EXTENDED BLOCH GROUP AND THE CHEEGER-CHERN-SIMONS CLASS

WALTER D. NEUMANN

Abstract. We define an extended Bloch group and show it is naturally isomorphic to $H_3(\text{PSL}(2, \mathbb{C})^\delta; \mathbb{Z})$. Using the Rogers dilogarithm function this leads to an exact simplicial formula for the universal Cheeger-Chern-Simons class on this homology group. It also leads to an independent proof of the analytic relationship between volume and Chern-Simons invariant of hyperbolic 3-manifolds conjectured in [15] and proved in [23], as well as effective formulae for the Chern-Simons invariant of a hyperbolic 3-manifold.

1. Introduction

There are several variations of the definition of the Bloch group in the literature; by [7] they differ at most by torsion and they agree with each other for algebraically closed fields. In this paper we shall use the following.

Definition 1.1. Let $k$ be a field. The pre-Bloch group $\mathcal{P}(k)$ is the quotient of the free $\mathbb{Z}$-module $\mathbb{Z}(k - \{0, 1\})$ by all instances of the following relation:

\begin{equation}
[x] - [y] - \frac{1 - x^{-1}}{1 - y^{-1}} + \frac{1 - x}{1 - y} = 0,
\end{equation}

This relation is usually called the five term relation. The Bloch group $\mathcal{B}(k)$ is the kernel of the map

$$\mathcal{P}(k) \to k^* \otimes_{\mathbb{Z}} k^*, \quad [z] \mapsto 2(z \otimes (1 - z)).$$

(In [14] the additional relations

$$[x] = [1 - \frac{1}{x}] = [\frac{1}{1 - x}] = -[\frac{1}{x}] = -[\frac{x - 1}{x}] = -[1 - x]$$

were used. These follow from the five term relation when $k$ is algebraically closed, as shown by Dupont and Sah [7]. Dupont and Sah use a different five term relation but it is conjugate to the one used here by $z \mapsto \frac{1}{z}$.)

There is an exact sequence due to Bloch and Wigner:

$$0 \to \mathbb{Q}/\mathbb{Z} \to H_3(\text{PSL}(2, \mathbb{C})^\delta; \mathbb{Z}) \to \mathcal{B}(\mathbb{C}) \to 0.$$

The superscript $\delta$ means “with discrete topology.” We will omit it from now on.

$\mathcal{B}(\mathbb{C})$ is known to be uniquely divisible, so it has canonically the structure of a $\mathbb{Q}$-vector space (Suslin [20]). It’s $\mathbb{Q}$-dimension is infinite and conjectured to be countable (the “Rigidity Conjecture,” equivalent to the conjecture that $\mathcal{B}(\mathbb{C}) = \mathcal{B}(\overline{\mathbb{Q}})$, where $\overline{\mathbb{Q}}$ is the field of algebraic numbers). In particular, the $\mathbb{Q}/\mathbb{Z}$ in the Bloch-Wigner exact sequence is precisely the torsion of $H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z})$, so any finite torsion subgroup is cyclic.

1991 Mathematics Subject Classification. 57M27; 19E99, 57T99.
In the present paper we define an extended Bloch group $\widehat{B}(\mathbb{C})$ by replacing $\mathbb{C} - \{0, 1\}$ in the definition of $\mathcal{B}(\mathbb{C})$ by an abelian cover and appropriately lifting the five term relation (1). Our main results are that we can lift the Bloch-Wigner map $H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z}) \to \mathcal{B}(\mathbb{C})$ to an isomorphism

$$\lambda: H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z}) \xrightarrow{\cong} \widehat{B}(\mathbb{C})$$

Moreover, the “Roger’s dilogarithm function” (see below) gives a natural map

$$R: \widehat{B}(\mathbb{C}) \to \mathbb{C}/\pi^2\mathbb{Z}.$$ 

We show that the composition

$$R \circ \lambda: H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z}) \to \mathbb{C}/\pi^2\mathbb{Z}$$

is the Cheeger-Chern-Simons class (cf [3]), so it can also be described as $i(\text{vol} + i\text{cs})$, where $\text{cs}$ is the universal Chern-Simons class. It has been a longstanding problem to provide such a computation of the Chern-Simons class. Dupont in [5] gave an answer modulo $\pi^2\mathbb{Q}$ and our computation is a natural lift of his.

