VANISHING CYCLES AND MONODROMY OF COMPLEX POLYNOMIALS

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1. Introduction

In this paper we describe the trivial summand for monodromy around a fibre of a polynomial map $\mathbb{C}^n \to \mathbb{C}$, generalising and clarifying work of Artal Bartolo, Cassou-Noguès and Dimca [2], who proved similar results under strong restrictions on the homology of the general fibre and singularities of the other fibres. They also showed a polynomial map $f\colon \mathbb{C}^2 \to \mathbb{C}$ has trivial global monodromy if and only if it is "rational of simple type" in the terminology of Miyanishi and Sugie. We refine this result and correct the Miyanishi-Sugie classification of such polynomials, pointing out that there are also non-isotrivial examples.

Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a non-constant polynomial map. The polynomial describes a family of complex affine hypersurfaces $f^{-1}(c)$, $c \in \mathbb{C}$. It is well-known that the family is locally trivial, so the hypersurfaces have constant topology, except at finitely many *irregular* fibres $f^{-1}(c_i), i = 1,\dots,m$ whose topology may differ from the generic or *regular* fibre of f.

Definition 1.1. If $f^{-1}(c)$ is a fibre of $f\colon \mathbb{C}^n \to \mathbb{C}$ choose ϵ sufficiently small that all fibres $f^{-1}(c')$ with $c' \in D^2_\epsilon(c) - \{c\}$ are regular and let $N(c) := f^{-1}(D^2_\epsilon(c))$. Let $F = f^{-1}(c')$ be a regular fibre in $N(c)$. Then

$$V_q(c) := \operatorname{Ker}\bigl(H_q(F;\mathbb{Z}) \to H_q(N(c);\mathbb{Z})\bigr)$$
$$V^q(c) := \operatorname{Cok}\bigl(H^q(N(c);\mathbb{Z}) \to H^q(F;\mathbb{Z})\bigr)$$

are the groups of *vanishing q-cycles* and *vanishing q-cocycles* for $f^{-1}(c)$. They have the same rank, which we call the *number of vanishing q-cycles for $f^{-1}(c)$*.

Choose a regular value c_0 for f and paths γ_i from c_0 to c_i for $i = 1,\dots,m$ which are disjoint except at c_0. We can use these paths to refer homology or cohomology of a regular fibre near one of the irregular fibres $f^{-1}(c_i)$ to the homology or cohomology of the "reference" regular fibre $F = f^{-1}(c_0)$.

The fundamental group $\Pi = \pi_1\bigl(\mathbb{C} - \{c_1,\dots,c_m\}\bigr)$ acts on the homology $H_*(F;\mathbb{Z})$ and cohomology $H^*(F;\mathbb{Z})$. If this action is trivial we say that f has "trivial global monodromy group". This action has the following generators.

Let $h_q(c_i)\colon H_q(F) \to H_q(F)$ and $h^q(c_i)\colon H^q(F) \to H^q(F)$ be the monodromy about the fibre $f^{-1}(c_i)$ (obtained by translating the fibre F along the path γ_i until close to the fibre $f^{-1}(c_i)$, then in a small loop around that fibre, and back along γ_i). We are interested in the fixed group $H^q(F)^{h^q(c_i)} = \operatorname{Ker}\bigl(1 - h^q(c_i)\bigr)$ of this local monodromy.

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Theorem 1.2. *For*[1] $q > 0$ *the maps* $H^q(F;\mathbb{Z}) \to V^q(c_i)$ *induce an isomorphism*

$$H^q(F;\mathbb{Z}) \cong \bigoplus_{i=1}^{m} V^q(c_i)$$

Under this isomorphism we have:

$$\operatorname{Ker}\bigl(1 - h^q(c_j)\bigr) = K^q(c_j) \oplus \bigoplus_{i \neq j} V^q(c_i),$$

for some $K^q(c_j) \subset V^q(c_j)$. *Hence, the subgroup of cohomology fixed under global monodromy is*

$$H^q(F;\mathbb{Z})^{\Pi} = \bigoplus_{i=1}^{m} K^q(c_i).$$

Finally, there is a natural short exact sequence

$$0 \to \operatorname{Cok}\bigl(1 - h^{q-1}(c)\bigr) \to H^{q+1}(N(c), \partial N(c)) \to K^q(c) \to 0.$$

