) which is residually finite $p$, that is, the intersection of all normal subgroups of $\Delta$ of $p$-power index is trivial.*

Before we prove this theorem, we point out that being residually finite $p$ is a strong property. Since finite $p$-groups are nilpotent, it implies that the lower central $p$-series of $\Delta$ intersects in the trivial group (this is the series of subgroups

$$\Delta = \Delta^0(p) \supset \Delta^1(p) \supset \Delta^2(p) \supset \dots,$$

where $\Delta^i(p)/\Delta^{i+1}(p)$ is defined inductively to be the largest exponent $p$ quotient of $\Delta^i(p)/[\Delta, \Delta^i(p)]$ ). Moreover

**Proposition 3.3.** *Every non-abelian subgroup of a residually finite $p$ group has a $\mathbb{Z}/p \times \mathbb{Z}/p$ quotient.*

*Proof.* The subgroup is residually finite $p$, so it has a non-abelian $p$-group quotient. But any non-abelian $p$-group $G$ has a $\mathbb{Z}/p \times \mathbb{Z}/p$ quotient—if $G/[G,G]$ were cyclic, then $G/[G,[G,G]]$ would be a central extension of a cyclic group, hence abelian, so $[G,G] = [G,[G,G]]$, which contradicts the fact that a finite $p$-group is nilpotent. □

*Proof of Theorem 3.2.* (Following **[KLS]**, see also **[Vig]**.) $\Gamma$ is represented into $A^1$, the group of elements of $A$ of reduced norm 1, and hence into $A^1_\nu$. The discrete valuation on the local field $k_\nu$ extends to one on $A_\nu$. Let $\mathcal{O}_\nu$ be the valuation ring (elements of non-negative valuation in $A_\nu$ ) and $\mathcal{P}_\nu$ its maximal ideal. Then $A^1_\nu$ is contained in the multiplicative group $\mathcal{O}^\times_\nu$ of $\mathcal{O}_\nu$ and this multiplicative group has a Hausdorff filtration

$$\mathcal{O}^\times_\nu \supset (1+\mathcal{P}_\nu) \supset (1+\mathcal{P}^2_\nu) \supset \dots$$

whose quotients after the first are $\mathbb{Z}/p$-vector spaces and whose first quotient is the multiplicative group of the residue class field $\overline{l_\nu}$ of a certain quadratic extension $l_\nu$ of $k_\nu$ (the unique unramified extension of $k_\nu$). $\overline{l_\nu}$ is an extension of degree $t \leq 2[k{:}\mathbb{Q}]$ of the field of $p$ elements, so its multiplicative group is cyclic of order $p^t - 1$. □

A result of G. Mess **[Me]** is interesting in reference to Theorem 3.2 and Proposition 3.3—he shows that a 3-manifold group with a $\mathbb{Z}/p \times \mathbb{Z}/p$ quotient must either have a finite index subgroup with infinite abelianization or the sizes of the successive quotients of the derived series must grow rapidly.

**3.2.** The trace field, invariant trace field, and invariant quaternion algebra are quite strong invariants. The following proposition is trivial (and also applies to the cusp field).

**Proposition 3.4.** *Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic orbifold. If $M$ is amphicheiral* (*has an orientation reversing symmetry*) *then $\mathbb{Q}(\operatorname{tr}\Gamma) = \overline{\mathbb{Q}(\operatorname{tr}\Gamma)}$, and if $M$ is commensurably amphicheiral* (*has an orientation reversing commensurability*) *then $k(\Gamma) = \overline{k(\Gamma)}$ and $A(\Gamma) = \overline{A(\Gamma)}$, where the bar is complex conjugation.* □

For arithmetic $\Gamma$, commensurable amphicheirality is equivalent to conjugation symmetry of the invariant quaternion algebra (and is also equivalent to the existence of Fuchsian subgroups—see **[R1]**), but this is probably false in general. Nevertheless, in practice the proposition gives an effective test for amphicheirality and commensurable amphicheirality.

## 4. Short geodesics in arithmetic orbifolds

The main results of this section are Theorem 4.6 and its Corollary, which set strong limits on the possible lengths of short geodesics in arithmetic cusped hyperbolic orbifolds, thus giving a necessary condition for arithmeticity. We also describe a conjectural statement of the same type for closed arithmetic orbifolds, dependent on the "Lehmer Conjecture" of number theory. We first recall some basic relevant facts about arithmeticity of Kleinian groups, including the algebraic characterization of arithmeticity given in **[MR]** and **[R1**, Chap. 2**]**.

**4.1.** Arithmetic Kleinian groups are obtained as follows (cf., **[Bo]** and **[Vig**, Chap. 4**]**, also **[Vig]** for relevant details on quaternion algebras).

Let $k$ be a number field with one complex place and $A$ a quaternion algebra over $k$ ramified at all real archimedean places of $k$ (this means $A \otimes_k \mathbb{R}$ is the algebra of hamiltonian quaternions for each real embedding $k \hookrightarrow \mathbb{R}$). Let $\mathcal{O}$ be an order of $A$ (a finitely generated subring of the set of $A$-integers, which contains the ring $R_k$ of $k$-integers and generates $A$ as a $k$-vectorspace) and $\mathcal{O}^1$ the group of elements of norm 1 in $\mathcal{O}$. The complex place $k \hookrightarrow \mathbb{C}$ induces an embedding $\rho\colon A \hookrightarrow M_2(\mathbb{C})$ which restricts to $\rho\colon \mathcal{O}^1 \hookrightarrow \mathrm{SL}_2\mathbb{C}$. Then $\mathrm{P}\rho(\mathcal{O}^1)$ is a Kleinian group of finite covolume. An ***arithmetic Kleinian group*** is one commensurable with a group of the type $\mathrm{P}\rho(\mathcal{O}^1)$. We say it is ***derived from a quaternion algebra*** if it is actually a subgroup of some $\mathrm{P}\rho(\mathcal{O}^1)$. We call

$M=\mathbb{H}^3/\Gamma$ arithmetic or derived from a quaternion algebra if $\Gamma$ is arithmetic or derived from a quaternion algebra.

The field $k$ and quaternion algebra $A$ associated as above with an arithmetic Kleinian group $\Gamma$ are the same as the invariant trace field $k(\Gamma)$ and invariant quaternion algebra $A(\Gamma)$ of the previous section (see Theorem 4.3 below and subsequent remarks). The arithmetic Kleinian group $\Gamma$ has cusps if and only if $A$ is $\mathrm{M}_2(k)$, which holds if and only if $A$ is unramified at all places (see e.g., **[Vig**, Chap. 3**]**). We thus obtain (see also **[H]**):

**Proposition 4.1.** *A non-cocompact Kleinian group $\Gamma$ of finite covolume is arithmetic if and only if $\Gamma$ is conjugate in $\mathrm{PSL}_2\mathbb{C}$ to a group commensurable with some "Bianchi group" $\mathrm{PSL}_2\mathcal{O}_d$, where $\mathcal{O}_d$ is the ring of integers in $\mathbb{Q}(\sqrt{-d})$.* □

**4.2.** We shall make use of the characterization theorems for arithmetic Kleinian groups given in **[MR]** and **[R1]** in the special case of cusped orbifolds. We first recall these results in the general case.

**Theorem 4.2.** *Let $\Gamma$ be a Kleinian group of finite covolume. Then $\Gamma$ is arithmetic if and only if $\Gamma^{(2)}$ is derived from a quaternion algebra.* □

**Theorem 4.3.** *With $\Gamma$ as in Theorem 4.2, $\Gamma$ is derived from a quaternion algebra if and only if the following conditions hold:*
(i) *$k=\mathbb{Q}(\operatorname{tr}\Gamma)$ has just one complex place;*
(ii) *the set $\operatorname{tr}\Gamma$ consists of algebraic integers;*
(iii) *for every real place $\sigma\colon k\hookrightarrow\mathbb{R}$ the set $|\sigma(\operatorname{tr}\Gamma)|$ is bounded.* □

**Remarks.** 1. The arguments of **[R1]** and **[Tak]** show that $|\sigma(\operatorname{tr}\Gamma)|$ bounded can be replaced by $|\sigma(\operatorname{tr}\Gamma)|\le 2$.
2. It follows that when $\Gamma$ is derived from a quaternion algebra, the associated quaternion algebra is defined over $\mathbb{Q}(\operatorname{tr}\Gamma)$.

When $\Gamma$ contains parabolic elements we may deduce from these theorems (see **[H]** and **[R1]** for details):

**Proposition 4.4.** *Let $\Gamma$ be a non-cocompact Kleinian group of finite covolume. Then:*
(a) *$\Gamma$ is arithmetic if and only if $k(\Gamma)=\mathbb{Q}(\sqrt{-d})$ for some square-free $d\in\mathbf{N}$ and $\operatorname{tr}\Gamma$ consists of algebraic integers.*
(b) *$\Gamma$ is derived from a quaternion algebra if and only if $\operatorname{tr}\Gamma\subset\mathcal{O}_d$ for some $d$.* □

**Corollary 4.5.** *If $\Gamma$ is arithmetic and satisfies the condition of Corollary 2.3* (*or, more generally, $\Gamma=\Gamma^{(2)}P$*)*, then $\Gamma$ is derived from a quaternion algebra. In particular, this holds if $M=\mathbb{H}^3/\Gamma$ is a link complement in a $\mathbb{Z}/2$-homology sphere.* □

**4.3.** The main result of this section is the following theorem and its corollary.

**Theorem 4.6.** *Let $M^3=\mathbb{H}^3/\Gamma$ be a cusped arithmetic hyperbolic orbifold. If $M$ contains a geodesic of length less than $\operatorname{arccosh}\frac{3}{2}=0.9624236501\ldots$ then $\Gamma$ is commensurable with $\mathrm{PSL}_2\mathcal{O}_d$ with $d\in\{1,2,3,7,11,15,19\}$ and the geodesic has length one-half a value listed in Table 1.*

*Moreover, if* $\Gamma$ *is derived from a quaternion algebra, then the above can be improved by a factor of* $2$*: a geodesic of length less than* $2 \operatorname{arccosh} \frac{3}{2} = 1.9248473002\ldots$ *has length listed in Table 1 and* $\Gamma$ *is commensurable with a* $\mathrm{PSL}_2 \mathcal{O}_d$ *as above.*

**Table 1**

| length | d |
|---|---|
| 0.8625546276620610 | 3 |
| 0.9624236501192069 | 1 |
| 1.0612750619050357 | 1 |
| 1.0870701449957391 | 3 |
| 1.2659486384018949 | 7 |
| 1.3169578969248167 | 2 |
| 1.4183161349689732 | 2 |
| 1.4657153519472905 | 1 |
| 1.4860221248769271 | 7 |
| 1.5343944365026389 | 11 |
| 1.5667992369724111 | 3 |
| 1.6628858910586211 | 3 |
| 1.7251092553241221 | 3 |
| 1.7365960799226493 | 11 |
| 1.7400216453048509 | 15 |
| 1.7627471740390861 | 1 or 2 |
| 1.8522660627003648 | 1 |
| 1.9079255392337773 | 19 |

**Corollary 4.7.** *If the cusped hyperbolic 3-orbifold* $M = \mathbb{H}^3/\Gamma$ *contains a geodesic of length less than* $0.431277313$ ($0.862554627$ *if* $\Gamma = \Gamma^{(2)} P$) *then* $M$ *is non-arithmetic.* □

**Remarks.** 1. Let $\mathcal{O}$ be any maximal order of $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$. Using the standard description of types of maximal orders in $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ (see **[Vig**, p. 100, ex. 4.7**]**), it follows that $\mathbb{H}^3/\mathbb{P}\mathcal{O}^1$ always contains a geodesic of length $2 \operatorname{arccosh} \frac{3}{2}$, so Theorem 4.6 cannot be improved.
2. The proof of Theorem 4.6 gives the same results for ***closed*** arithmetic 3-orbifolds $M = \mathbb{H}^3/\Gamma$ with invariant trace field $k(\Gamma) = \mathbb{Q}(\sqrt{-d})$.

*Proof of Theorem 4.6.* Since for any arithmetic $\Gamma$, $\Gamma^{(2)}$ is derived from a quaternion algebra, we need only consider the case that $\Gamma$ is derived from a quaternion algebra.

We first introduce some notation. Let $M^3 = \mathbb{H}^3/\Gamma$ have finite volume and let $\gamma \in \Gamma$ be a loxodromic or hyperbolic element with $\operatorname{tr}\gamma = \lambda + \lambda^{-1}$ with $|\lambda| > 1$.