We show that any complete hyperbolic 3-manifold $M$ of finite volume has a natural “fundamental class” in $H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z})$. For compact $M$ the existence of this class is easy and well known: $M = \mathbb{H}^3/\Gamma$ is a $K(\Gamma, 1)$-space, so the inclusion $\Gamma \to \text{Isom}^+(\mathbb{H}^3) = \text{PSL}(2, \mathbb{C})$ induces $H_3(M) = H_3(\Gamma) \to H_3(\text{PSL}(2, \mathbb{C}))$, and the class in question is the image of the fundamental class $[M] \in H_3(M)$. For non-compact $M$ I do not know a simple algebraic description; the existence is shown in Section 12. The Cheeger-Chern-Simons class applied to this class gives $i(\text{vol}(M) + i\text{cs}(M))$ where $\text{vol}(M)$ is hyperbolic volume and $\text{cs}(M)$ is the Chern-Simons invariant. This Chern-Simons invariant is defined modulo $\pi^2\mathbb{Z}$ and for compact $M$ it arises here as the Chern-Simons invariant for the flat $\text{PSL}(2, \mathbb{C})$-bundle over $M$. According to Dupont (private communication) this is the same as the Chern-Simons invariant for the Riemannian connection over $M$ (this is proved modulo 6-torsion in [6]). However, the Chern-Simons invariant for the Riemannian connection carries slightly more information, since it is defined modulo $2\pi^2$ rather than modulo $\pi^2$. For non-compact $M$ we show that the Chern-Simons invariant is the one defined by Meyerhoff [10], which is naturally only defined modulo $\pi^2$.

This fundamental class of $M$ in $H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z})$ determines an element $\tilde{\beta}(M) \in \widehat{B}(\mathbb{C})$. Our results describe $\tilde{\beta}(M)$ directly in terms of an ideal triangulation of $M$, as a lift of the Bloch invariant $\beta(M) \in \mathcal{B}(\mathbb{C})$ defined in [14]. We need a “true” ideal triangulation, which is more restrictive than the “degree 1” ideal triangulations that sufficed in [14]. The ideal triangulations resulting from Dehn filling that are used by the programs Snappea [21] and Snap [9] (see also [4]) are not true, but we describe $\tilde{\beta}(M)$ in terms of these “Dehn filling triangulations” in Theorems 12.4 and 12.6, and deduce practical simplicial formulae for Chern-Simons invariants of hyperbolic 3-manifolds.

The formula of Theorem 12.4 exhibits directly the analytic relationship, conjectured in [15] and proved in [23], between volume and Chern-Simons invariant of hyperbolic manifolds, and gives an independent proof of it. A similar formula for Chern-Simons invariant was derived from [15, 23] in [12], but that formula was only accurate up to an unknown constant on any given Dehn surgery space (the constant was conjectured to be a multiple of $\pi^2/6$, and our results now confirm this). Snappea and versions of Snap prior to 1.10.2 use versions of that formula, and use a bootstrapping procedure to discover the constant and obtain the Chern-Simons
invariant for many manifolds. Snap now implements the formula of this paper and can thus compute the Chern-Simons invariant for any hyperbolic manifold.

More generally, any flat $\text{PSL}(2, \mathbb{C})$ bundle over a closed oriented 3-manifold $M$ determines a class in $H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z})$ and our results give a simplicial computation of this class as an element of $\check{B}(\mathbb{C})$.

The main results of this paper were announced in [13] and partial proofs were in the preliminary preprint [16] (sections 1–7 of this paper are essentially the same as the preliminary preprint of [16] with the addition of Lemma 7.4 and some clarifications in Section 4). This paper corrects the tentative statement in [13] (also in [16]) that our map $\lambda$ is not an isomorphism but has a cyclic kernel of order 2. There is a related error in Section 6B of [4]: it is wrongly stated that the Cheeger-Chern-Simons class takes values in $\mathbb{C}/2\pi^2 \mathbb{Z}$ rather than $\mathbb{C}/\pi^2 \mathbb{Z}$ (and therefore that the fundamental class in $H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z})$ of a cusped hyperbolic 3-manifold has a mod 2 ambiguity).