The above exact sequence lets one compute $\dim(K^q(c))$ inductively in terms of numbers of vanishing cycles and betti numbers of $H^*(N(c), \partial N(c))$. The following theorem localises the computation of $H^*(N(c), \partial N(c))$ into the singular fibre.

Theorem 1.3. *Let* $H_*(f^{-1}(c), \infty)$ *denote* $H_*(f^{-1}(c), U)$, *where* U *is a regular neighbourhood of infinity (e.g.,* $U = \{z \in f^{-1}(c) : ||z|| > R\}$ *for large* R*). Then we have a natural isomorphism* $H^{q+1}(N(c), \partial N(c)) \cong H_{2n-q-1}(f^{-1}(c), \infty)$

Under the assumptions that F has homology only in dimension $(n-1)$ and that all singularities of fibres of f are isolated, Artal Bartolo, Cassou-Nogués, and Dimca [2] proved the dimension formulae for $\operatorname{Ker}\bigl(1 - h^{n-1}(c)\bigr)$ and $H^{n-1}(F;\mathbb{Z})^{\Pi}$ that follow from the above theorems. Polynomials $f(x_1, \dots, x_n) = x_1 g(x_2, \dots, x_n)$ are examples of polynomials with trivial global monodromy that do not satisfy their assumptions for $n > 2$.

The first displayed formula of Theorem 1.2 was proved (in homology) by Broughton [3], see also [4] and [18]. The homology version of the above results is:

Theorem 1.4. *For* $q > 0$ *the inclusions* $V_q(c_i) \to H_q(F;\mathbb{Z})$ *induce an isomorphism*

$$H_q(F;\mathbb{Z}) \cong \bigoplus_{i=1}^{m} V_q(c_i).$$

Moreover, $\operatorname{Im}\bigl(1 - h_q(c)\bigr) \subset V_q(c)$, *and there is a natural exact sequence*

$$0 \to \operatorname{Im}\bigl(1 - h_q(c)\bigr) \to V_q(c) \to H_{q+1}(N(c), \partial N(c)) \to \operatorname{Ker}\bigl(1 - h_{q-1}(c)\bigr) \to 0.$$

and an isomorphism $H_{q+1}(N(c), \partial N(c)) \cong H^{2n-q-1}(f^{-1}(c), \infty)$.

The groups $H_{2n-2}(f^{-1}(c), \infty)$ and $H^{2n-2}(f^{-1}(c), \infty)$ are freely generated by the fundamental classes of the irreducible components of $f^{-1}(c)$, so for $q = 1$ the above results give:

Corollary 1.5. *If* r_c *is the number of irreducible components of* $f^{-1}(c)$ *then* $K^1(c) \cong \mathbb{Z}^{r_c - 1}$ *and* $V_1(c)/\operatorname{Im}\bigl(1 - h_1(c)\bigr) \cong \mathbb{Z}^{r_c - 1}$. □

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[1] All results hold also for $q = 0$ if H^q and H_q are read as reduced (co)homology throughout.

In particular, the 1-dimensional monodromy with $\mathbb{Q}$-coefficients about a fibre $f^{-1}(c)$ is trivial if and only if the number of irreducible components of this fibre exceeds by one the number of its vanishing 1-cycles. For an irreducible fibre this says this monodromy is trivial if and only if the fibre has no vanishing 1-cycles.