**Notation.** The length of the geodesic in $M$ determined by $\gamma$ is $2 \log |\lambda|$ and will be denoted by $\ell_0(\gamma)$. It is the real part of the "complex length" $\ell(\gamma) = 2 \log \lambda$ whose imaginary part is the holonomy of the geodesic determined by $\gamma$.

Note that, with $\gamma$ as above, $\operatorname{tr}\gamma = \pm 2\cosh\frac{\ell(\gamma)}{2}$, so

$$\begin{aligned} |\operatorname{tr}\gamma| = \left|e^{\frac{1}{2}\ell(\gamma)} + e^{-\frac{1}{2}\ell(\gamma)}\right| &\le \left|e^{\frac{1}{2}\ell(\gamma)}\right| + \left|e^{-\frac{1}{2}\ell(\gamma)}\right| \\ &= e^{\frac{1}{2}\ell_0(\gamma)} + e^{-\frac{1}{2}\ell_0(\gamma)} \\ &= 2\cosh\frac{\ell_0(\gamma)}{2}. \end{aligned}$$

Hence, if $|\operatorname{tr}\gamma| \ge 2$,

$$\ell_0(\gamma) \ge 2\operatorname{arccosh}\frac{|\operatorname{tr}\gamma|}{2},$$

so that for $\gamma$ with $|\operatorname{tr}\gamma| \ge 3$ we have $\ell_0(\gamma) \ge 2\operatorname{arccosh}\frac{3}{2}$.

Now suppose $\Gamma$ contains an element $\gamma$ with $\ell_0(\gamma) < 2\operatorname{arccosh}\frac{3}{2}$. Then $|\operatorname{tr}\gamma| < 3$. But by Proposition 4.4 (a), $\operatorname{tr}\gamma \in \mathcal{O}_d$. Thus if $\operatorname{tr}\gamma$ where real, it would be in $\mathbb{Z}$, and $\gamma$ would be parabolic or elliptic, contrary to assumption. The existence of a non-real $\tau \in \mathcal{O}_d$ with $|\tau| < 3$ imposes strong conditions on $d$ and $\tau$, as we now describe.

If $d \not\equiv -1 \pmod 4$ then $\tau = a + b\sqrt{-d}$ with $a, b \in \mathbb{Z}$ and $b \ne 0$. Then $|\tau|^2 = a^2 + db^2$, whence $d < 9$. If $d \equiv -1 \pmod 4$ then $\tau = (a + b\sqrt{-d})/2$ with $a \equiv b \pmod 2$ and $b \ne 0$. In this case $|\tau|^2 = (a^2 + db^2)/4$, and it follows that $d < 35$. We thus get a finite list of possible $d$ and for each such $d$ there is a short finite list of $\tau \in \mathcal{O}_d$ with $|\tau| < 3$. It is thus a simple matter to list the $d$ and $\tau$ with $\operatorname{Re}(2\operatorname{arccosh}\frac{\tau}{2}) < 2\operatorname{arccosh}\frac{3}{2}$, giving Table 1. □

**4.4.** For closed hyperbolic 3-orbifolds we conjecture a similar result to Corollary 4.7:

**Conjecture 4.8.** *There is a universal lower bound for the lengths of geodesics in closed arithmetic hyperbolic orbifolds.* (*The current guess is approximately* $0.09174218$, *or twice this,* $0.18348436$, *if the orbifold is derived from a quaternion algebra, see Sect. 4.5 below.*)

This conjecture would be a consequence of a classical conjecture of number theory due to D.H. Lehmer **[Le]** (see also **[Boy]** for a discussion). Suppose $P(x)$ is a monic integral polynomial of degree $n$ with roots $\theta_1, \ldots, \theta_n$. The ***Mahler measure*** of $P$ is

$$M(P) = \prod_{i=1}^{n} \max(1, |\theta_i|).$$

**Lehmer's Conjecture.** *There exists* $m > 1$ *such that* $M(P) \ge m$ *for all non-cyclotomic* $P$. *The currently conjectured value of* $m$ *is approximately* $1.176780821$ (*see Sect. 4.5 below*).

That Lehmer's conjecture implies Conjecture 4.8 follows from the following lemma, part of which is implicit in **[R1]** and **[Tak]**.

**Lemma 4.9.** *Let* $\Gamma$ *be a Kleinian (or Fuchsian) group derived from a quaternion algebra. Let* $\gamma \in \Gamma$ *be a loxodromic or hyperbolic element, and write* $\operatorname{tr}\gamma = u + u^{-1}$ *with* $|u| > 1$. *Then* $u$ *is an algebraic integer and* $u^{-1}$ *is a conjugate of* $u$. *Moreover,*

(a) *if* $\gamma$ *is loxodromic, then* $u$ *is not real, and exactly four conjugates of* $u$ *lie off the unit circle;*
(b) *if* $\gamma$ *is hyperbolic, then* $u$ *is real, and exactly two conjugates of* $u$ *lie off the unit circle;*
(c) $\ell_0(\gamma) = \log M(P)$ *or* $2 \log M(P)$ *according as* $\gamma$ *is loxodromic or hyperbolic, where* $P = P(x)$ *is the monic integral polynomial of* $u$.

*Proof.* We prove the Kleinian case; the Fuchsian case is similar. Note that $u + u^{-1} = \operatorname{tr}\gamma \in k(\Gamma) = k$. On the other hand, $u$ is not in $k$: $u \in k$ implies that the quaternion algebra is a matrix algebra, so $k = \mathbb{Q}(\sqrt{-d})$ and $u$ is a unit in this field, which contradicts the assumption $|u| > 1$. Thus $u$ has degree $2$ over $k$ and $u^{-1}$ is the other root of the minimal polynomial $u^2 - u \operatorname{tr}\gamma + 1 = 0$.

For any real embedding $\sigma: k \hookrightarrow \mathbb{R}$, extend $\sigma$ to $\psi: k(u) \to \mathbb{C}$. We claim that $|\psi(u)| = 1$. Indeed, if $|\psi(u)| \neq 1$ then for sufficiently large positive integers $t$ we have $|\sigma(\operatorname{tr}\gamma^t)| = |\psi(u^t + u^{-t})| > 2$, which contradicts Theorem 4.3 and its following Remark 1. Parts (a) and (b) of the Lemma now follow. Moreover, $M(P) = |u|^2$ or $|u|$ according as $u$ is complex or real, so part (c) also follows. □

**4.5.** Lemma 4.9 has a converse, as pointed out to us by Ted Chinburg:

**Lemma 4.10.** *Suppose* $u$ *is an algebraic integer such that* $|u| > 1$, $u^{-1}$ *is a conjugate of* $u$, *and* $u$ *satisfies one of conditions* (a) *or* (b) *of Lemma 4.9. Then, in case* (a) (*resp. case* (b)) *there is a Kleinian* (*resp. Fuchsian*) *group* $\Gamma$ *derived from a quaternion algebra and a loxodromic* (*resp. hyperbolic*) *element* $\gamma \in \Gamma$ *with* $\operatorname{tr}\gamma = u + u^{-1}$. *Moreover, in case* (b) *we can require* $\Gamma$ *to be a subgroup of a Kleinian group* $\Gamma_0$ *derived from a quaternion algebra.*

Before we prove this converse, we describe its implications for short geodesics. The conjectured smallest Mahler measure of approximately $1.176780821$ is attained by an example $P_1(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$ due to Lehmer **[Le]** given in 1933 (see **[Boy]** for a discussion). It has just one real root outside the unit circle. The smallest known Mahler measure for a polynomial with a complex root outside the unit circle is approximately $1.2013961862$, attained by an example $P_2(x) = x^{18} + x^{17} + x^{16} - x^{13} - x^{11} - x^9 - x^7 - x^5 + x^2 + x + 1$ of D. W. Boyd **[Boy]**. Lemma 4.10 implies the existence of a hyperbolic 3-orbifold $M_1$ containing a hyperbolic 2-suborbifold $N_1$ with a hyperbolic geodesic of length $2 \log M(P_1) \approx 0.324715228$, and also a hyperbolic 3-orbifold $M_2$ with a loxodromic geodesic of length $\log M(P_2) \approx 0.18348436$; all these orbifolds being derived from quaternion algebras. By Lemma 4.9, these are the conjectural shortest lengths for hyperbolic resp. loxodromic geodesics in hyperbolic 2- or 3-orbifolds derived from quaternion algebras. The conjectural shortest lengths in arithmetic orbifolds would be half these (approx. $0.162357614$ resp. $0.09174218$), and can also be realized in examples by the following theorem of Chinburg (private communication), which we prove below.

**Theorem 4.11.** *Suppose* $\Gamma$ *is a Kleinian or Fuchsian group which is derived from a quaternion algebra and* $\gamma \in \Gamma$ *is a loxodromic or hyperbolic element. Then there is a group* $\Gamma_0$ *commensurable with* $\Gamma$ *containing an element* $\gamma_0$ *with* $\gamma_0^2 = \gamma$ *in* $\mathrm{PSL}_2\mathbb{C}$.

*Proof of Lemma 4.10.* As in the proof of Lemma 4.9 we deal with the Kleinian group case; the Fuchsian case is analogous. Thus assume that $u$ is not real and has exactly four conjugates off the unit circle. Since $u^{-1}$ is assumed to be a conjugate of $u$, the conjugates of $u$ which are not on the unit circle are exactly $u$, $u^{-1}$ and their complex conjugates. Hence the only non-real conjugates of $\theta = u + u^{-1}$ are itself and $\overline{u} + \overline{u}^{-1}$. In particular the field $l_0 = \mathbb{Q}(\theta)$ has exactly one complex place whilst the field $l = \mathbb{Q}(u)$ is totally imaginary as $u$ has no real conjugates. Also note that $l$ has degree $2$ over $l_0$. We now construct a quaternion algebra over $l_0$ in which $l$ embeds.

Let $S$ be the set of real places of $l_0$ and $\nu$ a finite place of $l_0$ which is inert in $l$. Now define the set $S_1$ to be $S$ when $|S|$ is even and $S \cup \{\nu\}$ otherwise. Since $|S_1|$ is even, the classification theorem for quaternion algebras over number fields (see for example **[Vig**, Chapter 3**]**) guarantees the existence of a quaternion algebra $A$ over $l_0$ with $S_1$ as ramification set. Moreover, by construction, $l$ embeds in $A$—since the real places of $l_0$ are ramified in $l$ and $\nu$ is inert in $l$ (see **[Vig**, Theorems 1.2.8 and 3.3.8**]**).

Thinking of $l$ as a subfield of $A$, $u$ is an algebraic integer in $l$. Thus, to complete the proof we need to find an order of $A$ containing $u$. By assumption $u^{-1}$ is the distinct conjugate of $u$ over $l_0$. Hence $u$ has $l/l_0$-norm $1$. Now $A$ is a 2-dimensional $l$-vector space, so let $\tau_1, \tau_2$ be a basis. Then $\Lambda = R_l\tau_1 \oplus R_l\tau_2$ is an ideal of $A$ (see **[Vig**, Chap. 1**]**), where $R_l$ is the ring of integers of $l$, so $\mathcal{O}_\Lambda = \{x \in A : x\Lambda \subset \Lambda\}$ (the ***left order*** of $\Lambda$ in the terminology of **[Vig**, Chap. 1**]**) is an order of $A$ which clearly contains $R_l$ and hence $u$. The final sentence of Lemma 4.10 follows from **[MR]** or **[R1]**. □

*Proof of Theorem 4.11 (Chinburg).* We may assume $\Gamma = \mathrm{P}\rho(\mathcal{O}^1)$, where $\mathcal{O}$ is an order in the quaternion algebra $A = A(\Gamma)$ (cf. Sects. 4.1 and 3.1). Let $\gamma = \mathrm{P}\rho(u)$ for some $u \in \mathcal{O}^1$. Since $\gamma$ is loxodromic or hyperbolic, $\operatorname{tr}\gamma = u + u^{-1}$, and we can assume $|u| > 1$. As in the proof of Lemma 4.9, $u$ has degree $2$ over the center $k$ of $A$. Since $u \in \mathcal{O}^1$, the $k(u)/k$-norm of $u$ is $1$. By Hilbert's Theorem 90, there is a $u_0 \in k(u)$ with $u_0/\sigma(u_0) = u$, where $\sigma$ is the non-trivial element of $\mathrm{Gal}(k(u)/k)$. By multiplying $u_0$ by a suitable element of $R_k$ (the ring of integers of $k$), we may assume $u_0 \in R_{k(u)}$. Now $\sigma(u_0^2/u) = (\sigma(u_0))^2/\sigma(u) = (u_0/u)^2/u^{-1} = u_0^2/u$, so $u_0^2/u \in k^*$. It will now suffice to find an arithmetic group $\Gamma_0$ commensurable with $\Gamma$ and containing $\gamma_0 = \mathrm{P}\rho(u_0)$, since $\gamma_0^2 = \gamma$ in $A^*/k^*$.