Acknowledgments. The definition of the extended Bloch group was suggested by an idea of Jun Yang, to whom I am grateful also for many useful conversations. I'm grateful to Benedetti and Baselier, whose use in [1] of the preliminary preprint [16] and subsequent correspondence led me to finish writing this paper. This research was supported by the Australian Research Council and by the NSF.

2. The extended Bloch group

We shall need a $\mathbb{Z} \times \mathbb{Z}$ cover $\hat{\mathbb{C}}$ of $\mathbb{C} \setminus \{0, 1\}$ which can be constructed as follows. Let $P$ be $\mathbb{C} \setminus \{0, 1\}$ split along the rays $(-\infty, 0)$ and $(1, \infty)$. Thus each real number $r$ outside the interval $[0, 1]$ occurs twice in $P$, once in the upper half plane of $\mathbb{C}$ and once in the lower half plane of $\mathbb{C}$. We denote these two occurrences of $r$ by $r + 0i$ and $r - 0i$. We construct $\hat{\mathbb{C}}$ as an identification space from $P \times \mathbb{Z} \times \mathbb{Z}$ by identifying

$$(x + 0i, p, q) \sim (x - 0i, p + 2, q) \quad \text{for each } x \in (-\infty, 0)$$

$$(x + 0i, p, q) \sim (x - 0i, p, q + 2) \quad \text{for each } x \in (1, \infty).$$

We will denote the equivalence class of $(z, p, q)$ by $(z; p, q)$. $\hat{\mathbb{C}}$ has four components:

$$\hat{\mathbb{C}} = X_{00} \cup X_{01} \cup X_{10} \cup X_{11}$$

where $X_{\epsilon_0, \epsilon_1}$ is the set of $(z; p, q) \in \hat{\mathbb{C}}$ with $p \equiv \epsilon_0$ and $q \equiv \epsilon_1 \ (\text{mod } 2)$.

We may think of $X_{00}$ as the Riemann surface for the multivalued function

$$\mathbb{C} \setminus \{0, 1\} \to \mathbb{C}^2, \quad z \mapsto (\log z, -\log(1 - z)).$$

Taking the branch $(\log z + 2p\pi i, -\log(1 - z) + 2q\pi i)$ of this function on the portion $P \times \{(2p, 2q)\}$ of $X_{00}$ for each $p, q \in \mathbb{Z}$ defines an analytic function from $X_{00}$ to $\mathbb{C}^2$. In the same way, we may think of $\hat{\mathbb{C}}$ as the Riemann surface for the multivalued function $(\log z + p\pi i, -\log(1 - z) + q\pi i)$, $p, q \in \mathbb{Z}$, on $\mathbb{C} \setminus \{0, 1\}$.

Consider the set

$$\text{FT} := \left\{ \left( x, y, \frac{y}{x}, \frac{1 - x^{-1}}{1 - y^{-1}}, \frac{1 - x}{1 - y} \right) : x \neq y, x, y \in \mathbb{C} \setminus \{0, 1\} \right\} \subset (\mathbb{C} \setminus \{0, 1\})^5$$

of 5-tuples involved in the five term relation (1). An elementary computation shows:
Lemma 2.1. The subset \( FT^+ \) of \( (x_0, \ldots, x_4) \in FT \) with each \( x_i \) in the upper half plane of \( \mathbb{C} \) is the set of elements of \( FT \) for which \( y \) is in the upper half plane of \( \mathbb{C} \) and \( x \) is inside the triangle with vertices 0, 1, \( y \). Thus \( FT^+ \) is connected (even contractible).

\[ \square \]

Definition 2.2. Let \( V \subset (\mathbb{Z} \times \mathbb{Z})^5 \) be the subspace
\[ V := \{ ((p_0, q_0), (p_1, q_1), (p_1 - p_0, q_2), (q_1 - q_0, q_2 - q_1)), (q_1 - q_0, q_2 - q_1 - p_0) : p_0, p_1, q_0, q_1, q_2 \in \mathbb{Z} \}. \]

Let \( \hat{FT}_0 \) denote the unique component of the inverse image of \( FT \) in \( \hat{C}^5 \) which includes the points \( ((x_0; 0, 0), \ldots, (x_4; 0, 0)) \) with \( (x_0, \ldots, x_4) \in FT^+ \), and define
\[ \hat{FT} := \hat{FT}_0 + V = \{ x + v : x \in \hat{FT}_0 \text{ and } v \in V \}, \]
where we are using addition to denote the action of \( (\mathbb{Z} \times \mathbb{Z})^5 \) by covering transformations on \( \hat{C}^5 \). (Although we do not need it, one can show that the action of \( 2V \) takes \( \hat{FT}_0 \) to itself, so \( \hat{FT} \) has \( 2^5 \) components, determined by the parities of \( p_0, p_1, q_0, q_1, q_2 \).