By Theorem 1.6 below, this generalises Michel and Weber's positive answer in [11] to Dimca's question whether the local monodromy around a reduced and irreducible irregular fibre of a polynomial $f\colon \mathbb{C}^2 \to \mathbb{C}$ must be non-trivial (the same answer is implicit in Theorem 1 of [2]). The conditions are needed here: one can find $f\colon \mathbb{C}^2 \to \mathbb{C}$ having an irregular fibre with trivial local monodromy, this fibre having any number of components. Such examples exist with non-reduced fibres or fibres of arbitrarily high genus.

Recall that a polynomial $f\colon \mathbb{C}^n \to \mathbb{C}$ is *primitive* if its regular fibres are irreducible; equivalently, it is not of the form $g \circ h$ with $g\colon \mathbb{C} \to \mathbb{C}$ and $h\colon \mathbb{C}^n \to \mathbb{C}$ polynomial maps and $\deg g > 1$.

Theorem 1.6. *A fibre of a primitive polynomial $f\colon \mathbb{C}^2 \to \mathbb{C}$ has no vanishing cycles if and only if it is regular.*

By [18] this result holds in any dimension for a fibre with "isolated W-singularities at infinity."

We shall prove all the above theorems in Section 2. When the fibre $f^{-1}(c)$ is reduced with isolated singularities, there is a quick alternative proof of Corollary 1.5 for homology. Namely, let F_0 be the "non-singular core" of $f^{-1}(c)$ obtained by intersecting $f^{-1}(c)$ with a very large ball and then removing small regular neighbourhoods of its singularities. Then F_0 can be isotoped into a nearby regular fibre F and it is not hard to see (e.g., [15]):

Proposition 1.7. *Under the above assumption, $H_q(F, F_0)$ is isomorphic to $V_q(c)$ by an isomorphism that fits in the commutative diagram*

$$\begin{array}{ccc} H_q(F,F_0) & \xrightarrow{\cong} & V_q(c) \\ \uparrow & & \downarrow \subseteq \\ H_q(F) & \xrightarrow{1-h_q} & H_q(F) \end{array}$$

Since F_0 has r_c topological components, Corollary 1.5 for homology follows in this case using $q = 1$ and the long exact homology sequence for the pair (F, F_0).

The following consequence of the monodromy results was proved by Artal Bartolo et al. [2]. The improvement of the second sentence is Dimca [7].

Theorem 1.8. *A primitive polynomial $f\colon \mathbb{C}^2 \to \mathbb{C}$ has trivial global monodromy group if and only if f is rational of simple type, in the sense of Miyanishi and Sugie* [12]. *The same conclusion already follows if only the monodromy $h_1(\infty)$ at infinity of f is trivial ($h(\infty)$ is the monodromy around a very large circle in $\mathbb{C}$; so $h_1(\infty)$ is the product of the $h_1(c_i)$).*

A polynomial $f\colon \mathbb{C}^2 \to \mathbb{C}$ is "rational" if its generic fibre is rational (i.e., genus zero). "Simple type" means that if we take a nonsingular compactification $Y = \mathbb{C}^2 \cup E$ of $\mathbb{C}^2$ such that f extends to a holomorphic map $\overline{f}\colon Y \to \mathbb{C}P^1$ then $\overline{f}$ is of degree 1 on each "horizontal" irreducible component of the compactification divisor E (E is a union of smooth rational curves $E_1, \dots, E_n$ with normal crossings and a component E_i is called *horizontal* if $\overline{f}|E_i$ is non-constant).

We discuss Theorem 1.8 and refinements of it in Section 3. In the final Section 4 we describe corrections to Miyanishi and Sugie's classification of rational polynomials of simple type. Details of this will appear elsewhere.

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2. Proofs of the main theorems

Proof of Theorem 1.2. The direct sum statement has been proved in [3] (see also [18]) but a proof is quick so we include it for completeness and notation.