As in the proof of Lemma 4.10, we can find a maximal order $\mathcal{O}$ in $A$ containing $u_0$. Let $R_f$ be the (finite) set of finite places of $k$ over which $A$ ramifies. Let $S$ be a finite set of places of $k$ such that $u_0$ is an $S$-unit. For $\nu \notin R_f \cup S$ and $\nu$ finite, $u_0$ is in the unit group of $\mathcal{O}_\nu = \mathcal{O} \otimes_{R_k} R_\nu$, where $R_\nu$ is the ring of integers of the completion $k_\nu$. Hence $u_0$ fixes the vertex $P_\nu$ of the Bruhat-Tits building at $\nu$ which is fixed by $\mathcal{O}_\nu^*$. Suppose now that $\nu \in S - R_f$. Because $u$ has reduced norm $1$, $u$ fixes the vertex $P_\nu$ in the building at $\nu$ which is fixed by $\mathcal{O}_\nu^*$, though $u_0$ need not fix $P_\nu$. However, since $u_0^2 = u$ in $A^*/k^*$, the action of $u_0$ on the building at $\nu$ exchanges the vertices $P_\nu$ and $u_0P_\nu$. Since the building is a tree, $u_0$ fixes the mid-point $T_\nu$ of the shortest path between these vertices. (Note that $T_\nu$ is either a vertex or a midpoint of an edge.) From Borel's description (**[Bo**, Prop. 4.4**]**) of the maximal elements in the commensurability class of $\Gamma$, there is a group $\Gamma_0$ in this commensurability class consisting of precisely those elements of $A^*/k^*$ which locally fix the same points as described above for $u_0$ on Bruhat-Tits buildings, i.e., they fix $P_\nu$ (resp. $T_\nu$) if $\nu$ is a finite place not in $R_f \cup S$ (resp. in $S - R_f$). □

## 5. Commensurators of chain links

Let $C(p,s)$ denote the complement of the link in $S^3$ consisting of a $p$-link chain with $s$ left half-twists (if $s$ is negative we mean $-s$ right half-twists, cf. Fig. 1 for $(p,s)=(6,-3)$ or in Fig. 2 for $(p,s)=(7,0)$). $C(p,s)$ has several symmetries: $\alpha$ the symmetry that rotates the chain clockwise, taking each link into the next; $\beta$ the rotation by $180°$ about the circular axis of Fig. 2; and $\gamma$ the $180°$ rotation about the horizontal axis. These elements generate a group of symmetries of order $4p$:

$$\begin{aligned} G(p,s) &= \langle \alpha,\beta,\gamma : \alpha^p = \beta^s, \beta^2 = 1, \gamma^2 = 1, \alpha\beta = \beta\alpha, \gamma\beta = \beta\gamma, \gamma^{-1}\alpha\gamma = \alpha^{-1} \rangle \\ &\cong D_{2p} \times C_2 \quad \text{if } s \text{ is even} \\ &\cong D_{4p} \quad \text{if } s \text{ is odd,} \end{aligned}$$

where $D_{2n}$ and $C_2$ denote the dihedral group of order $2n$ and the cyclic group of order 2.

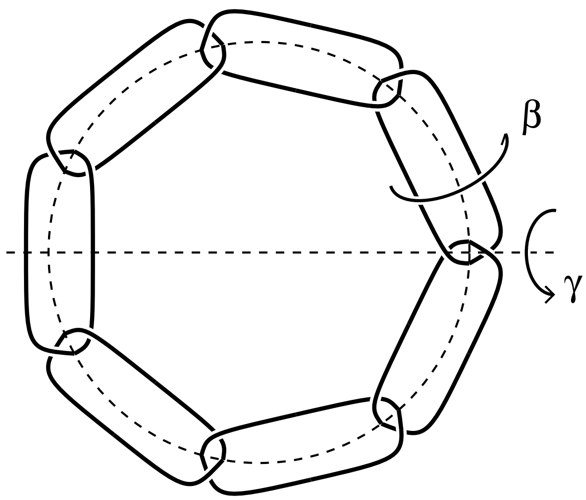


**Figure 2**

**Theorem 5.1.**

(i) *$C(p',s')$ is commensurable with $C(p,s)$ if $(p'+s',s') = \pm(p+s,s)$ or $\pm(-s,p+s)$ and is commensurable with $C(p,s)$ with reversed orientation if $(p'+s',s') = \pm(p+s,-s)$ or $\pm(s,p+s)$.*

(ii) *$C(p,s)$ has a hyperbolic structure (complete of finite volume) if and only if $\{|p+s|,|s|\} \not\subset \{0,1,2\}$.*

(iii) $C(p,s)$ *is arithmetic* (*and hence has infinitely many hidden symmetries*) *if and only if* $(|p+s|,|s|)$ *or* $(|s|,|p+s|)$ *is in* $(3,0)$, $(3,1)$, $(3,2)$, $(3,3)$, $(4,0)$, $(4,2)$, $(4,4)$, $(6,0)$. *The field of definition* $k(C(p,s))$ *is then* $\mathbb{Q}(\sqrt{-d})$ *with* $d$ *respectively* $7$, $1$, $3$, $1$, $3$, $7$, $2$, *and* $15$.

(iv) *If* $C(p,s)$ *is hyperbolic and non-arithmetic then it has no orientation preserving hidden symmetries and its full orientation preserving symmetry group is* $G(p,s)$ (*so* $\pi_1 C(p,s)$ *is normal in its orientation preserving commensurator with quotient* $G(p,s)$).

(v) *If* $C(p,s)$ *is hyperbolic and non-arithmetic then it has an orientation reversing commensurability if and only if* $p+s=\pm s$ *or* $p+s=0$ *or* $s=0$.

In the remainder of this section we give the topological part of the proof—part (i), the "only if" of (ii), and the "if" of (v). In the next section we describe the hyperbolic structures on chain link complements by means of Dehn surgery on a component of the Whitehead link and compute some of their arithmetic invariants; (ii) of the theorem is a side-product. In the subsequent two sections we prove parts (iii) and (iv) and (v).

We shall be using Dehn surgery a lot, so we briefly recall the basic terminology. Let $M$ be a 3-manifold with an end homeomorphic to $T^2\times[0,\infty)$, where $T^2$ is the 2-torus. We can cut off the end to get a 3-manifold $M_0=M-T^2\times(1,\infty)$ with a boundary component $T^2=T^2\times\{1\}$. Let $\mathbf{m},\mathbf{l}$ be some chosen basis of $H_1(T^2;\mathbb{Z})$ and let $(p,q)$ be a coprime integer pair. We can paste a solid torus onto $M_0$ at the boundary component $T^2$ to kill the homology class $p\mathbf{m}+q\mathbf{l}$. This process is called "$(p,q)$ Dehn filling" the chosen end of $M$. If the end of $M$ resulted by removing a knot $K$ from a manifold $N$, then we also speak of "$(p,q)$ Dehn surgery" on $K$.

We shall also need orbifold versions of Dehn filling and Dehn surgery. If $(p,q)$ is an integer pair that is not coprime, say $d=\gcd(p,q)$, then $(p,q)$ Dehn filling means the following: first perform $(p/d,q/d)$ Dehn filling and then give the core circle of the added solid torus a transverse angle of $2\pi/d$, so that it becomes the singular set of an orbifold structure with local group $C_d$ (the cyclic group of order $d$).

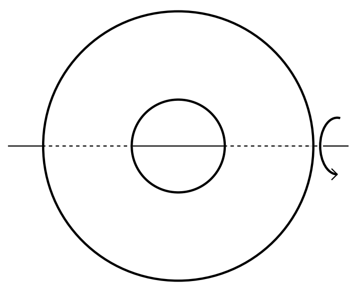

**Figure 3**

Let $\tau$ be the involution on $T^2$ which acts by multiplication by $-1$ on $H_1(T^2)$ (Fig. 3). The orbit space $P=T^2/\tau$ is a 2-sphere with four order 2 orbifold points (points with cone angle $\pi$). The involution $\tau$ extends to any solid torus $D^2\times S^1$ that $T^2$ bounds and $(D^2\times S^1)/\tau$ is the orbifold $Q$ of Fig. 4 with $\partial Q=P$. Thus, given a 3-orbifold $M$ with an end homeomorphic to $P\times[0,\infty)$, and a chosen homology basis

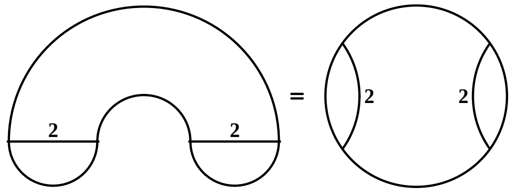


**Figure 4**

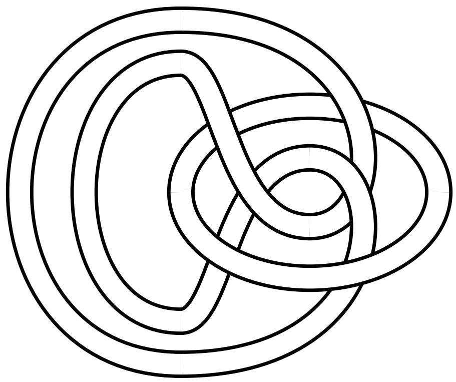

**Figure 5**

for the double cover $T^2$ of $P$, we can define $(p,q)$ Dehn filling the end of $M$ to mean the process of cutting off the end and replacing it by $Q$ in a way that induces $(p,q)$ Dehn filling by a solid torus in the double cover of the end.

*Proof of (i) and "if" of (v) of Theorem 5.1.* Let $W$ denote the complement of the Whitehead link pictured in Fig. 5. We can obtain $C(p,2q)$ by performing $(p,q)$ Dehn filling of $W$ at one component of the Whitehead link (with respect to the standard meridian/longitude pair) and then taking the $p$-fold cover of the resulting manifold or orbifold (it is an orbifold if $\gcd(p,q) \neq 1$). To obtain $C(p,s)$ this way for $s$ odd, one can give an interpretation of $(p,q)$ Dehn filling for $q$ half-integral, but instead we shall use a suitable quotient of $W$.

$W$ double covers the orbifold $W'$ pictured in Fig. 6, with one toral end and one end whose cross section is a sphere with four order 2 orbifold points. $(p,s)$ Dehn filling the toral end of $W'$ lifts to $(p,s/2)$ Dehn filling of one end of $W$. In fact, the result $W'(p,s)$ of this $(p,s)$ Dehn filling is simply the quotient of $C(p,s)$ by the subgroup $G_0 = \langle\alpha,\beta\rangle$ of $G(p,s)$. If we quotient $W'$ by the symmetry $\gamma$ we obtain the orbifold $W''$ pictured in Fig. 7. Its underlying space is $S^3$ with two balls removed, i.e., $S^2 \times I$. We have re-drawn $W''$ in Fig. 8 in a more symmetric fashion; it has an order 2 rotational symmetry about its vertical axis and reflection symmetries across appropriate vertical planes.

