Some examples of hyperbolic groups

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Abstract. We describe some examples of hyperbolic groups to compare the range of validity of different theories of JSJ-decompositions of groups.

1. Introduction

This note arose out of an attempt to understand the range of applicability of various theories of JSJ-decompositions of groups ([15], [14], [4], [5], [6]). The pioneering work of Sela for (word) hyperbolic groups uses the algebraic analogue of Dehn twists and needs the outer automorphism of the given one-ended hyperbolic group to be infinite. Bowditch's work ([4]) again for one-ended hyperbolic groups, uses Gromov boundary and its local cut point structure. The papers [5], [6] discuss similar decompositions for finitely presented groups. In this note, we produce examples of one-ended hyperbolic groups with finite outer automorphism groups which have JSJ-decompositions in the sense of Bowditch. Thus, it appears that Sela's techniques may not extend for all hyperbolic groups, let alone for finitely presented groups. These examples depend on the fact that there are two types of two-ended groups, namely those with infinite centre (e.g., $\mathbb{Z}$) and those with finite centre (e.g., $\mathbb{Z}_2 * \mathbb{Z}_2$) and that there are very few Dehn twists along the second type of groups (see Section 2 below). The geometric examples are described in Section 3 and the algebraic examples in Section 4. Since the term JSJ-decomposition in the case of groups seems to mean different things to different people, we briefly explain Bowditch's description for one-ended hyperbolic groups, which seems closest to Waldhausen's original conception of these decompositions.

Originally, the theory of canonical decompositions for orientable Haken manifolds with incompressible boundary was proposed by Waldhausen ([10] for summary and references), and was carried out more comprehensively by Johannson and Jaco-Shalen. This theory gives a decomposition of the Haken manifold M along two-sided annuli and tori and some of the resulting pieces are either Seifert fibered or I-bundles over surfaces. These are grouped into what is called a characteristic manifold of M. The decomposition is unique and a key property which almost characterizes the decomposition is that any properly embedded (resp. singular)

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two-sided annulus or torus can be isotoped (resp. homotoped) into the characteristic submanifold. Sela obtained similar decompositions for one-ended torsion free hyperbolic groups by using a group theoretic analogue of Dehn twists. He used an iteration technique which needs the outer automrphism group to be infinite. Bowditch obtains similar decompositions with strong uniqueness properties using the local cut point structure of the boundary; his approach does not need the group to be torsion free or the outer automorphism group to be infinite. His decomposition seems to capture all the features envisaged by Waldhausen and we take this to be a model for JSJ-decompositions for groups. Bowditch's decomposition for a one-ended hyperbolic group G is a graph of groups decomposition for G in which the vertex groups are either (1) two-ended or (2) maximal "hanging" Fuchsian groups or (3) non-elementary quasiconvex subgroups which are not of type (2). The three types are mutually exclusive and vertices of the same type are not adjacent. The edge groups which connect to any vertex of type (2) are peripheral subgroups of that vertex group. Any two-ended subgroup with more than two co-ends in G can be conjugated into a vertex group of type (1) or (2). Any hanging Fuchsian group which is non-elementary can be conjugated into a vertex group of type (2).

Dunwoody-Sageev [5], and Fujiwara-Papasoglu [6] studied similar decompositions for finitely presented groups over small groups like virtually polycyclic groups. But neither of these decompositions coincides with the classical JSJ-decompositions for 3-manifolds when the boundary is non-vacuous and they thus seem to capture only a part of the classical JSJ-decomposition. In [10] we gave a quick description of the JSJ-decomposition for 3-manifolds which was inspired by group theoretic ideas.

2. Dehn twists and $Out(G)$ of hyperbolic groups.

In this section, we recall Dehn twists for groups and some results of Paulin [11] Rips, and Bestvina-Feighn [3].