For each irregular value c_i we construct a neighbourhood $N_i = f^{-1}(D^2_\epsilon(c_i))$ of the corresponding irregular fibre as in Definition 1.1, with ϵ chosen small enough that the disks $D^2_\epsilon(c_i)$ are disjoint. Let c_0 be a regular value outside all these disks and choose disjoint paths γ_i joining c_0 to each disk $D^2_\epsilon(c_i)$. Let $P = \bigcup_{i=1}^m \gamma_i$ and $D = \bigcup_{i=1}^m D^2_\epsilon(c_i)$ so $K = P \cup D$ is the union of these paths and disks. Then $\mathbb{C}^n$ deformation retracts onto $f^{-1}(K)$. The Mayer-Vietoris sequence for $f^{-1}(K) = f^{-1}(P) \cup f^{-1}(D)$ gives

$$0 \to H^q(F) \oplus \bigoplus_{i=1}^m H^q(N_i) \to \bigoplus_{i=1}^m H^q(F) \to 0, \quad (q > 0). \tag{1}$$

Since the i-th summand of the sum $\bigoplus_1^m H^q(N_i)$ maps trivially to all but the i-th summand of $\bigoplus_1^m H_q(F)$, this shows:

Proposition 2.1. *$H^q(N_i) \to H^q(F)$ is injective with cokernel (by Definition 1.1) $V^q(c_i)$.* ☐

Thus, factoring source and target of the middle isomorphism of (1) by the subgroup $\bigoplus H^q(N_i)$ gives the desired isomorphism

$$H^q(F) \xrightarrow{\cong} \bigoplus_{i=1}^m V^q(c_i). \tag{2}$$

The long exact sequence for the pair (N_i, F) shows that we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^q(N_i) \arrow[r] \arrow[d, equal] & H^q(F) \arrow[r] \arrow[d, equal] & H^{q+1}(N_i, F) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & H^q(N_i) \arrow[r] & H^q(F) \arrow[r] & V^q(c_i) \arrow[r] & 0\,.
\end{tikzcd}$$

We now claim that we can identify the long exact sequence of the triple $(N_i, \partial N_i, F)$ as follows:

$$\begin{tikzcd}
H^q(\partial N_i, F) \arrow[r] \arrow[d, "\cong"] & H^{q+1}(N_i, \partial N_i) \arrow[r] \arrow[d, equal] & H^{q+1}(N_i, F) \arrow[r] \arrow[d, "\cong"] & H^{q+1}(\partial N_i, F) \arrow[d, "\cong"] \\
H^{q-1}(F) \arrow[r] & H^{q+1}(N_i, \partial N_i) \arrow[r] & V^q(c_i) \arrow[r] & H^q(F)\,.
\end{tikzcd}$$

The first and fourth vertical isomorphisms are seen by thickening F within ∂N and then using excision and the Künneth formula:

$$H^q(\partial N_i, F) \cong H^q(F \times I, F \times \partial I) \cong H^{q-1}(F).$$

We have already shown the third vertical isomorphism. Thus the above diagram is proved.

Consider now the composition $H^q(F) \to V^q(c_i) \to H^q(F)$ where the second map is the map of the above diagram. Tracing the definitions, we see it is the composition: $H^q(F) \to H^{q+1}(\partial N_i, F) \to H^q(F)$, where the first map is boundary map for the pair. We claim this composition is $1-h^q(c_i)$. Indeed, ∂N_i is isomorphic to the mapping torus $F \times_{h(c_i)} S^1$, so there is a map $F \times I \to N_i$ which identifies the ends of $F \times I$ by $h(c_i)$. The map $H^{q+1}(\partial N_i, F) \to H^q(F)$ is induced by the map of chain groups $C_q(F) \to C_{q+1}(N_i)$ which takes a q-chain A in F to the $(q+1)$-chain $A \times I$ in $F \times I$ mapped to N_i. The boundary map $H^q(F) \to H^{q+1}(N_i, F)$ is induced by the map which lifts a $(q+1)$ chain in (N_i, F) to the corresponding chain in $F \times I$ and then takes its boundary. Composing these maps of chains clearly gives the chain map $1 - h(c_i)_\sharp$. The induced composition in q-cohomology is thus $1 - h^q(c_i)$, as claimed.