Let $W''(p,s)$ be the corresponding quotient of $W'(p,s)$ (so $W''(p,s)$ results by $(p,s)$ Dehn filling one end of $W'$—the end is filled by the orbifold of Fig. 4). By drawing

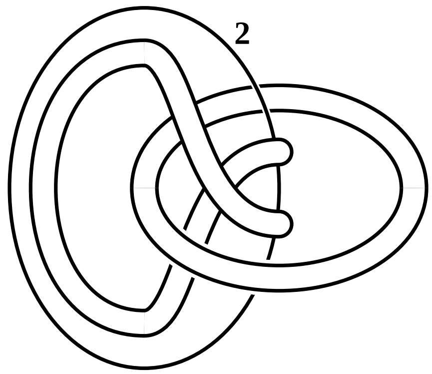


**Figure 6**

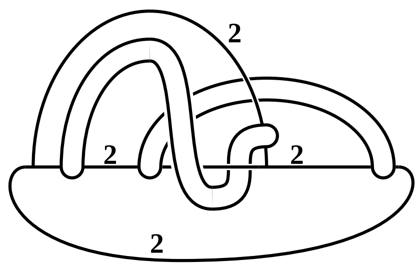


**Figure 7**

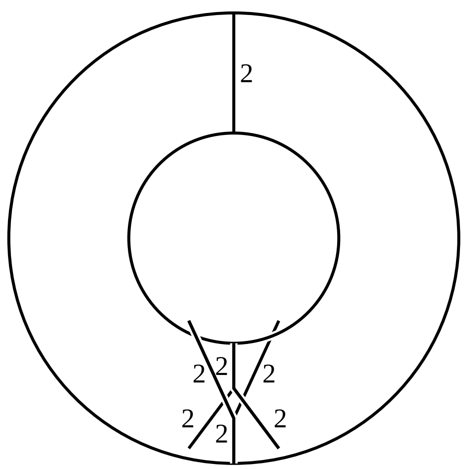


**Figure 8**

the curves killed by the Dehn filling (or by a homological calculation) it is not hard to see that the rotational symmetry of $W''$ takes $(p,s)$ Dehn filling to $(p+2s,-p-s)$ Dehn filling, so $W''(p,s) \cong W''(p+2s,-p-s)$. Note that $(p,s)$ and $(-p,-s)$ Dehn filling mean the same thing, and $(p',s') = (p+2s,-p-s)$ implies $(p'+s',s') = (s,-p-s)$, so the orientation preserving commensurability of the theorem is proved. The reflection symmetries of $W''$ take $(p,s)$ Dehn filling to $(p,-p-s)$ and $(p+2s,-s)$ Dehn filling, so $W''(p,s) \cong W''(p,-p-s)$ and $W''(p+2s,-s)$. This gives the orientation reversing commensurabilities of the theorem. □

If $W(p_1,q_1;\, p_2,q_2)$ denotes the result of Dehn filling at both cusps of $W$ (that is, $(p_1,q_1)$ and $(p_2,q_2)$ Dehn surgery on the two components of the Whitehead link) and similarly for $W''$ then the same argument shows:

**Proposition 5.2.** *$W''(p_1,s_1;\, p_2,s_2) \cong -W''(p_1,-p_1-s_1;\, p_2+2s_2,-s_2)$, where the minus sign signifies reversed orientation, so $W(p_1,q_1;\, p_2,q_2)$ is commensurable with $-W(p_1,-p_1/2-q_1;\, p_2+4q_2,-q_2)$.* □

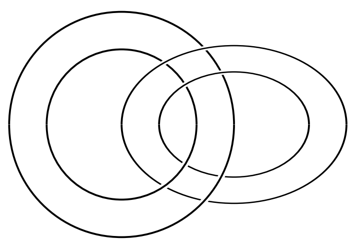

**Figure 9**

*Proof of "only if" of part (ii) of Theorem 5.1.*. We must show non-hyperbolicity for the cases claimed. For this it suffices, by the commensurability statement of part (i), to consider the five cases $(p,s) = (1,0)$, $(2,0)$, $(2,-1)$, $(3,-1)$, and $(4,-2)$. The $(p,s)$ chain link is then respectively the unknot, the $(4,2)$ cable link, the two component unlink, the $(3,3)$ cable link, and the link pictured in Fig. 9, which has an essential torus separating the pairs of parallel components. None of these admits a complete finite volume hyperbolic structure. □

## 6. Surgery on one component of the Whitehead link complement

**6.1.** In **[T]** Thurston describes how to obtain the complement $W$ of the Whitehead link (Fig. 5) by identifying faces of an ideal octahedron in pairs. The identification matches face $A$ with $A'$, $B$ with $B'$, etc., in Fig. 10, so as to respect the labeling of the edges.

If the octahedron is taken to be a regular ideal octahedron in hyperbolic space, then one obtains the complete finite volume hyperbolic structure on $W$. As described in **[T]** (see also **[NZ]**), by deforming the octahedron to differently shaped ideal hyperbolic octahedra one obtains incomplete hyperbolic structures on $W$, whose metric completions are hyperbolic Dehn surgeries and generalized Dehn surgeries on the Whitehead link.

We refer the reader to **[T]** and **[NZ]** for details on generalized hyperbolic Dehn surgery. The essential point is that a generalized Dehn surgery on $W$ is parameterized by

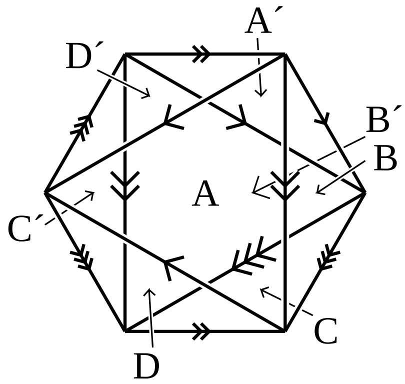


**Figure 10**

a "Dehn surgery parameter" $(p_i, q_i) \in R^2 \cup \{\infty\}$ for each cusp of $W$. The result, denoted $W(p_1, q_1; p_2, q_2)$, of this generalized hyperbolic Dehn surgery is defined if the $(p_i, q_i)$ are sufficiently close to $\infty$. Moreover, if $(p_i, q_i) = \infty$ for $i = 1$ or $2$ then the corresponding cusp of $W$ is complete—still a cusp—in $W(p_1, q_1; p_2, q_2)$, and if $(p_i, q_i) \in \mathbb{Z}^2$ then the cusp has been filled in by a geodesic along which $W(p_1, q_1; p_2, q_2)$ has the structure of a hyperbolic orbifold, or manifold if $gcd(p_i, q_i) = 1$, and the underlying topology is that of topological Dehn surgery, described in the previous section. (Precisely, if $(p_i, q_i) = (rp'_i, rq'_i)$ with $(p'_i, q'_i)$ a coprime integer pair and $r > 0$, then the cusp is filled in by a geodesic with a transverse cone angle of $2\pi/r$, while if $p_i/q_i \notin \mathbb{Q}$ then the cusp is filled by a single point at which $W(p_1, q_1; p_2, q_2)$ is not topologically a manifold.)

As in the above references, it is convenient to use an ideal triangulation of $W$ to discuss the deformations, since the shape of an ideal hyperbolic tetrahedron is determined by a single complex parameter in the upper half plane. By subdividing the octahedron of Fig. 10 as in Fig. 11, we obtain an ideal triangulation of $W$ with four simplices.

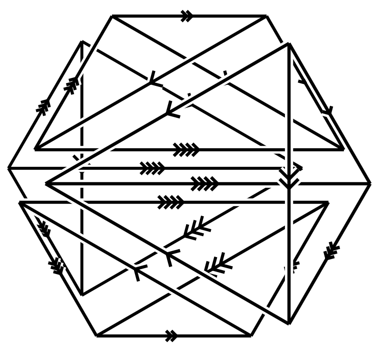

**Figure 11**

By cutting off the ends of $W$ one obtains a compact manifold-with-boundary $\overline{W}$ which can be obtained by identifying truncated tetrahedra as in Fig. 12. The two boundary tori of $\overline{W}$ are triangulated as in Fig. 13, where the vertices are labeled according to the

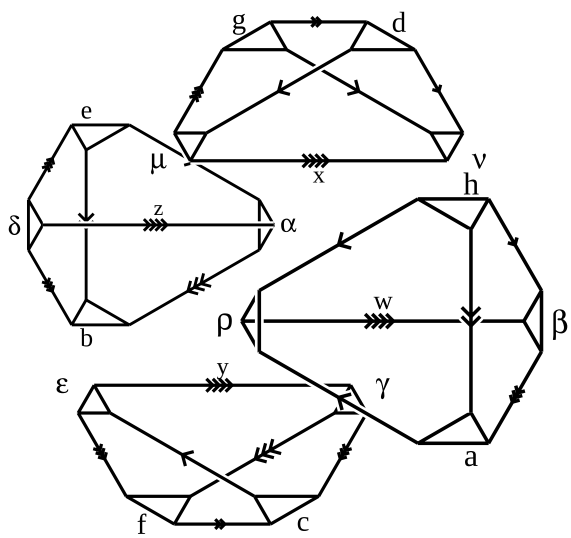


**Figure 12**

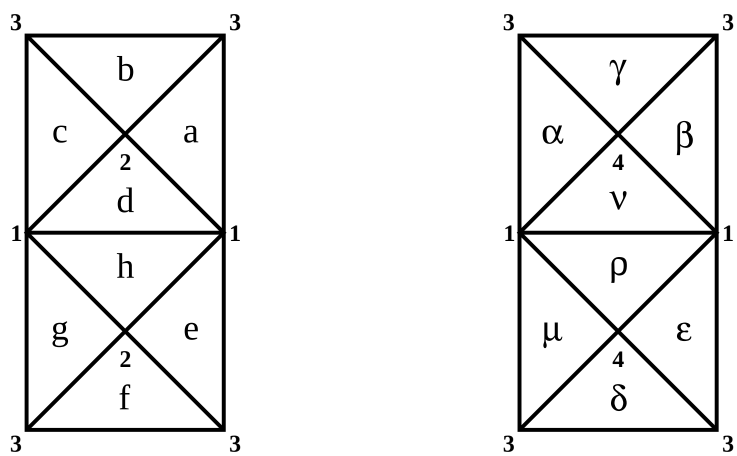


**Figure 13**

edges of the triangulation of $W$. Careful inspection shows that the standard topological meridian and longitude of each component of the Whitehead link are as indicated in Fig. 14, where we have also included labels for the complex parameters of the four tetrahedra.

**A note on orientations.** In the discussion of Dehn surgery on the figure eight knot complement in **[T]**, and also throughout **[NZ]**, a non-standard orientation convention was used for the longitude/meridian pair at a cusp, due to the cusp torus being viewed from inside the manifold rather than outside (this pair is drawn with standard orientation

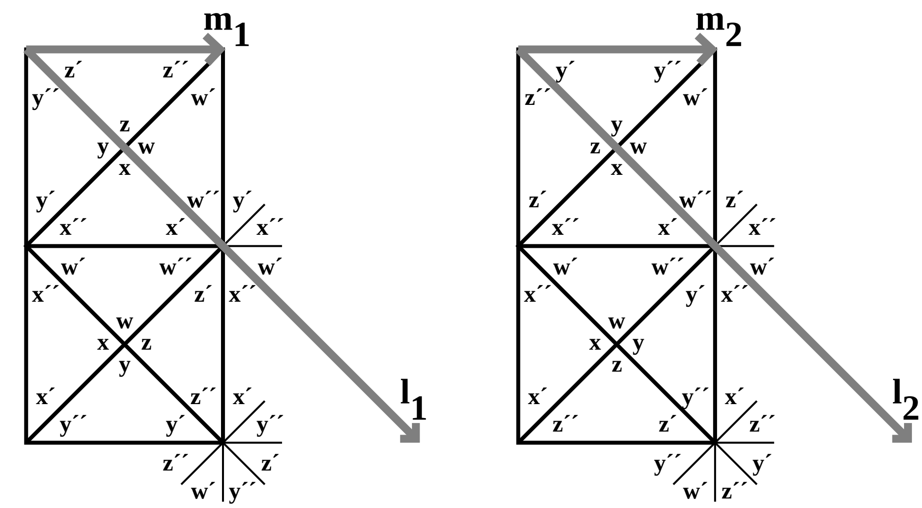


**Figure 14**

in Fig. 18 of **[NZ]**, but this was inconsistent with the text). Thus $(p,q)$ Dehn surgery in those discussions would be $(p,-q)$ Dehn surgery in the convention which we follow here, affecting some signs in some formulae.

We can read off from Fig. 14 the consistency relations (see **[NZ]** or **[T]**) at the four edges:

$$\begin{aligned}
\log w'' + \log z' + \log x'' + \log w' + \log x'' + \log y' + \log w'' + \log x' &= 2\pi i \\
\log w + \log x + \log y + \log z &= 2\pi i \\
\log z'' + \log w' + \log y'' + \log z' + \log y'' + \log x' + \log z'' + \log y' &= 2\pi i \\
\log w + \log x + \log z + \log y &= 2\pi i.
\end{aligned}$$

Here $\log$ denotes the standard branch of natural log on the complex plane split along $(-\infty, 0]$. Since $x' = \frac{x-1}{x}$ and $x'' = \frac{1}{1-x}$ and similarly for $w,\ y,\ z$, these simplify to the two relations:

$$\log w + \log x + \log y + \log z = 2\pi i \qquad (6.1a)$$

$$\log(1-w) + \log(1-x) - \log(1-y) - \log(1-z) = 0 \qquad (6.1b)$$

Similarly, we can read off the parameters $u_1,\ v_1,\ u_2,\ v_2,$ which describe the holonomy of the meridians and longitudes $\mathbf{m}_1,\ \mathbf{l}_1,\ \mathbf{m}_2,\ \mathbf{l}_2,$ at the two cusps. They are

$$\begin{aligned} u_1 &= \log y'' + \log z' + \log z'' + \log w' - \pi i \\ v_1 &= \log y'' + \log y + \log x + \log x' + \log w'' + \log z' + \log x'' \\ &\quad + \log x + \log y + \log y' + \log z'' + \log w' - 4\pi i \\ u_2 &= \log z'' + \log y' + \log y'' + \log w' - \pi i \\ v_2 &= \log z'' + \log z + \log x + \log x' + \log w'' + \log y' + \log x'' \\ &\quad + \log x + \log z + \log z' + \log y'' + \log w' - 4\pi i \end{aligned}$$

which, written in terms of $x, y, z, w$ and simplified by (6.1), becomes

$$u_1 = \log(w-1) + \log x + \log y - \log(y-1) - \pi i \tag{6.2$a$}$$
$$v_1 = 2\log x + 2\log y - 2\pi i \tag{6.2$b$}$$
$$u_2 = \log(w-1) + \log x + \log z - \log(z-1) - \pi i \tag{6.2$c$}$$
$$v_2 = 2\log x + 2\log z - 2\pi i. \tag{6.2$d$}$$

$(u_1, u_2)$ can be taken as the "analytic Dehn surgery parameter," as discussed in **[T]** and **[NZ]**, in which case $w,\ x,\ y,\ z,$ constrained by equations (6.1), become complex analytic functions of this parameter.