As mentioned in the introduction, there are two types of two-ended groups (equivalently virtually cyclic groups). Those with infinite centre admit the infinite cyclic group as quotient and those with finite centre admit the infinite dihedral group as quotient; the second characterization follows from Stallings description of finitely generated groups with two ends. Suppose that H is a two-ended group and G has a decomposition $G_1 *_H G_2$. If t is a non-trivial element in the centre of H, we define the Dehn twist $D_t : G \to G$ by $D_t(g_1) = g_1, g_1 \in G_1$, $D_t(g_2) = tgt^{-1}, g_2 \in G_2$. If G is an HNN-extension $G = G_1 *_H$ with stable letter s (so there is an injective homomorphism $\phi\colon H \to G$ and G is generated by G_1 and s with relations $shs^{-1} = \phi(h), h \in H$), we define D_t by $D_t(g_1) = g_1, g_1 \in G_1$ and $D_t(s) = st$. It is easy to see that D_t is not an inner automorphism in the second case. We assume that t has infinite order and examine D_t in the first case. If $D_t(g) = hgh^{-1}, g \in G$, then h must be in the centre of G_1. We also should have

$t^{-1}h$ in the centre of G_2 which implies that $h \in H$. Since h,t commute and since t has infinite order, one of $h, t^{-1}h$ should be of infinite order. Thus unless one of G_1, G_2 has infinite centre, D_t is not an inner automorphism. Similarly, no power of D_t is an inner automorphism of G. Thus:

Proposition 2.1. *Let $G = G_1 *_H G_2$ or $G_1 *_H$ where H is two-ended and has infinite centre. Then D_t defines an element of infinite order when $G = G_1 *_H$ and also in the case when $G = G_1 *_H G_2$ unless one of G_1, G_2 has infinite centre.*

A partial converse follows from the deep work of Rips, Paulin and Bestvina-Feighn (see Corollary 1.3 of [3])

Theorem 2.2. *If G is a one-ended hyperbolic group with $Out(G)$ infinite, then G splits over a two-ended subgroup.*

We feel that the exact converse of the Proposition 2.1 is true but we do not have a complete proof and the arguments we have in mind are very different from the rest of the arguments of this paper. We therefore leave it as an open problem.

3. Geometric examples

We next describe examples of one-ended hyperbolic groups G with $Out(G)$ finite and having non-trivial JSJ-decompositions in the sense of Bowditch. Moreover these groups have subgroups of finite index with infinite outer automorphism groups. Since Sela's argument uses infinitely many non-conjugate automorphisms obtained by iterating Dehn twists along crossing splittings, these decompositions cannot be obtained directly by Sela's methods. Here, the word "crossing" is used in the following sense. Suppose that C_1 and C_2 are two virtually cyclic groups along which G splits and T_1 and T_2 are the corresponding Bass-Serre trees for the action of G. We say that C_1 and C_2 cross if the action of C_1 is hyperbolic on T_2 and the action of C_2 is hyperbolic on T_1. Sela shows that otherwise both the actions are elliptic. The above notion does not depend on the splittings as long as there are some splittings along C_1 and C_2. In our examples there are such subgroups for which the splittings cross. But these subgroups are dihedral and hence there are no Dehn twists to produce $\mathbb{R}$-trees as in Sela's approach.

The picture for the examples is the plane with the origin corresponding to the trivial group, the four half-axes corresponding to $\mathbb{Z}_2$'s and filling up the quadrants by suitable rigid groups. Thus, we start with F which is the free product of four copies of $\mathbb{Z}_2$: $F = \mathbb{Z}_2^{(1)} * \mathbb{Z}_2^{(2)} * \mathbb{Z}_2^{(3)} * \mathbb{Z}_2^{(4)}$ and let $D_i = \mathbb{Z}_2^{(i)} * \mathbb{Z}_2^{(i+1)}$, $1 \leq i \leq 4$, $\mathbb{Z}_2^{(5)} = \mathbb{Z}_2^{(1)}$. The kernel K' of the homomorphism to $\mathbb{Z}_2$ which identifies all the $\mathbb{Z}_2$'s can be identified with the fundamental group of a sphere with four holes D and K is the orbifold fundamental group of the quotient of D by a reflection in a plane which cuts each hole in half. Thus K is (non-orientable) Fuchsian with D_i, $1 \leq i \leq 4$ corresponding to the peripheral subgroups. We next glue four groups

G_i to F along the D_i. In these examples the dihedral subgroups $H_i \subset G_i$ along which we glue will be malnormal in G_i, and thus the resulting G is hyperbolic by a theorem of Bestvina and Feighn [2]. In the first set of examples the G_i have index two subgroups which are fundamental groups of closed hyperbolic 3-manifolds. We can take the G_i to be isomorphic or all different. We next describe these examples (another set of examples is given at the end).