Since the composition $H^q(F) \to V^q(c_i) \to H^q(F)$ is $1 - h^q(c_i)$ and $H^q(F) \to V^q(c_i)$ is surjective with kernel $\bigoplus_{j \neq i} V^q(c_j)$, it follows that $\operatorname{Ker}\bigl(1 - h^q(c_i)\bigr)$ contains $\bigoplus_{j \neq i} V^q(c_j)$. It hence has the form $K^q(c_i) \oplus \bigoplus_{j \neq i} V^q(c_j)$ in terms of the isomorphism of (2). Thus the second statement of Theorem 1.2 follows. The final exact sequence of the theorem then follows by replacing the first term of the bottom sequence of the above diagram by its image and the last arrow by its kernel. □

Proof of Theorem 1.3. Let N_i^0 be $f^{-1}(D^2_\epsilon(c_i)) \cap D^{2n}$ where D^{2n} is first chosen of large enough radius that $f^{-1}(c_i)$ is transverse to the boundary of it and all larger disks (for the existence of such a radius, even if $f^{-1}(c_i)$ has non-isolated singularities, see e.g., Proposition 2.3.1 of [1]), and ϵ is then re-chosen small enough that ∂D^{2n} is transverse to $f^{-1}(c_i')$ for all $c_i' \in D^2_\epsilon(c_i)$. Put $\partial_0 N_i^0 := \partial N_i \cap N_i^0$ and $F_i^0 := f^{-1}(c_i) \cap D^{2n}$ and $C_i := f^{-1}(c_i) - \operatorname{int}(F_i^0)$. Then the inclusion of $N_i - C_i$ in N_i is a homotopy equivalence and the inclusion of ∂N_i into $\partial N_i \cup (N_i - N_i^0 - C_i)$ is a homotopy equivalence, so we have: $H^{q+1}(N_i, \partial N_i) \cong H^{q+1}(N_i - C_i, \partial N_i \cup (N_i - N_i^0 - C_i))$. Excision then shows this is isomorphic to $H^{q+1}(N_i^0, \partial N_i^0 - \partial F_i^0)$, and this equals $H^{q+1}(N_i^0, \partial_0 N_i^0)$ by homotopy equivalence. Putting $\partial_1 N_i^0 := \partial N_i^0 - \operatorname{int}(\partial_0 N_i^0)$, Poincaré-Lefschetz duality gives $H^{q+1}(N_i^0, \partial_0 N_i^0) \cong H_{2n-q-1}(N_i^0, \partial_1 N_i^0)$. But the pair $(N_i^0, \partial_1 N_i^0)$ is homotopy equivalent to $(F_i^0, \partial F_i^0)$. By excision $H_*(F_i^0, \partial F_i^0) = H_*(f^{-1}(c_i), \infty)$. □

The proof of the homology versions of these results is essentially the same so we omit it.

Proof of Theorem 1.6. Since f is primitive, any non-reduced fibre of of f has more than one component and thus has vanishing cycles by Corollary 1.5. For a reduced fibre, on the other hand, the desired result is implicit in several places in the literature. For instance, it follows immediately from the result that for primitive $f\colon \mathbb{C}^2 \to \mathbb{C}$ the Euler characteristic of an irregular fibre always exceeds that of the regular fibre (Proposition 1 of Suzuki[2] [20]), together with the following easy facts:

1. The number of vanishing cycles for $f^{-1}(c_i)$ is $\chi(N_i) - \chi(F)$.
2. For an irregular fibre in any dimension $\chi(f^{-1}(c_i)) = \chi(N_i)$ (see e.g., [2], comments preceding Théorème 3).

[2] The case of reduced fibre, which is all we need here, is already in [19]. Reference [9] is often cited for this, but only seems to prove the case of a non-singular fibre.