Define \( \hat{P}(\mathbb{C}) \) as the free \( \mathbb{Z} \)-module on \( \hat{C} \) factored by all instances of the relations:

\[ \sum_{i=0}^{4} (−1)^i (x_i; p_i, q_i) = 0 \quad \text{with} \quad ((x_0; p_0, q_0), \ldots, (x_4; p_4, q_4)) \in \hat{FT} \]

and

\[ (x; p, q) + (x; p', q') = (x; p, q') + (x; p', q) \quad \text{with} \quad p, q, p', q' \in \mathbb{Z} \]

We shall denote the class of \( (z; p, q) \) in \( \hat{P}(\mathbb{C}) \) by \( [z; p, q] \).

We call relation (2) the lifted five term relation. We shall see that its precise form arises naturally in several contexts. In particular, we give it a geometric interpretation in Sect. 3.

We call relation (3) the transfer relation. It is almost a consequence of the lifted five term relation, since we shall see that the effect of omitting it would be to replace \( \hat{P}(\mathbb{C}) \) by \( \hat{P}(\mathbb{C}) \oplus \mathbb{Z}/2 \), with \( \mathbb{Z}/2 \) generated by an element \( \kappa := [x; 1, 1] + [x; 0, 0] − [x; 1, 0] − [x; 0, 1] \) which is independent of \( x \).

Lemma 2.3. There is a well-defined homomorphism
\[ \nu : \hat{P}(\mathbb{C}) \twoheadrightarrow \mathbb{C} \wedge \mathbb{C} \]
defined on generators by \( [z; p, q] \mapsto \log z + p\pi i) \wedge (−\log(1 − z) + q\pi i). \)

Proof. We must verify that \( \nu \) vanishes on the relations that define \( \hat{P}(\mathbb{C}) \). This is trivial for the transfer relation (3). We shall show that the lifted five term relation is the most general lift of the five term relation (1) for which \( \nu \) vanishes. If one applies \( \nu \) to an element \( \sum_{i=0}^{4} (−1)^i [x_i; p_i, q_i] \) with \( (x_0, \ldots, x_4) = (x, y, \ldots) \in FT^+ \) one obtains after simplification:

\[ ((q_0 − p_2 − q_2 + p_3 + q_3) \log x + (p_0 − q_1 + q_4) \log(1 − x) + (−q_1 + q_2 − q_3) \log y + (+p_1 + p_0 + q_3 + q_4) \log(1 − y) + (p_2 − p_3 + p_4) \log(x − y)) \wedge \pi i. \]

An elementary linear algebra computation shows that this vanishes identically if and only if \( p_2 = p_1 − p_0, p_3 = p_1 − p_0 + q_1 − q_0, q_3 = q_2 − q_1, p_4 = q_1 − q_0, \) and
Define $q_4 = q_2 - q_1 - p_0$, as in the lifted five term relation. The vanishing of $\nu$ for the general lifted five term relation now follows by analytic continuation. \qed

**Definition 2.4.** Define $\tilde{B}(\mathbb{C})$ as the kernel of $\nu: \hat{\mathcal{P}}(\mathbb{C}) \to \mathbb{C} \wedge \mathbb{C}$.

Define

$$R(z; p, q) = \mathcal{R}(z) + \frac{\pi i}{2} (p \log(1-z) + q \log z) - \frac{\pi^2}{6}$$

where $\mathcal{R}$ is the Rogers dilogarithm function

$$\mathcal{R}(z) = \frac{1}{2} \log(z) \log(1-z) - \int_0^z \frac{\log(1-t)}{t} dt.$$ 

Then

**Proposition 2.5.** $R$ gives a well defined map $R: \hat{\mathbb{C}} \to \mathbb{C}/\pi^2 \mathbb{Z}$. The relations which define $\hat{\mathcal{P}}(\mathbb{C})$ are functional equations for $R$ modulo $\pi^2$ (the lifted five term relation is in fact the most general lift of the five term relation (1) with this property). Thus $R$ also gives a homomorphism $R: \hat{\mathcal{P}}(\mathbb{C}) \to \mathbb{C}/\pi^2 \mathbb{Z}$.