Let τ be an involution of S^3 with two fixed points. For example, if we think of S^3 as the one-point compactification of R^3 then τ may be given by $\tau(x,y,z) = -(x,y,z)$.

Let $L \subset S^3$ be a link that does not pass through either fixed point, is mapped to itself by τ, and has hyperbolic complement. An example is the chain of $2p$ links in a circle with alternate links lying in the xy-plane or crossing this plane at right angles, which has hyperbolic complement for $p \geq 3$ (see Neumann-Reid [9]; the link in question is called $C(2p,-p)$ there). Let $M_0 = (S^3 - L)/\tau$. This is a hyperbolic orbifold with exactly two orbifold points, at which the local structure is cone on RP^2. By Thurston's Dehn surgery theorem almost any filling of the cusps of M_0 results in a compact hyperbolic 3-orbifold M. The fundamental group $\pi_1(M)$ is then word-hyperbolic. Let γ be any geodesic in M connecting the two orbifold points (there are infinitely many choices, since every homotopy class of paths joining the points contains a geodesic). Then γ is a 1-dimensional sub-orbifold of M with fundamental group $H \cong \mathbb{Z}/2 * \mathbb{Z}/2 \subset \pi_1(M)$. We may take each G_i, H_i pair to be such a pair $\pi_1(M)$, H.

Consider the action of this G_i on $\mathbb{H}^3$ given by the hyperbolic structure on M. The geodesic γ is covered by a family of disjoint complete geodesic lines in $\mathbb{H}^3$ and H_i is the group that preserves one of them, say $\overline{\gamma} \subset \mathbb{H}^3$. Any conjugate gH_ig^{-1} not equal to H_i preserves a different one of these lines $g\overline{\gamma}$. A non-trivial element common to H_i and gH_ig^{-1} would have to preserve both lines and therefore be the rotation by π about their common perpendicular. But this element is not in G_i since it has positive dimensional fixed point set. It follows that H_i is malnormal in G_i. We can thus glue four of these G_i's to F as described above to get a desired example G.

Note that M has two-sheeted cover which is the fundamental group of a closed hyperbolic 3-manifold and γ is covered by a geodesic. Thus G has a subgroup G' of index 2 which is obtained by glueing the fundamental groups of four hyperbolic 3-manifolds along maximal cyclic subgroups to the fundamental group of a disc with four holes along the peripheral subgroups. Since a closed hyperbolic 3-manifold group does not split over a (finite or infinite) virtually cyclic group, it follows that G_i does not split over a virtually cyclic group. By Bowditch [4], the obvious graph of groups decomposition of G with five vertices and four edges with F corresponding to the vertex in the centre is the JSJ-decomposition of G. The subgroups corresponding to $\mathbb{Z}_2^{(1)} * \mathbb{Z}_2^{(3)}$ and $\mathbb{Z}_2^{(2)} * \mathbb{Z}_2^{(4)}$ give crossing splittings (hyperbolic-hyperbolic in Sela's terminology [15]) of G. That $\mathrm{Out}(G)$ is finite follows as in the proof of Theorem 4.2 below. Since G' has splittings along $\mathbb{Z}$ with neither side elementary, Proposition 2.1 implies that $\mathrm{Out}(G')$ is infinite. Since the G_i do not split over finite or two-ended subgroups, if G splits over a finite group then F should split over a finite group relative to its peripheral subgroups, which is clearly impossible. Thus G is one-ended. In [15] and [14], JSJ-decompositions are obtained by utilizing an infinity of non-conjugate Dehn twists (thus $\mathrm{Out}(G)$ should be infinite) to produce actions of G on $\mathbb{R}$-trees and analysing the actions. Since $\mathrm{Out}(G)$ is finite in our case it seems unlikely that a theory using Dehn twists can produce the above JSJ-decompositions.