It also follows from the more general result of Siersma and Tibăr [18] that a polynomial fibre in any dimension with "isolated W-singularities at infinity" is regular if and only if it has no vanishing cycles, since a reduced fiber of $f\colon \mathbb{C}^2 \to \mathbb{C}$ satisfies this condition. □

3. Discussion of Theorem 1.8

In this section we assume f is primitive; this is clearly implied by global trivial monodromy. We give a mild improvement (implicit in [7]) of Theorem 1.8.

Proposition 3.1. 1. *The global monodromy on the closed fibre $\overline{F}$ is trivial if and only if f has rational generic fibres.*

2. *If we consider the subgroup $B \subset H_1(F)$ generated by small loops around the punctures of F, then the global monodromy restricted to B is trivial if and only if $\overline{f}$ is degree 1 on all horizontal curves.*

3. *The above two statements also hold with "global monodromy" replaced by "monodromy at infinity."*

Proof. Let $Y = \mathbb{C}^2 \cup E$ be as described just after Theorem 1.8. E is a union of smooth rational curves $E_1, \dots, E_n$ with normal crossings. Since $H_1(F)$ and $H_1(\overline{F})$ are torsion free, triviality of monodromy for homology or cohomology are equivalent.

1. Deligne's invariant cycle theorem [6] gives an epimorphism $H^1(Y) \to H^1(\overline{F})^{\Pi}$. But $H^1(Y) = 0$, so if the global monodromy is trivial then $H^1(\overline{F}) = 0$. The converse is trivial.

2. Note that if $\overline{f}$ is degree > 1 on some horizontal curve E then the homology classes represented by the punctures where $\overline{F}$ meets E get permuted non-trivially as we circle a branch point of $\overline{f}|E$.

3. This follows as in [7] by the same proofs as above if we replace Y by a neighbourhood Y_0 of the fiber $\overline{f}^{-1}(\infty)$ and apply the invariant cycle theorem of [5] ([7] cites Theorem 7.13 in [8]), which says that $H^1(Y_0) \to H^1(\overline{F})^{h(\infty)}$ is surjective. $H^1(Y_0) = 0$ since Y_0 retracts onto $\overline{f}^{-1}(\infty)$, which is a simply connected union of some of the rational curves E_i. □

We can refine the proof of the second part of the proposition to obtain a stronger result. Let $p_{i1}, \dots, p_{ik_i}$ be the points where $\overline{f}^{-1}(c_i)$ meets horizontal curves and for each $j = 1, \dots, k_i$ let δ_{ij} be the degree of $\overline{f}$ on a small neighbourhood of the point p_{ij} in its horizontal curve. Thus, the generic fibre F near $f^{-1}(c_i)$ has δ_{ij} punctures near p_{ij} that are cyclically permuted by the monodromy around c_i. It follows that the restriction of $1 - h_1(c_i)$ to the subgroup B of the above proposition has image of dimension $\sum_{j=1}^{k_i}(\delta_{ij} - 1)$. Denote

$$\begin{aligned} e_{c_i} :&= \dim \operatorname{Im}\big(1 - h_1(c_i)\big) - \dim \operatorname{Im}\big(\big(1 - h_1(c_i)\big)|B\big) \\ &= \dim \operatorname{Im}\big(1 - h_1(c_i)\big) - \sum_{j=1}^{k_i}(\delta_{ij} - 1). \end{aligned}$$

This measures the "extra" part of $\operatorname{Im}\big(1 - h_1(c_i)\big)$ that does not arise from the homology at infinity.

It is clear that if $e_{c_i} = 0$ then the local monodromy $\overline{h}_1(c_i)\colon H_1(\overline{F}) \to H_1(\overline{F})$ of the closed fibre around $\overline{f}^{-1}(c_i)$ is trivial. The converse is *not* true for arbitrary maps of a surface, but the following theorem implies that it is for our local monodromy map.

