Kleinian Groups Generated by Rotations

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Abstract. We discuss which Kleinian groups are commensurable with Kleinian groups generated by rotations, with particular emphasis on Kleinian groups that arise from Dehn surgery on a knot.

Introduction

In a problem session, organized by A. Kim, of the 1991 German-Korean-SEAMS Conference on Geometry, E. Vinberg asked for a cocompact Kleinian group which is not commensurable with a group generated by rotations (elements of finite order). Examples are, in fact, not hard to find. In this note we describe in some detail which Kleinian groups have this property among the Kleinian groups that occur as fundamental groups of Dehn surgeries on knots. We also briefly discuss some related questions.

In the following Γ and Λ will always denote Kleinian groups of finite covolume, that is, discrete subgroups of $PSL(2,\mathbb{C}) = Isom^+(\mathbb{H}^3)$ such that the orbifold $\mathbb{H}^3/\Gamma$ or $\mathbb{H}^3/\Lambda$ has finite volume. They are *commensurable* if they have isomorphic subgroups of finite index. By Mostow-Prasad rigidity this is equivalent to the condition that they can be conjugated within $PSL(2,\mathbb{C})$ so their intersection has finite index in each.

Kleinian groups commensurable with groups generated by reflections (rather than rotations) have been studied by E. Vinberg [**V**] and E. M. Andreev [**An**]. They are a very restricted class of Kleinian groups. For example, as is pointed out in [**NR**], the invariant trace field of such a Kleinian group — indeed, of any Kleinian group commensurable with a non-orientation-preserving subgroup of $Isom(\mathbb{H}^3)$ — has to be preserved by complex conjugation (and the same for the invariant quaternion algebra), which is a rare occurrence. For a Kleinian group generated by rotations I know of no similar restriction on the invariant trace field, although the invariant trace field severely restricts the possible orders of the rotations involved (cf. Theorem 3 below).

In the following section we discuss Dehn surgery on knots and state and prove our main theorem (Theorem 1). In Section 2 we discuss the case of knot complements themselves: a hyperbolic knot complement is commensurable with a $\mathbb{H}^3/\Lambda$ with Λ generated by rotations if the knot is invertible and conjecturally in just one other case (Theorem 2). In Section 3 we make some additional comments about what rotations can be contained in Kleinian groups commensurable with a given Kleinian group Γ. If such rotations do not occur in Γ itself, it is reasonable to call them "hidden rotations" of Γ. We show that "most" Kleinian groups have no hidden rotations of orders other than $2, 3, 4$, or 6.

1. Dehn surgery on knots

Let (S^3, K) be a hyperbolic knot (i.e., $M = S^3 - K$ admits a complete finite-volume hyperbolic structure). Thurston's hyperbolic Dehn surgery theorem ([**T**], [**N-Z**]) implies that by excluding finitely many integer pairs (p, q) we may assume the result $M(p, q)$ of (p, q)-Dehn-Surgery on (S^3, K) is hyperbolic. Say $M(p, q) = \mathbb{H}^3/\Gamma(p, q)$. Group-theoretically, $\Gamma(p, q)$ is the result of factoring $\pi_1(S^3 - K)$ by the normal closure of the element $m^p l^q$, where $m, l \in \pi_1(S^3 - K)$ are represented by a meridian and a longitude in a torus neighbourhood boundary of K in S^3.

Theorem 1.
(i) *If (S^3, K) is non-invertible and does not branched cyclic cover a torus knot, then for all but finitely many values of p/q the above group $\Gamma(p, q)$ is not commensurable with a group generated by rotations.*
(ii) *If (S^3, K) is an invertible knot, then each $\Gamma(p, q)$ has index 2 in a group generated by rotations.*
(iii) *If (S^3, K) is a d-fold cyclic cover of the unknot, then each $\Gamma(p, q)$ is commensurable with a group generated by rotations.*
(iv) *If (S^3, K) is not in one of the above cases, then infinitely many $\Gamma(p, q)$ are commensurable with groups generated by rotations and infinitely many are not.*

Example. Figure 1 shows a knot 3-fold cyclic covering an unknot. This knot (and its obvious generalizations) provides an example for case (iii). However, in some sense "most" knots have no symmetries at all, so they satisfy case (i).

Proof. It is well known (cf. [**Ar**]) that $\Lambda \subset Isom^+(\mathbb{H}^3)$ is generated by rotations if and only if the space $\mathbb{H}^3/\Lambda$ is simply connected.

Let $M(p, q) = \mathbb{H}^3/\Gamma(p, q)$ be as in the theorem. We shall abbreviate $\Gamma = \Gamma(p, q)$. We first assume (S^3, K) is non-invertible.

Thurston's Dehn surgery theorem (loc. cit.) also says $\mathrm{vol}(M(p, q)) < \mathrm{vol}(M)$, and since, by Borel [**B**], only finitely many hyperbolic orbifolds with volume below a given bound are arithmetic, we may, by excluding finitely many (p, q), assume $M(p, q)$ is non-arithmetic. Then by Margulis (cf. [**Z**], ch. 6) there exists a maximal element Γ_0 in the commensurability class of Γ. Denote $M_0(p, q) = \mathbb{H}^3/\Gamma_0$. The map $M(p, q) \to M_0(p, q)$ is a covering map of orbifolds. We shall show first that for (p, q) sufficiently large it is a cyclic covering.