The real Dehn surgery parameters $(p_1, q_1)$ and $(p_2, q_2)$ are determined by the equations

$$\begin{aligned} p_1u_1 + q_1v_1 &= 2\pi i \\ p_2u_2 + q_2v_2 &= 2\pi i. \end{aligned} \tag{6.3}$$

For the complete structure on $W$ we have $u_1 = v_1 = u_2 = v_2 = 0,$ which, with (6.1) and (6.2), easily implies $x = y = z = w = i$ (this is also clear from the regularity of the octahedron for the complete structure). We shall thus deform from these values of $x, y, z, w,$ maintaining the consistency relations (6.1).

**6.2.** We are only interested in generalized Dehn surgeries on one cusp of $W,$ so we always have $(p_1, q_1) = \infty.$ The requirement that the first cusp is complete means $u_1 = v_1 = 0.$ From $v_1 = 0$ and equation (6.2b) we deduce $x = -y^{-1}.$ Equation (6.1a) then gives $z = -w^{-1}$ and (6.1b) then gives $y = w,$ so

$$(x, y, z, w) = (x, -x^{-1}, x, -x^{-1}).$$

Conversely, $(x, y, z, w) = (x, -x^{-1}, x, -x^{-1})$ satisfies the consistency relations (6.1) and the cusp relations $u_1 = v_1 = 0$ and thus gives a structure on $W$ which is complete at the first cusp. We thus just have the one complex parameter $x,$ in the upper half plane, describing our deformations.

We abbreviate $(u_2, v_2) = (u, v)$ and $(p_2, q_2) = (p, q),$ and we shall write $W(p, q)$ for $W(\infty;\, p, q).$ The formulae (6.2) simplify to

$$\begin{aligned} u &= \log x + \log(x+1) - \log(x-1) \\ v &= 4\log x - 2\pi i \end{aligned} \tag{6.4}$$

and the real Dehn surgery parameter $(p, q)$ is determined by the equation

$$pu + qv = 2\pi i. \tag{6.5}$$

Thurston's complex analytic parameter for hyperbolic Dehn surgery on one component of the Whitehead link is $u$. We shall use the parameter $x$ instead. Since $x$ describes the shape of an ideal simplex, which degenerates for $x$ real, the natural domain for the parameter $x$ is the complex upper half plane $U$. The maps $x \mapsto u$ and $x \mapsto v$ take $U$ biholomorphically to the domains $\mathcal{U}$ and $\mathcal{V}$, where $\mathcal{U}$ is the complex plane split along the rays $(-\infty, 2\log(\sqrt{2}-1)]$ and $[2\log(\sqrt{2}+1), \infty)$ and $\mathcal{V} = \{v \in \mathbb{C} : -2\pi i < \operatorname{Im} v < 2\pi i\}$. Thus $x \in U$, $u \in \mathcal{U}$, and $v \in \mathcal{V}$ are all equally good complex analytic parameters for generalized hyperbolic Dehn surgery on one component of the Whitehead link.

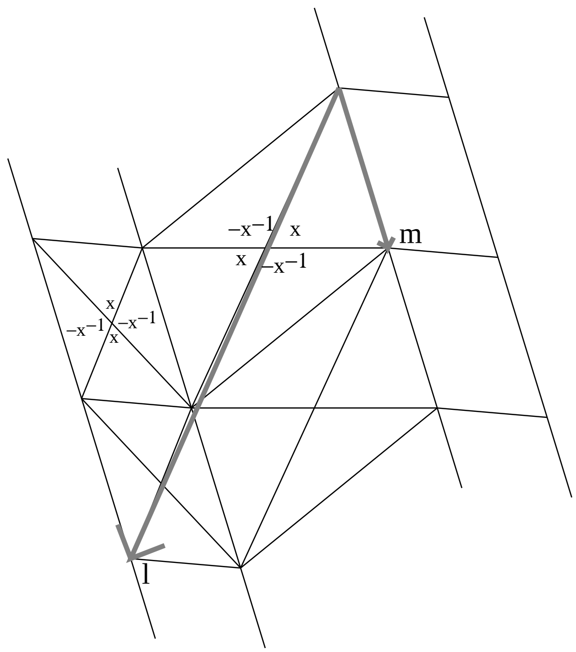


**Figure 15**

The triangulation of the cusp torus of $W(p, q)$ is shown in Fig. 15 with the meridian $\mathbf{m}$ and longitude $\mathbf{l}$ drawn in. Its isomorphism type as a complex torus $\mathbb{C}^2/\Lambda$ is described by the complex parameter $\tau = \lambda_2/\lambda_1$ with positive imaginary part, after choosing an oriented base $\lambda_1, \lambda_2$ for the lattice $\Lambda$. We can consider $\mathbf{m}$ and $\mathbf{l}$ to represent generating translations in the universal cover of the cusp torus, and we define:

**Definition.** The ***cusp parameter*** of $W(p, q)$ is the parameter $\tau$ of the cusp torus with respect to the oriented basis $\mathbf{m}, -\mathbf{l}$. We denote the cusp parameter for $W(p, q)$ by $\tau(W(p, q))$. (We make this choice of basis for consistency with **[T]** and **[NZ]**, as well as because of Theorem 6.3 below, which says this $\tau$ is often an algebraic integer; the seemingly "more natural" choice $\mathbf{l}, \mathbf{m}$ replaces $\tau$ by $-\tau^{-1}$.)

Our next theorem says that $\tau(W(p,q))$ determines $W(p,q)$. To state it we need some notation. Let $\mathcal{N}$ denote the closed parallelogram in $\mathbb{R}^2$ with vertices $\pm(-4,1), \pm(0,1)$ (Fig. 16).

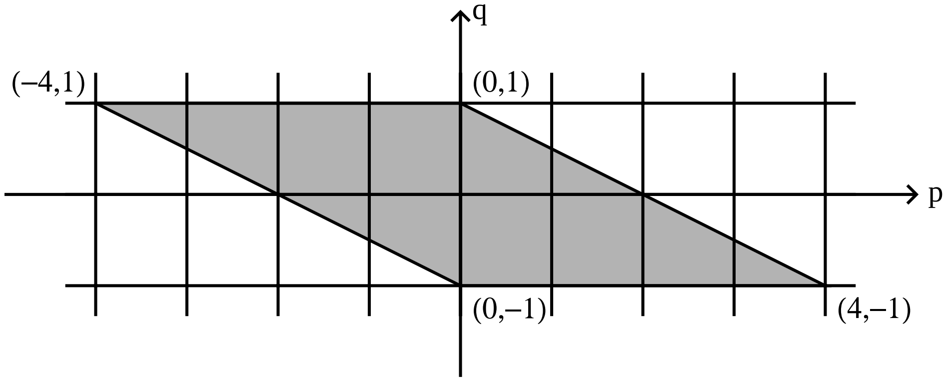


**Figure 16**

It is known (and we shall show) that this is the "non-hyperbolic" domain in the space of real Dehn surgery parameters. Since $(p,q)$ Dehn surgery and $(-p,-q)$ Dehn surgery give the same result, the space of real surgery parameters for hyperbolic Dehn surgeries is

$$\mathcal{H} = (\mathbb{R}^2 \cup \{\infty\} - \mathcal{N})/\pm 1.$$

**Theorem 6.1.** *The map $(p,q) \mapsto \tau(W(p,q))$ is a homeomorphism $\mathcal{H} \to U$, where $U$ is the complex upper half plane.*

*Proof.* By elementary calculation from Fig. 15 one obtains $\tau$ in terms of $x$ as

$$\tau = \frac{4x}{1-x^2} - 2. \tag{6.6}$$

By solving equation (6.6) for $x$ in terms of $\tau$ one verifies that this map $x \mapsto \tau$ is a 2-fold branched cover $U \to U$, branched at $x = i$, and that $x$ and $-x^{-1}$ both map to the same $\tau$. Note that replacing $x$ by $-x^{-1}$ replaces $u$ by $-u$, $v$ by $-v$, and hence $(p,q)$ by $(-p,-q)$. To prove the theorem we must thus show that the map $x \mapsto (p,q)$ is a homeomorphism of $U$ to $\mathbb{R}^2 \cup \{\infty\} - \mathcal{N}$. We shall denote the map $x \mapsto (p,q)$ by $\phi: U \to \mathbb{R}^2 \cup \{\infty\}$.

Thurston's hyperbolic Dehn surgery theorem (cf., **[T]** or **[NZ]**) asserts that $\phi$ is a homeomorphism from a neighborhood of the parameter value $x = i$ corresponding to the complete structure on $W$ to a neighborhood of $\infty$. Elementary computation shows that $\phi$ takes the boundary $\mathbb{R} \cup \{\infty\}$ of $\bar{U}$ homeomorphically to the boundary of $\mathcal{N}$ (specifically, the intervals $[-\infty,-1]$, $[-1,0]$, $[0,1]$, $[1,\infty]$ go to the segments $[(4,-1),(0,1)]$, $[(0,1),(-4,1)]$, $[(-4,1),(0,-1)]$, $[(0,-1),(4,-1)]$ respectively). It therefore suffices to show that $\phi$ has non-vanishing Jacobian at all $x \in U - \{i\}$.

Denote

$$a(x) = \log x, \qquad b(x) = \log \frac{x-1}{x+1}.$$

Since $a$ is a biholomorphic map $a\colon U \to \mathcal{A} = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}$, we may show instead that $\phi \circ a^{-1}\colon a \mapsto (p,q)$ has non-vanishing Jacobian away from $a(i) = \pi i/2$. In other words, we must show that $\partial a/\partial p$ and $\partial a/\partial q$ are linearly independent over $\mathbb{R}$ for $a \neq \pi i/2$. The defining equation for $p$ and $q$ is

$$p(a-b) + q(4a - 2\pi i) = 2\pi i.$$

Implicit differentiation of this equation with respect to $p$ and $q$ gives

$$\frac{\partial a}{\partial p}(p - p\frac{\partial b}{\partial a} + 4q) = b - a$$
$$\frac{\partial a}{\partial q}(p - p\frac{\partial b}{\partial a} + 4q) = 2\pi i - 4a,$$

so we must show that $b-a$ and $2\pi i - 4a$ are linearly independent over $\mathbb{R}$. However the map $b \circ a^{-1}\colon a(x) \mapsto b(x)$ is conjugate by $\log$ to the order 4 Möbius transformation $x \mapsto (x-1)/(x+1)$ and it is thus easy to see that if one divides $\mathcal{A}$ into four "quadrants" by the vertical and horizontal lines through $\pi i/2$, then $a \mapsto b$ rotates each quadrant clockwise into the next. It follows that the line from $b$ to $a$ never points in the same direction as the line from the center $\pi i/2$ of rotation to $a$ (unless $a = \pi i/2$). That is, $b-a$ and $\pi i/2 - a$ are independent over $\mathbb{R}$, as was to be shown. □

We now restrict to the case that $(p,q) \notin \mathcal{N}$ and $p \in \mathbb{Z}$ and $q \in \frac{1}{2}\mathbb{Z}$. Recall that $W$ covers the orbifolds $W'$ and $W''$ of Figs. 6 and 7 with covering groups $C_2$ and $C_2 \times C_2$. $W(p,q)$ thus branched covers $W'(p,2q)$ and $W''(p,2q)$ in the notation of Sect. 5, and these are hyperbolic orbifolds under our assumptions on $p$ and $q$.

Note that $(p,q) \mapsto (p+2q, 2q)$ transforms the domain $\mathcal{N}$ into the square with vertices $(\pm 2, \pm 2)$, so part (ii) of Theorem 5.1 is proven. We next compute some arithmetic invariants of our orbifolds.