**Proof.** If one follows a closed path from $z$ that goes anti-clockwise around the origin it is easily verified that $R(z; p, q)$ is replaced by $R(z; p, q) + \pi i \log(1-z) - q \pi^2 = R(z; p + 2, q) - q \pi^2$. Similarly, following a closed path clockwise around 1 replaces $R(z; p, q)$ by $R(z; p, q + 2) + p \pi^2$. Thus $R$ modulo $\pi^2$ is well defined on $\hat{\mathbb{C}}$ (and $R$ modulo $2 \pi^2$ is well defined on $X_{00}$; in fact $R$ itself is well defined on a $\mathbb{Z}$ cover of $X_{00}$ which is the universal nilpotent cover of $\mathbb{C} - \{0,1\}$).

It is well known that $L(z) := \mathcal{R}(z) - \frac{\pi^2}{6}$ satisfies the functional equation

$$L(x) - L(y) + L\left(\frac{y}{x}\right) - L\left(\frac{1-x^{-1}}{1-y^{-1}}\right) + L\left(\frac{1-x}{1-y}\right) = 0$$

for $0 < y < x < 1$. Since the 5-tuples involved in this equation are on the boundary of $\text{FT}^+$, the functional equation

$$\sum (-1)^i R(x; 0, 0) = 0$$

is valid by analytic continuation on the whole of $\text{FT}^+$. Now

$$\sum (-1)^i R(x; p_i, q_i)$$

differs from this by

$$\frac{\pi i}{2} \sum (-1)^i (p_i \log(1-x_i) + q_i \log x_i).$$

For $(x_0, \ldots, x_4) = (x, y, \ldots) \in \text{FT}^+$ this equals

$$\frac{\pi i}{2} ((q_0 - p_2 - q_2 + p_3 + q_3) \log x + (p_0 - q_2 + q_4) \log(1-x) + (-q_1 - q_2 + q_3) \log y + (-p_1 + p_3 + q_3 - p_4 - q_4) \log(1-y) + (p_2 - p_3 + p_4) \log(x-y))$$

and as in the proof of Lemma 2.3, this vanishes identically on $\text{FT}^+$ if and only if the $p_i$ and $q_i$ are as in the lifted five term relation. Thus the lifted five-term relation gives a functional equation for $R$ when $(x_0, \ldots, x_4) \in \text{FT}^+$. By analytic continuation, it is a functional equation for $R$ mod $\pi^2$ in general. The transfer relation is trivially a functional equation for $R$. \qed

Our main result is...
Theorem 2.6. There exists an isomorphism \( \lambda : H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z}) \to \hat{\mathcal{B}}(\mathbb{C}) \) such that the composition \( \lambda \circ R : H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z}) \to \mathbb{C}/\pi^2 \mathbb{Z} \) is the characteristic class given by \( i(\text{vol} + i \text{cs}) \).

To describe the map \( \lambda \) we must give a geometric interpretation of \( \hat{\mathcal{C}} \).

3. Parameters for ideal hyperbolic simplices

In this section we shall interpret \( \hat{\mathcal{C}} \) as a space of parameters for what we call “combinatorial flattenings” of ideal hyperbolic simplices. We need this to define the above map \( \lambda \). It also gives a geometric interpretation of the lifted five term relation.

We shall denote the standard compactification of \( \mathbb{H}^3 \) by \( \mathbb{H}^3 = \mathbb{H}^3 \cup \mathbb{C}P^1 \). An ideal simplex \( \Delta \) with vertices \( z_0, z_1, z_2, z_3 \in \mathbb{C}P^1 \) is determined up to congruence by the cross-ratio

\[
z = [z_0 : z_1 : z_2 : z_3] = \frac{(z_2 - z_1)(z_3 - z_0)}{(z_2 - z_0)(z_3 - z_1)}.
\]

Permuting the vertices by an even (i.e., orientation preserving) permutation replaces \( z \) by one of

- \( z \), \( z' = 1 - \frac{1}{z} \), or \( z'' = 1 - \frac{1}{\overline{z}} \).