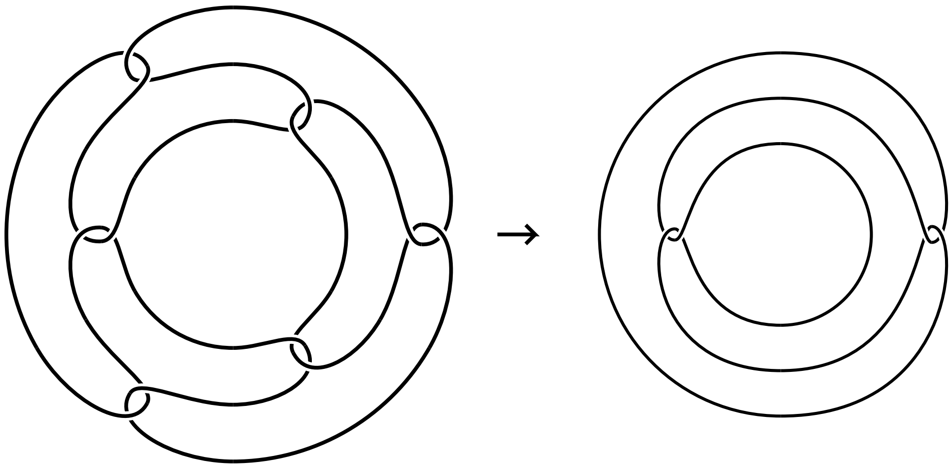

Figure 1

There is a bound on the degree of $M(p,q) \to M_0(p,q)$, independent of (p,q), since a complete hyperbolic orbifold has volume above some fixed positive bound (Margulis). If (p,q) is sufficiently large, then the geodesic γ added to M by Dehn surgery is much shorter than the other closed geodesics of $M(p,q)$. It follows that, for (p,q) sufficiently large, the image γ_0 of γ in $M_0(p,q)$ is the shortest closed geodesic of $M_0(p,q)$, and that no other geodesic of $M(p,q)$ could cover γ_0. Then $M = M(p,q) - \gamma$ covers $M_0(p,q) - \gamma_0$. We exclude the finitely many (p,q) for which this is not the case.

Now, consider this covering restricted to the boundary T of a solid torus neighborhood of γ. The image T_0 in $M_0(p,q) - \gamma_0$ is an orbifold covered by the torus. Let A and A_0 be the fundamental groups of T and T_0. If A is not normal in A_0, that is the covering $T \to T_0$ is not regular, then T_0 must be a triangle orbifold of type $(2,4,4)$, $(2,3,6)$, or $(3,3,3)$, since these are the only orbifolds covered by a torus for which the orbifold fundamental group A_0 contains non-normal torsion-free subgroups. However, such a covering cannot extend to a solid torus and the covering $T \to T_0$ extends to a tubular neighborhood of γ. Hence the covering $T \to T_0$ is regular. As described for instance in [**R**], it follows that the covering $M = M(p,q) - \gamma \to M_0(p,q) - \gamma_0$ is regular, since $\pi_1(T)$ normally generates $\pi_1(M)$ (M is a knot complement); we repeat the argument for completeness.

We show Γ is normal in Γ_0 by showing that it is the normal closure of a suitable subgroup. If we identify A_0 with its image in Γ_0 then $A = A_0 \cap \Gamma$. The degree of our covering is on the one hand equal to $|A_0/A| = |A_0/(A_0 \cap \Gamma)| = |A_0\Gamma/\Gamma|$ and on the other hand equal to $|\Gamma_0/\Gamma|$, so $A_0\Gamma = \Gamma_0$. The normal closure of A in $\Gamma_0 = A_0\Gamma$ therefore equals the normal closure of A in Γ, which is Γ itself, as pointed out above.

The covering transformation group G for $M \to M_0(p,q) - \gamma_0$ is $G = \Gamma_0/\Gamma = A_0/A$, so it is the same as for $T \to T_0$. Since the longitude of the knot complement $M(p,q) - \gamma = S^3 - K = M$ generates the kernel of $H_1(T) \to H_1(M)$, it is preserved by the G-action up to sign. Hence the same is true for the meridian, so the G-action extends to an action on S^3. By the solution to the Smith Conjecture (cf. [**MB**]) G is cyclic or dihedral. But dihedral is excluded by our assumption that $K \subset S^3$ is a non-invertible knot, so G is cyclic.

Now suppose that $\Lambda \subset \Gamma_0$ exists so that Λ is generated by rotations, i.e., the space $\mathbb{H}^3/\Lambda$ is simply connected (cf. first sentence of this section). Then $\mathbb{H}^3/\Gamma_0$ must have finite fundamental group. But $\mathbb{H}^3/\Gamma_0$ is the result of a Dehn filling of M/G, where G is the above cyclic group. By [**BH**], since we assumed M/G is not a torus knot complement, such a Dehn surgery can give finite fundamental group for at most 24 values of p/q. Thus by excluding these (p,q) we can avoid this, so part (i) of the theorem is proved.