**Theorem 6.2.** *Denote* $z = x - x^{-1}$ (*so* $\tau = -4/z - 2$ *and* $\mathbb{Q}(z) = \mathbb{Q}(\tau)$). *The invariant trace fields and trace fields for* $W(p,q)$, $W'(p,2q)$, *and* $W''(p,2q)$ *are as follows:*

$$\left.\begin{aligned} k(\pi_1 W(p,q)) &= \mathbb{Q}(z) \\ \mathbb{Q}(\operatorname{tr}(\pi_1 W(p,q))) &= \mathbb{Q}(\sqrt{z}) \end{aligned}\right\} q \in \mathbb{Z},$$

$$\left.\begin{aligned} k(\pi_1 W'(p,2q)) = k(\pi_1 W''(p,2q)) &= \mathbb{Q}(z) \\ \mathbb{Q}(\operatorname{tr}(\pi_1 W'(p,2q))) = \mathbb{Q}(\operatorname{tr}(\pi_1 W''(p,2q))) &= \mathbb{Q}(\sqrt{z}, i) \end{aligned}\right\} q \in \frac{1}{2}\mathbb{Z}.$$

*In particular, the cusp field equals the invariant trace field for all these examples.*

**Remark.** By Corollary 2.3 it follows that $\mathbb{Q}(z) = \mathbb{Q}(\sqrt{z})$ if $q \in \mathbb{Z}$ and $p$ is odd. In the examples we have checked, $\mathbb{Q}(z) \neq \mathbb{Q}(\sqrt{z})$ otherwise.

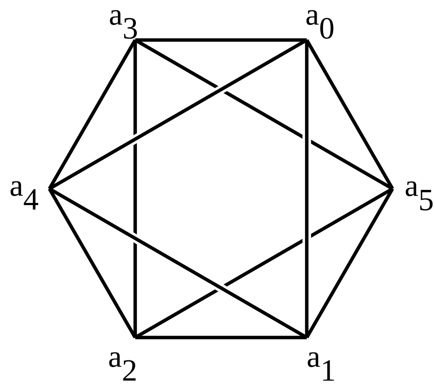


**Figure 17**

*Proof.* Denote the vertices of our original octahedron $O$ by $a_0, \dots, a_5$ as in Fig. 17. If we position $O$ in $\mathbb{H}^3$ with these vertices at $0,\ 1,\ \infty,\ -1,\ x,\ x^{-1}$, respectively, then it is easy to compute that the four constituent tetrahedra (Fig. 12) have parameters $x,\ -x^{-1},\ x,\ -x^{-1}$, so $O$ has the correct shape for the value $x$ of our deformation parameter.

Denote by $a$ the element of $\mathrm{PGL}_2\mathbb{C} = \mathrm{Isom}(\mathbb{H}^3)$ which translates $O$ across its face $A$, taking face $A'$ to $A$. Thus $a$ takes $0,\ -1,\ x^{-1}$ to $1,\ 0,\ x$ respectively. Solving for an element of $\mathrm{PGL}_2\mathbb{C}$ shows that $a$ can be represented by the matrix

$$a = \begin{pmatrix} 1 & 1 \\ 1-z & 1 \end{pmatrix} \in \mathrm{PGL}_2\mathbb{C}.$$

Similarly, if $b,\ c,\ d$ are the translations across faces $B,\ C,\ D$ respectively, one computes that

$$b = \begin{pmatrix} 1 & 1 \\ 1 & 1+z \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 1-z \\ 1 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1+z & 1 \\ 1 & 1 \end{pmatrix}.$$

If $q \in \mathbb{Z}$ then $\pi_1 W(p,q)$ is generated by $a,\ b,\ c$, and $d$. $\pi_1 W'(p,2q)$ is generated by the above elements and the additional element

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which rotates the octahedron about the axis through $a_1$ and $a_3$.

The vertices $a_0, \dots, a_3$ lie at the cusp of $W'(p,2q)$ and are all in $\mathbb{Q}(z) \cup \{\infty\}$ and the generators $a,\ b,\ c,\ d,\ \beta$, of $\pi_1 W'(p,2q)$ have entries in $\mathbb{Q}(z)$. As in the proof of Lemma 2.5, it follows that all lifts of the cusp are in $\mathbb{Q}(z)$, so $k(\pi_1 W'(p,2q)) \subset \mathbb{Q}(z)$. But $k(\pi_1 W'(p,2q))$ must contain $\tau$, which generates $\mathbb{Q}(z)$, so it equals this field.

Suppose $q \in \mathbb{Z}$. To compute $\mathbb{Q}(\mathrm{tr}\,\pi_1 W(p,q))$ we must put the generators $a,\ b,\ c,\ d$ in $\mathrm{SL}_2\mathbb{C}$ by dividing each by the square root of its determinant, namely $\sqrt{z}$. Their traces then evidently generate $\mathbb{Q}(\sqrt{z})$, so $\mathbb{Q}(\mathrm{tr}\,\pi_1 W(p,q))$ contains $\mathbb{Q}(\sqrt{z})$. On the other hand our representation of $\pi_1 W(p,q)$ then has image in $\mathrm{PSL}_2\mathbb{Q}(\sqrt{z})$, so $\mathbb{Q}(\mathrm{tr}\,\pi_1 W(p,q))$ is at most $\mathbb{Q}(\sqrt{z})$.

The additional generator $\beta$ of $\pi_1 W'(p,2q)$ is represented by the matrix $\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right) \in \mathrm{PSL}_2\mathbb{Q}(\sqrt{z}, i)$, so $\mathbb{Q}(\operatorname{tr}\pi_1 W'(p,2q)) \subseteq \mathbb{Q}(\sqrt{z}, i)$. But $a$ and $\beta b$ have traces $1/\sqrt{z}$ and $i\sqrt{z}$, so the trace field contains $\sqrt{z}$ and $i$. The same argument applies to $\pi_1 W''(p,2q)$, since the additional involution needed to generate $\pi_1 W''(p,2q)$ is the $180°$ rotation about the axis $a_0 a_2$, which is

$$\gamma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathrm{PSL}_2\mathbb{Q}(\sqrt{z}, i).$$

(Remark: $\gamma$ does not respect the octahedron after Dehn surgery.) $\square$

**Theorem 6.3.** *Suppose $(p,q) \in \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ is not in $\mathcal{N}$. $\tau = \tau(W(p,q))$ is not an algebraic integer if and only if $p = 0$ and $|q|$ is an odd prime power. $\pi_1 W'(p,2q)$ fails to have integral traces if and only if $p = 0$ or $-4q$ and $|q|$ is an odd prime power.*

*Proof.* Since $\tau = -4/z - 2$, the integrality of $\tau$ is equivalent to that of $4/z$. Moreover, to check if $\operatorname{tr}\pi_1 W'(p,2q)$ consists of algebraic integers, it suffices to check a set of generators and their pairwise products (cf. **[Mag**, p. 148**]**). Using the generators in the above proof one obtains traces $2/\sqrt{z}$, $\sqrt{z} + 2/\sqrt{z}$, multiples of these by $i$, and rational integral linear combinations of $1$, $z$, and $4/z$. So $\operatorname{tr}\pi_1 W(p,q)$ consists of algebraic integers if and only if $z$ and $4/z$ (and hence also their square roots) are algebraic integers.

By (6.4) and (6.5), $x$ satisfies the equation

$$\left(\frac{x(x+1)}{x-1}\right)^p x^{4q} = 1, \tag{6.7}$$

which multiplies out to a monic polynomial with rational integer coefficients and constant term $\pm 1$ unless $p + 4q = 0$. Thus $x$ is a unit, whence $z = x - x^{-1}$ is an algebraic integer, if $p + 4q \neq 0$. On the other hand, if $x$ satisfies equations (6.4) and (6.5) for $(p,q)$ then one checks that $(1-\bar{x})/(1+\bar{x})$ satisfies (6.4) and (6.5) for $(p+4q, -q)$. But replacing $x$ by $(1-\bar{x})/(1+\bar{x})$ replaces $z = x - x^{-1}$ by $4/\bar{z}$. Thus $4/\bar{z}$—and hence also $4/z$—is an algebraic integer if $p \neq 0$.

To complete the proof it suffices to show that when $p = 0$, $z$ is an algebraic integer divisor of $4$ if and only if $|q|$ is not an odd prime power. (Recall, that since $q \in \frac{1}{2}\mathbb{Z}$, $2q$ is an arbitrary integer.) Assume $p = 0$. Then (6.4) and (6.5) imply that $x/i = \exp(2\pi i/4q)$, so $z/i = \exp(2\pi i/4q) + \exp(-2\pi i/4q)$. Now $z$ will be a divisor of $4$ if and only if $\zeta := (z/i)\exp(2\pi i/4q)$ is. But $\zeta$ divides $4$ if and only if its norm divides $4$. The norm of $\zeta = \exp(2\pi i/2q) + 1$ is $\Phi(1)$, where $\Phi$ is the minimal polynomial for $-\exp(2\pi i/2q)$. Now, $-\exp(2\pi i/2q)$ is a primitive $r$-th root of unity with $r = |4q|$, $|q|$, or $|2q|$, according as $2q$ is an odd integer, twice an odd integer, or divisible by $4$. Thus $\Phi$ is the $r$-th cyclotomic polynomial, and it is known that $\Phi(1) = 1$ unless $r$ is the power of a prime, in which case $\Phi(1)$ equals this prime (cf. e.g., **[W]**). Thus $\Phi(1)$ fails to divide $4$ if and only if $r$ is an odd prime power, in which case $q = \pm r$. $\square$

An interesting example is $W(0,3)$ (or $W(12,-3)$, which is commensurable with it). The invariant trace field $k(\pi_1 W(0,3))$ is $\mathbb{Q}(\sqrt{-3})$, but $\pi_1 W(0,3)$ has non-integral traces, so it cannot be commensurable with $\mathrm{PSL}_2\mathcal{O}_3$. However its volume turns out to be

$20$ times the volume of $\mathbb{H}^3/\mathrm{PSL}_2\mathcal{O}_3$—this is $10/3$ times the volume of a regular ideal tetrahedron.

## 7. Arithmeticity of chain links

*Proof of part (iii) of Theorem 5.1.* Recall that $C(p,s)$ is a cover of $W'(p,s)$ and if $W'(p,s)$ is hyperbolic, then it is double branched covered by $W(p,q)$ with $s=2q$. We claim that the geodesic $\gamma(p,s)$ added by hyperbolic Dehn filling $W'$ to form $W'(p,s)$ has length

$$\ell_0(\gamma(p,s)) = -\frac{d}{s}\,\mathrm{Re}(u) = \frac{d}{2p}\,\mathrm{Re}(v), \quad d = \gcd(p,s), \tag{7.1}$$

where $u$ and $v$ are as in equations (6.4) and (6.5). Indeed, if $\ell(\gamma(p,s))$ is the complex length of the added geodesic then, by equation (6.5) and by **[Th]** or **[NZ]**, we have equations

$$pu + s\frac{v}{2} = 2\pi i$$
$$p'u + s'\frac{v}{2} = \ell(\gamma(p,s)),$$

where $p'$ and $s'$ are integers satisfying $ps' - sp' = d$. Multiplying the first of these two equations by $s'$ and the second by $s$ and subtracting gives $du = 2\pi i s' - s\ell(\gamma(p,s))$, from which the first equality of (7.1) follows by taking real part; the second equality follows similarly.

The computations that we now describe were computer-aided, so we omit some details. We wish to find the $(p,s)$ for which $\ell_0(\gamma(p,s)) \geq 0.431277313$ (cf. Corollary 4.7). We use equations (7.1). First, by plotting level curves in the $(p+s,s)$-plane for $\mathrm{Re}(u)$ and $\mathrm{Re}(v)$ one finds that the conditions $|\mathrm{Re}(u)| \geq 0.43$ and $|\mathrm{Re}(v)/2| \geq 0.43$ force $(p+s,s)$ to lie in a bounded domain (which lies entirely within the circle of radius $15$). It is then quick to verify that the $(p,s)$ for which $\ell_0(\gamma(p,s)) \geq 0.431277313$ are those with $(|p+s|,|s|)$ or $(|s|,|p+s|)$ in $\{(n,0) : 3 \leq n \leq 14\} \cup \{(n,n) : 3 \leq n \leq 7\} \cup \{(3,1),\,(3,2),\,(4,2)\}$. Thus, only in these cases might $W'(p,s)$ be arithmetic.