We can obtain examples G with similar properties which have splittings along a $\mathbb{Z}$ and a $\mathbb{Z}_2 * \mathbb{Z}_2$ which cross as follows. Consider just two pairs (G_1, H_1) and (G_2, H_2) as above. By identifying the two $\mathbb{Z}_2$'s in H_i we obtain HNN-extensions G_1' and G_2' with stable letters t_1 and t_2 say. Now amalgamate G_1' to G_2' along t_1 and t_2 and let G be the resulting group and T the cyclic subgroup generated by $t_1 = t_2$. Then G admits crossing splittings along $\mathbb{Z}$ and $\mathbb{Z}_2 * \mathbb{Z}_2$. The dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$ is the obvious one obtained from the $\mathbb{Z}_2$ in each G_i' and the splitting is HNN. This time the Fuchsian group is $\mathbb{Z} * \mathbb{Z}_2 * \mathbb{Z}_2$ which is the fundamental group of the orbifold obtained from a torus with two holes by factoring by an involution that fixes two arcs. The underlying space of the orbifold is an annulus and each circle boundary component is divided into two intervals, one of which is boundary of the orbifold and the other of which is a mirror line. The groups G_i are attached to this along the two boundary components to get G. The $\mathbb{Z}$ splitting of G is along the core circle of the orbifold and the dihedral splitting is along a line joining the two mirror arcs of the orbifold.

4. Algebraic examples

The following construction follows [8] (see also [7]). Let C_n be the cyclic group of order $n > 1$. Define $U(p,q) = \langle x, y \mid x^p = y^q = 1 \rangle$, which is the ordinary free product $C_p * C_q$, where we will assume p and q are distinct large primes, say $> 10^6$. Let

$$D = \langle a, b \mid a^2 = b^2 = 1 \rangle \cong C_2 * C_2$$

be the infinite dihedral group and form the ordinary free product $F = U(p,q) * D = C_p * C_q * D$. For each $i > 1$ define the word v_i on $\{x, y\}$ by

$$v_i = \prod_{j=80(i-1)+1}^{80i} \{(xy)^i xy^2\}.$$

For example, v_1 is the word $xyxy^2(xy)^2xy^2 \cdots xy^2(xy)^{80}xy^2$.

Let $R = \{a^{-1}v_1, b^{-1}v_2\}$ and let $N = R^F$ be its normal closure and $\Gamma = F/N$. With respect to the free product length in $F = C_p * C_q * D$, the presentation of Γ satisfies the metric cancellation condition $C'(\frac{1}{10})$. In [8] it is shown that Γ has the following properties:

1. Γ is generated by x and y which have orders p and q respectively.

2. Γ is complete (that is, Γ is centreless and $\mathrm{Out}(\Gamma) = 1$), hopfian and cohopfian.
3. Every element of finite order in Γ is conjugate to an element in one of the cyclic subgroups $\langle a\rangle$, $\langle b\rangle$, $\langle x\rangle$ or $\langle y\rangle$.
4. Two elements in D are conjugate in Γ if and only if they are already conjugate in D. In particular a and b are not conjugate in Γ.
5. Distinct elements in $\langle x\rangle$ and $\langle y\rangle$ are not conjugate.

The basic tools used in the proof are from the theory of small cancellation quotients of ordinary free products (see also [7]).

We now observe that by using Tietze transformations to eliminate the generators a and b, the presentation for Γ can be rewritten as $\Gamma \cong \langle x, y \mid x^p = y^q = v_1^2 = v_2^2 = 1\rangle$. Since p and q are distinct large primes, one easily checks this presentation satisfies the metric cancellation condition $C'(\frac{1}{10})$ with respect to the free group metric. Hence in particular Γ is a word hyperbolic group. Moreover, non-trivial elements of Γ have cyclic centralizers [16]. From this and the previously stated facts about conjugacy, it follows that D is malnormal in Γ. Also each of the cyclic subgroups $\langle a\rangle$,$\langle b\rangle$,$\langle x\rangle$ and $\langle y\rangle$ is malnormal.