(We note that this final exclusion may be necessary. For example, if (S^3, K) is the $(-2,3,7)$-pretzel knot then $(r,1)$-Dehn surgery gives a manifold with finite fundamental group for $r = 17, 18, 19$ — see [**BH**] — so in this case if $\Gamma(p,q)$ is a Kleinian group with $p/q = 17, 18, 19$ then it has a subgroup of finite index generated by rotations.)

For part (iii) of the theorem suppose that $(S^3, K)/G$ is the unkot. Then the underlying space of $M_0(p,q) = M(p,q)/G$ is the result of (p, dq)-Dehn-surgery on this unknot with $d = |G|$. That is, $M_0(p,q)$ has underlying space the lens space $L(p, dq)$, which has a simply-connected d-fold covering, proving part (iii). The proof of (iv) is similar: in this case $M_0(p,q)$ has underlying space equal to the result of (p, dq) Dehn surgery on a torus knot, which is a Seifert fibered manifold with infinite fundamental group for infinitely many (p,q) and with finite fundamental group for infinitely many (p,q).

Finally, for part (ii), suppose (S^3, K) is invertible. That is, there is an involution of S^3 which reverses K. The quotient S^3/C_2 by this involution, as a space, is S^3, while the quotient of a tubular neighborhood of K is a ball. Dehn surgery just replaces this ball by another ball with different orbifold structure, so $M(p,q)/C_2$, as a space, is still S^3. The corresponding C_2-extension of Γ is thus generated by rotations. □

2. Hyperbolic knot complements and groups generated by rotations

One can also ask when a hyperbolic knot complement $M = S^3 - K$ is itself commensurable with a $\mathbb{H}^3/\Lambda$ with Λ generated by rotations. Again, this means $\mathbb{H}^3/\Lambda$ has simply connected underlying space. If the knot is invertible, then the quotient of $M = S^3 - K$ by the inversion has underlying space an open disk, so the answer is "yes." Otherwise, M is non-arithmetic by Reid [**R**] (who shows that the figure-eight knot is the only knot with arithmetic hyperbolic complement; the figure-eight knot is invertible) and we can again argue that the "orientable commensurator quotient" M_0 of M (i.e., the quotient of $\mathbb{H}^3$ by the largest Kleinian group Γ_0 containing $\Gamma = \pi_1(M)$) would have underlying space with finite fundamental group. This could only happen if Γ_0 is larger than the normalizer $N(\Gamma)$ of Γ in $PSL(2,\mathbb{C})$ (since $N(\Gamma)/\Gamma$ is cyclic, by the same argument as in Sect. 1, so $\mathbb{H}^3/N(\Gamma)$ still has infinite homology). As described in [**NR, Sect. 9**], this is an exceedingly rare phenomenon which quite possibly only happens for the figure-eight knot and the two "dodecahedral knots" of Aitcheson and Rubinstein [**AR**]. One of the dodecahedral knots is invertible (this knot is number 5 in the series of knots of which number 3 is shown in Fig. 1). The other dodecahedral knot is non-invertible and its $\mathbb{H}^3/\Gamma_0$ is contractible. Thus summarizing:

Theorem 2. *Let (S^3, K) be a hyperbolic knot. Then $\Gamma = \pi_1(S^3 - K)$ is commensurable with a Kleinian group generated by rotations if (S^3, K) is invertible or is the non-invertible dodecahedral knot. Any other example would have to have the normalizer $N(\Gamma)$ of Γ not equal to the commensurator Γ_0 and conjecturally there are no further examples of this.* □

3. Hidden rotations in Kleinian groups

One can ask whether a group Λ commensurable with a given Kleinian group $\Gamma \subset PSL(2,\mathbb{C})$ can contain rotations than Γ does not contain — we call these "hidden rotations" for Γ.

Of course, Kleinian groups can have hidden rotations — any torsion-free subgroup Γ of a group Λ with torsion does, for example. But we can exclude a lot of possible hidden rotations too.

Let $k(\Gamma)$ be the invariant trace field of Γ. That is, it is the field generated by the traces of squares of elements of Γ, cf. [**NR**].

Theorem 3. *If a group commensurable with Γ contains a $(2\pi/p)$-rotation, then $\cos(2\pi/p) \in k(\Gamma)$.*

Proof. Suppose g is a $(2\pi/p)$-rotation in a group Λ commensurable with Γ. Then the trace of g^2 is $2\cos(2\pi/p)$, so $2\cos(2\pi/p) \in k(\Lambda)$. Since $k(\Lambda) = k(\Gamma)$ (cf. [**NR**]), the Theorem follows. □

Note that $\cos(2\pi/p)$ is rational for $p \le 4$ and $p = 6$, so Theorem 2 never excludes elements of order ≤ 4 or of order 6. However, "most" fields will not contain $\cos(2\pi/p)$ for $p = 5$ or $p > 6$, so "most" Kleinian groups Γ will admit no hidden rotations of these orders.

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