Theorem 3.2. *With* $\overline{V}_1(c_i) := \operatorname{Ker}(H_1(\overline{F}) \to H_1(\overline{N}_i))$, *we have*

$$\operatorname{Im}\bigl(1 - \overline{h}_1(c_i)\bigr) \subset \overline{V}_1(c_i)$$

and both these groups have rank e_{c_i}. *Moreover*

$$\sum_{i=1}^{m} e_{c_i} \geq 2\,\operatorname{genus}(F).$$

Proof. The inclusion $\operatorname{Im}(1 - \overline{h}_1(c_i)) \subset \overline{V}_1(c_i)$ is clear, while the fact that they have the same dimension is shown in part 2c) of section III of [2]. We sketch the argument. The dimension equality in question is, in fact, generally true for a neighbourhood $N = \overline{f}^{-1}(D)$ of a singular fiber of a fibration $\overline{f}$ of a projective surface over a curve. Here D is a disk about a point c and $\overline{F}_c = \overline{f}^{-1}(c)$ is the only singular fibre over this disk. The ingredients are:

- $\overline{N}$ deformation retracts onto $\overline{F}_c$;
- $H_1(\overline{F}) \to H_1\bigl(\overline{F}_c\bigr) = H_1(\overline{N})$ is surjective mod torsion (actually strictly surjective in our case since f is primitive); thus $\dim V_1(c) = \dim \operatorname{Ker}\bigl(H_1(\overline{F}) \to H_1(\overline{N})\bigr) = \dim H_1(\overline{F}) - \dim H_1\bigl(\overline{F}_c\bigr)$;
- $\dim(\operatorname{Im}(1 - \overline{h}_1(c))$ also equals $\dim H_1(\overline{F}) - \dim H_1\bigl(\overline{F}_c\bigr)$ (this is a consequence of the fact that the nullity of the intersection form of N is 1).

We leave the details of each of these ingredients to the reader (or see [2] and [10]).

As in [2], following Kaliman [10], we denote the above number:

$$k_{c_i} := \dim \operatorname{Im}(1 - \overline{h}_1(c_i)) = \dim \overline{V}_1(c_i) = \dim H_1(\overline{F}) - \dim H_1\bigl(\overline{f}^{-1}(c_i)\bigr)$$

We must show $k_{c_i} = e_{c_i}$.

We have a short exact sequence

$$0 \to B \to H_1(F) \to H_1(\overline{F}) \to 0 \tag{3}$$

and taking the image of $1 - h(c_i)$ applied to this sequence gives a sequence

$$0 \to \mathbb{Z}^{\sum_j(\delta_{ij}-1)} \to \operatorname{Im}\bigl(1 - h_1(c_i)\bigr) \to \operatorname{Im}\bigl(1 - \overline{h}_1(c_i)\bigr) \to 0. \tag{4}$$

This sequence is exact except possibly at its middle term (this holds for a homomorphic image of any short exact sequence). The cokernel of $\mathbb{Z}^{\sum_j(\delta_{ij}-1)} \to \operatorname{Im}\bigl(1 - h_1(c_i)\bigr)$ has dimension, by definition, e_{c_i}. Since the sequence induces a surjection of this cokernel to $\operatorname{Im}\bigl(1 - \overline{h}_1(c_i)\bigr)$ we see:

$$e_{c_i} \geq k_{c_i}. \tag{5}$$

That this is, in fact, equality, follows from Corollaire 5(ii) of [2], but we give a proof since we need one of its ingredients. Corollary 1.5 implies:

$$\dim V_1(c_i) = \dim \operatorname{Im}\bigl(1 - h_1(c_i)\bigr) + (r_{c_i} - 1) = \sum_{j=1}^{k_i} (\delta_{ij} - 1) + e_{c_i} + r_{c_i} - 1.$$