Lemma 4.1. *Let $D \subset \Gamma$ be as a above, and let H be any group containing an isomorphic copy D' of D, but not containing elements of orders p or q. Form the amalgamated free product $G = \Gamma *_{D=D'} H$ identifying the two copies of D. If $\psi \in$ $\mathrm{Aut}(G)$, then ψ composed with a suitable inner automorphism is a automorphism θ which is the identity on Γ.*

Proof. Since $\psi(y)$ has order q and the only elements of order q lie in conjugates of $\langle y\rangle$, following ψ by a conjugation we may suppose $\psi(y) = y^m$ for some $1 \le m < q$. Also the only elements of order p are in conjugates of x, so we can assume $\psi(x) = wx^k w^{-1}$ for some $w \in G$.

First suppose $w \notin \Gamma$. We may assume $w = \gamma_0 h_1 \cdots h_n \gamma_n$ is in normal form in the amalgamated free product and that $n \ge 1$. Then we must have

$$1 = \psi(v_1^2) = wx^k w^{-1} y^m wx^k w^{-1} y^{2m} \cdots (wx^k w^{-1} y^m)^{80} wx^k w^{-1} y^{2m}.$$

Then by the normal form theorem for amalgamated free products either $\gamma_n x^k \gamma_n^{-1} \in D$ or $\gamma_0^{-1} y^m \gamma_0 \in D$. But both of these are clearly false since D contains no elements of order p or q. This contradiction shows we must have $w \in \Gamma$.

Then since Γ is cohopfian, $\psi|_\Gamma$ is an automorphism which followed by an inner automorphism (which must actually be conjugation by a power of y) is the identity. Hence composing the original ψ with a conjugation gives an automorphism θ which is the identity on Γ $\square$

Let $D_i \subset \Gamma_i$ be 4 such pairs of groups constructed using large primes p_i, q_i for $i = 1, \ldots, 4$ where we assume the 8 primes involved are all distinct. Let $L = C_2^{(1)} * C_2^{(2)} * C_2^{(3)} * C_2^{(4)}$ be the free product of 4 copies of C_2. Define $D_i' = C_2^{(i)} * C_2^{(i+1)}$ for $i = 1, 2, 3$ and define $D_4' = C_2^{(4)} * C_2^{(1)}$. Let $\hat{G}$ be the graph of groups formed by identifying each D_i with the corresponding D_i'. (Note that this could equivalently be done by forming 4 successive free products with amalgamation.) As the D_i are malnormal in Γ_i and the D'_i malnormal in L, the results of Bestvina and Feign [2] imply that $\hat{G}$ is word hyperbolic. As before, $\hat{\Gamma}$ has crossing splittings along dihedral groups.

Theorem 4.2. *The word hyperbolic group $\hat{G}$ with crossing splittings along dihedral groups constructed above has* $\mathrm{Out}(\hat{G}) = 1$.

Proof. We begin by observing that because of malnormality and the normal form theorem for amalgamated free products, each of the $C_2^{(i)}$ is its own centraliser in $\hat{G}$. Let $\psi \in \mathrm{Aut}(\hat{G})$. Note that for each $i = 1, \ldots, 4$, the group $\hat{G}$ van be viewed as the amalgamated free product of Γ_i with another group H_i amalgamating along $D_i = D'_i$. By the previous lemma there are elements $w_i \in \hat{G}$ for $i = 1, \ldots, 4$ such that $\psi|_{\Gamma_i}$ is conjugation by w_i. So if we follow ψ by conjugation by w_1 we obtain an automorphism θ which is the identity on Γ_1 and acts as conjugation by $u_i = w_1^{-1} w_i$ on Γ_i for $i = 2, 3, 4$. Now θ fixes Γ_1 and hence fixes $C_2^{(2)}$, so that u_2 must centralise $C_2^{(2)}$ which implies $u_2 = 1$. But then similarly u_3 must centralise $C_2^{(3)}$ which implies $u_3 = 1$. Hence finally u_4 must centralise $C_2^{(4)}$ which implies $u_4 = 1$. So θ is the identity as desired. □

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