The table of Theorem 4.6 of possible short geodesic lengths in cusped arithmetic orbifolds now eliminates all cases but those of part (iii) of Theorem 5.1, namely $(|p+s|,|s|)$ or $(|s|,|p+s|)$ equal to one of $(3,0)$, $(3,1)$, $(3,2)$, $(3,3)$, $(4,0)$, $(4,2)$, $(4,4)$, $(6,0)$. (We could also have eliminated all but these and $(6,6)$ by the observation that a cusped arithmetic orbifold can have torsion only of order 2, 3, 4, or 6, since the trace of the square of the torsion element must be in some $\mathcal{O}_d$.)

To show that the above cases do give arithmetic orbifolds we use the characterization of arithmeticity in Proposition 4.4. We must thus verify that in these cases the invariant trace field $k(\pi_1 W'(p,s))$ is $\mathbb{Q}(\sqrt{-d})$ with $d$ respectively $7,\ 1,\ 3,\ 1,\ 3,\ 7,\ 2,\ 15,$ and that $W'(p,s)$ has integral traces. By Theorem 6.2 and the proof of Theorem 6.3 we must check that $z$ and $4/z$ lie in the appropriate $\mathcal{O}_d$. This is evident from Table 2, which includes also the non-arithmetic case $(6,6)$. $\square$

**Table 2.** $z$ **and** $4/z$

| $\{\|p+s\|, \|s\|\}$ | $z$ | $4/z$ |
|---|---|---|
| $\{3,0\}$ | $\frac{-3+\sqrt{-7}}{2}$ | $\frac{-3-\sqrt{-7}}{2}$ |
| $\{3,1\}$ | $-1+\sqrt{-1}$ | $2-2\sqrt{-1}$ |
| $\{3,2\}$ | $\frac{-1+\sqrt{-3}}{2}$ | $-2-2\sqrt{-3}$ |
| $\{3,3\}$ | $+\sqrt{-1}$ | $-4\sqrt{-1}$ |
| $\{4,0\}$ | $-1+\sqrt{-3}$ | $-1-\sqrt{-3}$ |
| $\{4,2\}$ | $\frac{-1+\sqrt{-7}}{2}$ | $-1-\sqrt{-7}$ |
| $\{4,4\}$ | $+\sqrt{-2}$ | $-2\sqrt{-2}$ |
| $\{6,0\}$ | $\frac{-1+\sqrt{-15}}{2}$ | $\frac{-1-\sqrt{-15}}{2}$ |
| $\{6,6\}$ | $+\sqrt{-3}$ | $\frac{-4\sqrt{-3}}{3}$ |

It is remarkable that the arithmetic $W'(p,s)$ are picked out exactly by Theorem 4.6. We had conjectured the following alternate characterization of the arithmetic cases, and T. Chinburg showed us a proof, which we sketch.

**Proposition 7.1.** *$k(\pi_1 W'(p,s))$ is a quadratic imaginary field if and only if $(p,s)$ is one of the arithmetic cases above or $(|p+s|,|s|)=(6,6)$ (for which $W'(p,s)$ does not have integral traces).*

*Sketch Proof.* Suppose $\mathbb{Q}(z)$ is imaginary quadratic. Since $z = x - x^{-1}$, $x$ has degree at most $4$ over $\mathbb{Q}$. If $p=0$ or $p+4q=0$ then equation (6.7) shows that $x$, respectively $(x+1)/(x-1)$, is a root of unity of degree at most $4$ over $\mathbb{Q}$. This gives finitely many cases to check. If neither $p$ nor $p+4q$ is zero, then the proof of Theorem 6.3 shows that $z$ is an algebraic integer divisor of 4 in an imaginary quadratic number field. This again gives finitely many cases to check. □

We have computed the minimal polynomial for the generating element $z$ of $k(\pi_1 W'(p,s))$ whenever $|p+s|$ and $|s|$ are at most $13$. The degree is always at most $\max\{|p+s|,|s|\}-1$, as is not hard to prove, but is often smaller, as Table 3 shows.

## 8. Commensurators of chain links: proofs

In this section we prove parts (iv) and (v) of Theorem 5.1.

**Lemma 8.1.** *Let $(p,s)$ be such that $C(p,s)$ is non-arithmetic. The cusp parameter $\tau$ is not in $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, unless $(|p+s|,|s|)=(6,6)$.*

*Proof.* Since $\tau+2=-4/z$ we must just show that $z$ is not in $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}\sqrt{-3})$. If $p=0$ then equations (6.4) and (6.5) give $x=\exp((s+2)\pi i/2s)$, and since $z=x-x^{-1}$, it is easy to see that $z$ is in $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$ if and only if $|s|=3,\ 4,$ or $6$. The

**Table 3.** $[k(W'(p,s)):\mathbb{Q}]$

| $\lvert p+s\rvert$ \ $\lvert s\rvert$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | 2 | 2 | 4 | 2 | 6 | 4 | 6 | 4 | 10 | 4 | 12 |
| 1 | | | | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 2 | | | | 2 | 2 | 4 | 3 | 6 | 4 | 8 | 5 | 10 | 6 | 12 |
| 3 | 2 | 2 | 2 | 2 | 3 | 4 | 4 | 6 | 7 | 6 | 9 | 10 | 8 | 12 |
| 4 | 2 | 3 | 2 | 3 | 2 | 4 | 3 | 6 | 4 | 8 | 5 | 10 | 6 | 12 |
| 5 | 4 | 4 | 4 | 4 | 4 | 4 | 5 | 6 | 7 | 8 | 8 | 10 | 11 | 12 |
| 6 | 2 | 5 | 3 | 4 | 3 | 5 | 2 | 6 | 4 | 6 | 5 | 10 | 4 | 12 |
| 7 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 8 | 4 | 7 | 4 | 7 | 4 | 7 | 4 | 7 | 4 | 8 | 5 | 10 | 6 | 12 |
| 9 | 6 | 8 | 8 | 6 | 8 | 8 | 6 | 8 | 8 | 6 | 9 | 10 | 8 | 12 |
| 10 | 4 | 9 | 5 | 9 | 5 | 8 | 5 | 9 | 5 | 9 | 4 | 10 | 6 | 12 |
| 11 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 11 | 12 |
| 12 | 4 | 11 | 6 | 8 | 6 | 11 | 4 | 11 | 6 | 8 | 6 | 11 | 4 | 12 |
| 13 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 |

cases $|s| = 3$ or $4$ are arithmetic, so the only case is $|s| = 6$. If $p + 2s = 0$ then $C(p,s)$ is commensurable with $C(0,s)$ by part (i) of Theorem 5.1, so again, the only non-arithmetic case with $z$ in $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$ is $|s| = 6$. Finally, if $p \neq 0$ and $p + 2s \neq 0$ then the proof of Theorem 6.3 shows that $z$ is an algebraic integer divisor of $4$. The only algebraic integer divisors of $4$ with positive imaginary part in $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$ are $z = i,\ 2i,\ 4i,\ 1+i,\ -1+i,\ 2+2i,\ -2+2i,\ \omega,\ 2\omega,\ 4\omega,\ -\bar{\omega},\ -2\bar{\omega}$, and $-4\bar{\omega}$, where $\omega = \exp(2\pi i/3)$. One computes directly that these correspond to $(p+s,s) = (3,3),\ (\infty,\infty),\ (3,-3),\ (1,3),\ (3,1),\ (1,-3),\ (3,-1),\ (2,3),\ (0,4),\ (2,-3),\ (3,2),\ (4,0),\ (3,-2)$, respectively, which are arithmetic cases, and thus excluded in the lemma. $\square$

*Proof of Theorem 5.1 (iv).* An orbifold cusp is called ***rigid*** if its fundamental group is the non-abelian split extension of $\mathbb{Z}/n$ by $(\mathbb{Z} \oplus \mathbb{Z})$ with $n = 3,\ 4$, or $6$. In this case the parameter $\tau$ of a flat torus covering a horosphere section of the cusp must be in $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$.

Suppose $C(p,s)$ is nonarithmetic and $(|p+s|,|s|) \neq (6,6)$. By Lemma 8.1, $C(p,s)$ cannot cover an orbifold with a rigid cusp. On the other hand, by the Thurston/Jørgenson theory (cf., **[T**, Chapter 5**]**) the quotient $W''(p,s)$ of $C(p,s)$ has volume less than $\mathrm{vol}(W'') \approx 0.918$. If $W''(p,s)$ non-trivially covered the orientable commensurator quotient of $C(p,s)$, then this commensurator quotient would have volume at most half this value, namely $<\approx 0.459$. But this volume bound puts the commensurator quotient on Adams' list of smallest orientable orbifolds with non-rigid cusps (**[A1]**), and these orbifolds are all arithmetic. Thus $W''(p,s)$ is the orientable commensurator quotient.

Finally, suppose $|(p+s|,|s|) = (6,6)$. By Table 2, the cusp parameter of $W''(p,s)$ lies in $\mathbb{Q}(\sqrt{-3})$. Now, $W''(p,s)$ cannot cover an orbifold with a non-rigid cusp by

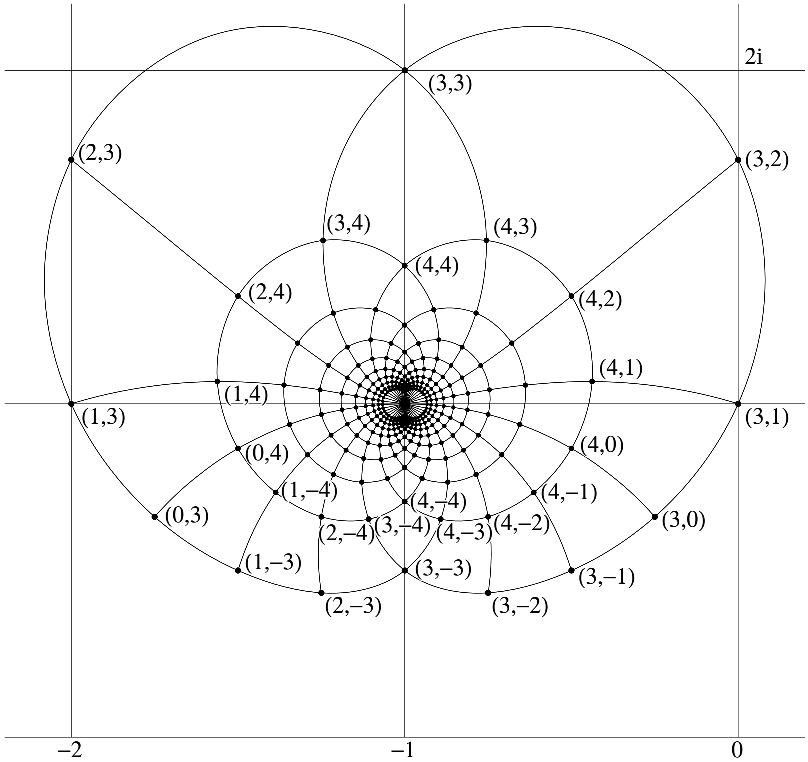


**Figure 18**

the argument of the previous paragraph, so if it non-trivially covers some orbifold, that orbifold must have a $(2,3,6)$ or $(3,3,3)$ cusp. But $W''(p,s)$ has volume $5v_0/6$, where $v_0$ is the volume of the regular ideal tetrahedron, and Adams, in these Proceedings (**[A2]**), shows that the four orbifolds with rigid cusps of type $(2,3,6)$ or $(3,3,3)$ and with volume less than $v_0/4$ have volumes $v_0/12$, $v_0/6$, $v_0/6$, or $5v_0/24$. $W''(p,s)$ cannot cover any of these because they are arithmetic (cf. **[NR2]**). Thus the orientable commensurator quotient of $W''(p,s)$ has volume at least $v_0/4$, so $\pi_1 W''(p,s)$ has index $1$, $2$, or $3$ in its orientation preserving commensurator. The cusp of $W''(p,s)$ cannot 2-fold cover a $(2,3,6)$ or $(3,3,3)$ cusp, because of the 3-torsion. If the commensurator had index $3$ then either $W''(p,s)$ or some $2$-fold cover of $W''(p,s)$ would have a $3$-fold symmetry. By considering the set of $3$-orbifold points, it is not hard to eliminate the existence of such a symmetry. □

*Proof of Theorem 5.1 (v).* Recall that we have already shown that the $C(p,s)$ of part (v) of Theorem 5.1 have orientation reversing commensurabilities, so we must just show that the others do not. If $C(p,s)$ has an orientation reversing commensurability, then its orientable commensurator quotient $W''(p,s)$ admits an orientation reversing symmetry, so the cusp of $W''(p,s)$ also does. The result thus follows from the following lemma. □