Summing this over i and applying the consequence $\sum \dim V_1(c_i) = \dim H_1(F) = 1 - \chi(F)$ of Theorem 1.4 on the left and the Riemann-Hurwitz formula on the right gives

$$1-\chi(F)=\sum(d_E-1)+\sum_{i=1}^{m}(e_{c_i}+r_{c_i}-1),$$

where the first sum on the right is over all horizontal curves E and d_E is the degree of $\overline{f}$ on E. Since $\sum d_E$ is the number of punctures of F this simplifies to

$$2\,\mathrm{genus}(F)=1-\delta+\sum_{i=1}^{m}(e_{c_i}+r_{c_i}-1), \tag{6}$$

where δ is the number of horizontal curves. But Kaliman proves this equation in [10] with e_{c_i} replaced by k_{c_i}, so the inequalities (5) must be equalities. The final inequality of the theorem follows from (6) and the formula

$$\delta-1\geq\sum_{i=1}^{m}(r_{c_i}-1),$$

of Kaliman [10]. □

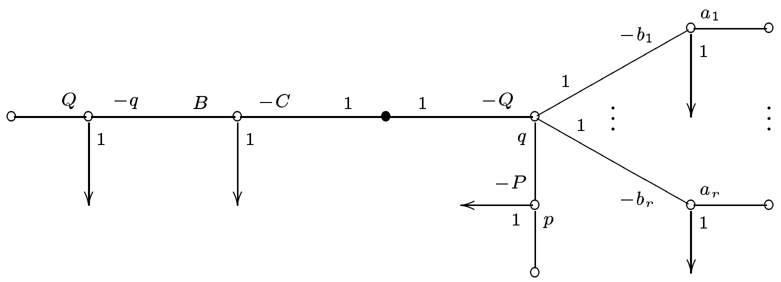
A consequence of the above proof is the exactness of the kernel sequence (and hence also the image sequence (4)) of $1-h(c_i)$ applied to the short exact sequence (3). Indeed, if we replace each group A in (3) by the chain complex $0\to A\xrightarrow{1-h}A\to 0$, then the resulting short exact sequence of chain complexes has long exact homology sequence $0\to\mathrm{Ker}(1-h_1|B)\to\mathrm{Ker}(1-h_1)\to\mathrm{Ker}(1-\overline{h}_1)\to\mathrm{Cok}(1-h_1|B)\to\mathrm{Cok}(1-h_1)\to\mathrm{Cok}(1-\overline{h}_1)\to 0$. The equality in (5) implies that the middle map of this sequence has rank 0, and hence is the zero map since $\mathrm{Cok}(1-h_1|B)$ is free abelian.

4. Classification of rational polynomials of simple type

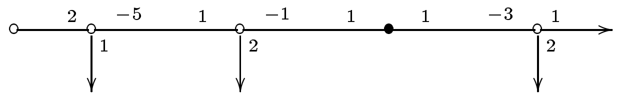
The classification in [12] mistakenly assumed isotriviality (all regular fibres of f are conformally isomorphic to each other) at one stage in the proof (page 346, lines 10–11). There are in fact also many non-isotrivial 2-variable rational polynomials of simple type, the simplest being $f(x,y)=x(1+xy)(1+axy)+xy$ of degree 5, whose regular fibres $f^{-1}(c)$ are 4-punctured $\mathbb{C}P^1$'s such that the cross-ratio of the punctures varies linearly with c.

In this section we list the non-isotrivial rational polynomials of simple type. We list their regular splice diagrams (see [13], [14]), since this gives a useful description of the topology. For each case there are several possible topologies for the irregular fibres, depending on additional parameters. We have a proof that these examples complete the classification but it is tedious and not yet written down in full detail, so the result should be considered tentative.

Let p,q,P,Q be positive integers with $Pq-pQ=1$ and let r and $a_1,\dots,a_r$ be positive integers. Let $A=\sum_{i=1}^{r}a_i$, $B=AQ+P-Q$, $C=Aq+p-q$, and $b_i=qQa_i+1$ for $i=1,\dots,r$. Then the following is the regular splice diagram of a rational polynomial of simple type.



There is one further degree 8 example that does not fall in the above family. The splice diagram is



In all these examples the curve obtained by filling the puncture corresponding to the second arrowhead from the left has constant conformal type as we vary the regular fibre $f^{-1}(c)$, and that puncture varies linearly with $c \in \mathbb{C}$.

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