The parameter \( z \) lies in the upper half plane of \( \mathbb{C} \) if the orientation induced by the given ordering of the vertices agrees with the orientation of \( \mathbb{H}^3 \). But we allow simplices whose vertex ordering does not agree with their orientation. We also allow degenerate ideal simplices whose vertices lie in one plane, so the parameter \( z \) is real. However, we always require that the vertices are distinct. Thus the parameter \( z \) of the simplex lies in \( \mathbb{C} - \{0, 1\} \) and every such \( z \) corresponds to an ideal simplex.

There is another way of describing the cross-ratio parameter \( z = [z_0 : z_1 : z_2 : z_3] \) of a simplex. The group of orientation preserving isometries of \( \mathbb{H}^3 \) fixing the points \( z_0 \) and \( z_1 \) is isomorphic to \( \mathbb{C}^* \) and the element of this \( \mathbb{C}^* \) that takes \( z_2 \) to \( z_3 \) is \( z \) (equivalently: if we position \( z_0, z_1 \) at 0, \( \infty \) in the upper half-space model of \( \mathbb{H}^3 \) then \( z_3 = zz_2 \)). Thus the cross-ratio parameter \( z \) is associated with the edge \( z_0z_1 \) of the simplex\(^1\). The parameter associated in this way with the other two edges \( z_0z_3 \) and \( z_0z_2 \) out of \( z_0 \) are \( z' \) and \( z'' \) respectively, while the edges \( z_2z_3 \), \( z_1z_2 \), and \( z_1z_3 \) have the same parameters \( z \), \( z' \), and \( z'' \) as their opposite edges. See Fig. 1.

\[\text{Figure 1.}\]

\(^1\)It is associated with the unoriented edge since there is an orientation preserving symmetry of the simplex exchanging \( z_0 \) and \( z_1 \) (it also exchanges \( z_2 \) and \( z_3 \) and acts on \( \mathbb{C}^* \) by \( z \mapsto 1/z \)).
Note that \( zz'z'' = -1 \), so the sum
\[
\log z + \log z' + \log z''
\]
is an odd multiple of \( \pi i \), depending on the branches of \( \log \) used. In fact, if we use standard branch of \( \log \) then this sum is \( \pi i \) or \(-\pi i \) depending on whether \( z \) is in the upper or lower half plane.

The imaginary parts of \( \log z, \log z' \), and \( \log z'' \) are the dihedral angles of the ideal simplex (resp. their negatives if the vertex ordering does not agree with the orientation the simplex inherits from \( \mathbb{H}^3 \)). We now consider certain “adjustments” of these angles by multiples of \( \pi \).

**Definition 3.1.** We shall call any triple of the form
\[
w = (w_0, w_1, w_2) = (\log z + p\pi i, \log z' + q\pi i, \log z'' + r\pi i)
\]
with \( p, q, r \in \mathbb{Z} \) and \( w_0 + w_1 + w_2 = 0 \)
a **combinatorial flattening** for our simplex.

Each edge \( E \) of \( \Delta \) is assigned one of the components \( w_i \) of \( w \), with opposite edges being assigned the same component. We call \( w_i \) the **log-parameter** for the edge \( E \) and denote it \( l_E(\Delta, w) \).

This combinatorial flattening can be written
\[
\ell(z; p, q) := (\log z + p\pi i, -\log(1 - z) + q\pi i, \log(1 - z) - \log z - (p + q)\pi i),
\]
and \( \ell \) is then a map of \( \hat{C} \) to the set of combinatorial flattenings of simplices.

**Lemma 3.2.** This map \( \ell \) is a bijection, so \( \hat{C} \) may be identified with the set of all combinatorial flattenings of ideal tetrahedra.

**Proof.** We must show that we can recover \((z; p, q)\) from \((w_0, w_1, w_2) = \ell(z; p, q)\). It clearly suffices to recover \( z \). But \( z = \pm e^{w_0} \) and \( 1 - z = \pm e^{-w_1} \), and the knowledge of both \( z \) and \( 1 - z \) up to sign determines \( z \). \( \square \)