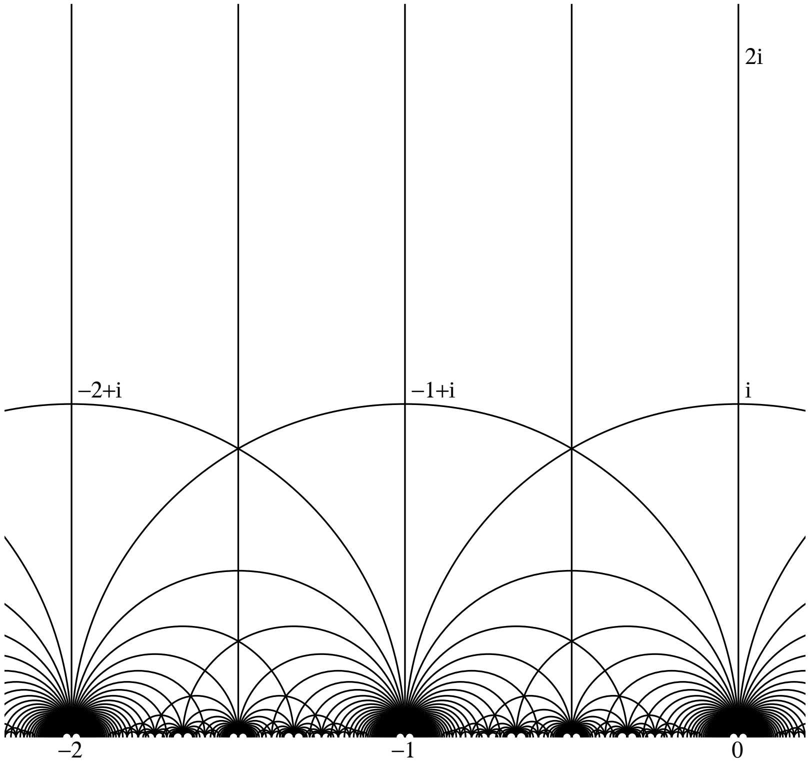


**Figure 19**

**Lemma 8.2.** *The cusp of* $W''(p,s)$ *has an orientation reversing symmetry if and only if* $p+s=0$, $s=0$, $p+s=\pm s$, *or* $W''(p,s)$ *is arithmetic.*

*Proof.* The complex parameter of the cusp of $W''(p,s)$ is $\tau(p,q)/2$ with $q = s/2$ (by the parameter for a flat $(2,2,2,2)$-orbifold we mean the parameter of its unique 2-fold torus covering space). In Fig. 18 we show the image in $(\tau/2)$-space of the integer lattice in $(p+s,s)$-space (excluding the non-hyperbolic region $\mathcal{N}$). The moduli space of flat $(2,2,2,2)$-orbifolds is the moduli space of flat tori, namely $U/\mathrm{PSL}_2\mathbb{Z}$, where $U$ is complex upper half-space. Orientation reversal induces an involution on $U/\mathrm{PSL}_2\mathbb{Z}$ (induced by $\tau \mapsto -\overline{\tau}$) whose fixed point set gives the set of moduli of orbifolds which admit an orientation reversing isometry. The lift of this set to $U$ is shown in Fig. 19. Fig. 20 superimposes the previous two figures—it is clear that the only $(p+s,s)$ lattice points for which the cusp of $W''(p,s)$ admits an orientation reversing isometry are those with $p+s=0$, $s=0$, or $p+s=\pm s$, or $\{|p+s|,|s|\} \in \{\{3,1\},\{3,2\},\{4,2\}\}$. By Theorem 5.1 part (iii), this proves the lemma. □

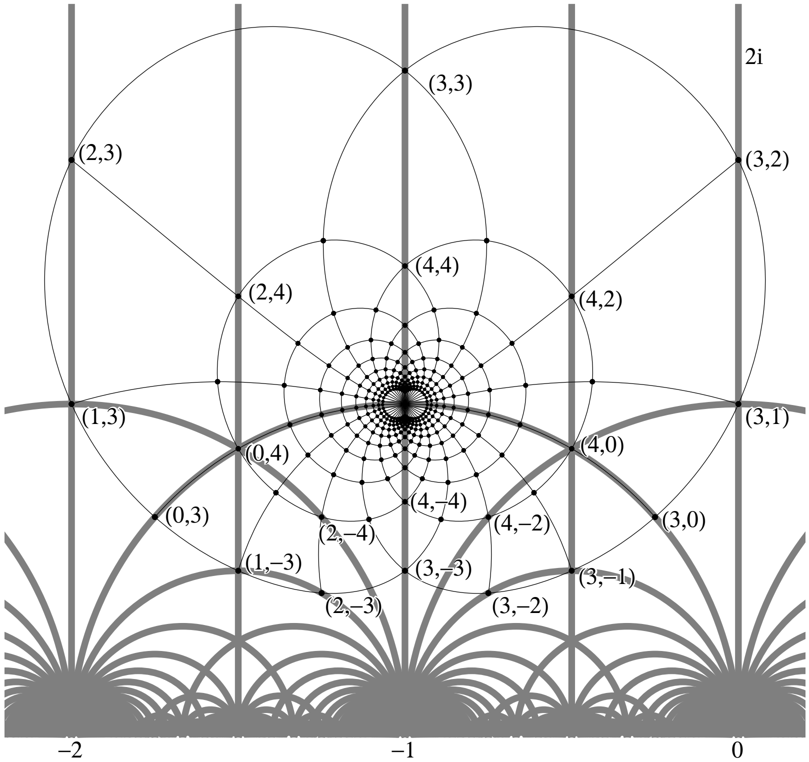


**Figure 20**

## 9. The dodecahedral knots of Aitchison and Rubinstein

**9.1.** One of the original motivating questions for this work was the question: does there exist a hyperbolic knot other than the figure-eight knot whose complement has hidden symmetries (i.e., the commensurator of its fundamental group is larger than its normalizer)?

Of course an arithmetic knot complement would have hidden symmetries, but by **[R2]**, the figure-eight knot complement is the only such.

**Proposition 9.1.** *The following are equivalent for a hyperbolic knot complement other than the figure-eight knot complement:*
(1) *it has hidden symmetries;*
(2) *its orientable commensurator quotient has a rigid cusp;*
(3) *the knot complement non-normally covers some orbifold.*

*Proof.* Clearly (1) is equivalent to (3). A hyperbolic knot complement (even in a rational homology sphere) has a homologically determined longitude, which must be preserved up to sign by any isometry. Since the orientable isometry group must act effectively on the cusp torus, it can only be cyclic or $\mathbb{Z}/2$ extended by a cyclic group, so the quotient has a non-rigid cusp. Thus (2) implies (1). If a knot complement covers an orbifold with a non-rigid cusp then the covering must be normal by Lemma 4 of **[R2]**, so (1) implies (2). □

We now describe a pair of knot complements, constructed by Aitchison and Rubinstein in their paper **[AR]** in these Proceedings, which are mutually commensurable and which do have hidden symmetries.

Let $D$ denote the regular ideal dodecahedron in hyperbolic 3-space. All its dihedral angles are $\pi/3$. As described in **[AR]** there exist two distinct hyperbolic knot complements $D_f$ and $D_s$ obtained by identifying faces of two copies of $D$. Let $\Gamma$ be the group generated by the order 120 group of symmetries of the ideal dodecahedron $D$ and all reflections in faces of $D$. This $\Gamma$ is clearly the tetrahedral group determined by the tetrahedron $T[5,2,2;2,3,6]$, which is a fundamental domain for the symmetry group of $D$. Let $\Gamma^+$ be the orientation preserving subgroup of $\Gamma$. Clearly, the hyperbolic 3-orbifold $\mathbb{H}^3/\Gamma^+$ has a rigid cusp of type $(2,3,6)$ and is covered by $D_f$ and $D_s$. It is in fact the orientable commensurator quotient of $D_f$ and $D_s$ (and $\mathbb{H}^3/\Gamma$ is therefore the non-orientable commensurator quotient), for the following reason: by **[Mey]**, $\mathbb{H}^3/\Gamma^+$ has volume approximately $0.343003$, and if there were a smaller quotient, it would have to have volume at most half this, which would contradict the results of Adams **[A2]**.

The question at the beginning of this section may now be updated:

**Question 1.** *Does there exist a hyperbolic knot other than the figure-eight knot and the two dodecahedral knots with hidden symmetries*?

As pointed out in Sect. 8, if a hyperbolic manifold covers an orbifold with a rigid cusp then the cusp parameter is in $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$. The only knot complements with this property that we know are the complements of the above three knots.

**Question 1′.** *Is there a knot other than one of the above three whose complement is hyperbolic with cusp parameter in* $\mathbb{Q}(\sqrt{-1})$ *or* $\mathbb{Q}(\sqrt{-3})$?

As described above, the dodecahedral knot groups are subgroups of a polyhedral group, so they also answer the question studied by Y. W. Lee in **[L1]** and **[L2]**: can a polyhedral group contain a knot group of finite index other than the figure-eight knot group (which is in the polyhedral group $\mathrm{PGL}_2\mathcal{O}_3$)? One can ask if these three knots are the only examples of this phenomenon.

## 10. Further comments

**10.1.** We first discuss hyperbolic knot complements.

**Question 2.** *When is the trace field equal to the cusp field for hyperbolic knot complements*?

The only known examples where this fails are the two dodecahedral knots. They have cusp field $\mathbb{Q}(\sqrt{-3})$ and trace field $\mathbb{Q}(\sqrt{-3},\sqrt{5})$. Indeed, the cusp field is obvious. The trace field certainly contains $\sqrt{5}$ because of the element of order $5$ in $(\Gamma^+)^{(2)}$ and is no bigger than $\mathbb{Q}(\sqrt{-3},\sqrt{5})$ because $(\Gamma^+)^{(2)}$ can be generated by the three matrices

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad \begin{pmatrix}0 & -\omega^{-1}\\ \omega & (1+\sqrt{5})/2\end{pmatrix},\quad \begin{pmatrix}0 & -\omega\\ \omega^{-1} & 0\end{pmatrix},$$

where $\omega = e^{\pi i/3}$.

**Question 3.** *Do hyperbolic knot complements always have integral traces*?

We know no counterexamples. R. Riley has informed us that none of the examples he has computed gives a counterexample. In many cases Bass' theorem **[Ba2]**, quoted in the introduction, forces integral traces. Bass' theorem does not apply to the knot $8_{17}$ (this knot complement has a closed incompressible surface of genus 2, cf. **[Lo]**), but Riley informs us that it nevertheless has integral traces. Note that, by Theorem 6.3, the *link* complements $C(p,2q)$ with $p=-4q$ and $q$ an odd prime power have non-integral traces. These chain links belong to a general class of links, called "star links" in **[O]**, which also includes knots. Thus the class of star knots might be a place to look for counterexamples, but we doubt this.

If the answer to both questions 2 and 3 is positive for a particular knot complement other than the figure-eight complement, then this knot complement has no hidden symmetries, for if it did, then Proposition 4.4(a) would imply arithmeticity by the comments preceding Question $1'$.

In contrast to the situation for knot complements, it is easy to construct many 1-cusp orbifolds with cusp field smaller than the invariant trace field. For example, take an orbifold with two non-rigid cusps which are separated by a 2-sphere with three orbifold points and perform Dehn filling of one of the cusps. This will not affect the cusp parameter of the other one, so all the Dehn fillings will have the same cusp field, but their invariant trace fields will differ. See **[NR1]** for more details.

**10.2.** We do not yet know an example of a cocompact Kleinian group $\Gamma$ with invariant quaternion algebra $A(\Gamma)$ equal to the matrix algebra $\mathrm{M}_2(k(\Gamma))$, but they probably exist.

We have already seen an example of non-commensurable groups with the same $A(\Gamma)$, namely, $\pi_1 W''(0,6)$ is non-arithmetic but has $A(\Gamma)=\mathrm{M}_2(\mathbb{Q}(\sqrt{-3}))$, the same as $\mathrm{PGL}_2\mathcal{O}_3$. However, knot complement examples of this exist also. Namely, the knot $5_2$ and the $(-2,3,7)$-pretzel knot both have invariant trace field equal to $\mathbb{Q}(\theta)$ where $\theta^3-\theta^2+2\theta-1=0$, but they are non-commensurable because they have the same volume, $\approx 2.82812208$, but different cusp volumes (cf. **[HMW]**; commensurable non-arithmetic one-cusp orbifolds of the same volume must have the same cusp volume because they have a mutual orbifold quotient—the commensurator). J. Weeks has pointed out that they are cut-and-paste equivalent by cutting and pasting along an immersed geodesic thrice punctured sphere.

## References


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