We can give a geometric interpretation of the choice of parameters in the five term relation (2). If \( z_0, \ldots, z_4 \) are five distinct points of \( \partial \mathbb{H}^3 \), then each choice of four of five points \( z_0, \ldots, z_4 \) gives an ideal simplex. We denote the simplex which omits vertex \( z_i \) by \( \Delta_i \). The cross ratio parameters \( x_i = [z_0 : \ldots : z_i : \ldots : z_4] \) of these simplices can be expressed in terms of \( x := x_0 \) and \( y := x_1 \) as
\[
\begin{align*}
x_0 &= [z_1 : z_2 : z_3 : z_4] =: x \\
x_1 &= [z_0 : z_2 : z_3 : z_4] =: y \\
x_2 &= [z_0 : z_1 : z_3 : z_4] = \frac{y}{x} \\
x_3 &= [z_0 : z_1 : z_2 : z_4] = \frac{1 - x^{-1}}{1 - y^{-1}} \\
x_4 &= [z_0 : z_1 : z_2 : z_3] = \frac{1 - x}{1 - y}
\end{align*}
\]
The lifted five term relation has the form
\[
\sum_{i=0}^{4} (-1)^i(x_i; p_i, q_i) = 0
\]
with certain relations on the \( p_i \) and \( q_i \). We will give a geometric interpretation of these relations.

Using the map of Lemma 3.2, each summand in this relation (4) represents a choice \( \ell(x_i; p_i, q_i) \) of combinatorial flattening for one of the five ideal simplices. For each 1-simplex \( E \) connecting two of the points \( z_i \) we get a corresponding linear combination

\[
(5) \sum_{i=0}^{4} (-1)^i l_E(\Delta_i, \ell(x_i; p_i, q_i))
\]

of log-parameters (Definition 3.1), where we put \( l_E(\Delta_i, \ell(x_i; p_i, q_i)) = 0 \) if the line \( E \) is not an edge of \( \Delta_i \). This linear combination has just three non-zero terms corresponding to the three simplices that meet at the edge \( E \). One easily checks that the real part is zero and the imaginary part can be interpreted (with care about orientations) as the sum of the “adjusted angles” of the three flattened simplices meeting at \( E \).

**Definition 3.3.** We say that the \((x_i; p_i, q_i)\) satisfy the flattening condition if each of the above linear combinations (5) of log-parameters is equal to zero. That is, the adjusted angle sum of the three simplices meeting at each edge is zero.

**Lemma 3.4.** Relation (4) is an instance of the lifted five term relation (2) if and only if the \((x_i; p_i, q_i)\) satisfy the flattening condition.

**Proof.** We first consider the case that \((x_0, \ldots, x_4) \in \text{FT}^+\). Recall this means that each \( x_i \) is in \( \mathbb{H} \). Geometrically, this implies that each of the above five tetrahedra is positively oriented by the ordering of its vertices. This implies the configuration of

![Figure 2.](image_url)

Fig. 2 with \( z_1 \) and \( z_3 \) on opposite sides of the plane of the triangle \( z_0z_2z_4 \) and the line from \( z_1 \) to \( z_3 \) passing through the interior of this triangle. Denote the combinatorial flattening of the \( i^{th} \) simplex by \( \ell(x_i; p_i, q_i) \). If we consider the log-parameters at the edge \( z_3z_4 \) for example, they are \( \log x + p_0 \pi i \), \( \log y + p_1 \pi i \), and \( \log(y/x) + p_2 \pi i \) and the condition is that \( (\log x + p_0 \pi i) - (\log y + p_1 \pi i) + (\log(y/x) + p_2 \pi i) = 0 \).

This implies \( p_2 = p_1 - p_0 \). Similarly the other edges lead to other relations among the \( p_i \) and \( q_i \), namely:
\[ \begin{align*}
    z_0 z_1 & : \quad p_2 - p_3 + p_4 = 0 \\
    z_1 z_2 & : \quad p_0 - q_3 + q_4 = 0 \\
    z_2 z_3 & : \quad q_0 - q_1 + p_3 = 0 \\
    z_3 z_4 & : \quad p_0 - p_1 + p_2 = 0 \\
    z_4 z_0 & : \quad -q_1 + q_2 - q_3 = 0 \\
\end{align*} \]

Elementary linear algebra verifies that these relations are equivalent to the equations \( p_2 = p_1 - p_0, p_3 = p_1 - p_0 + q_1 - q_0, q_3 = q_2 - q_1, p_4 = q_1 - q_0, \) and \( q_4 = q_2 - q_1 - p_0, \) as in the lifted five term relation (2). The lemma thus follows for \((x_0, \ldots, x_4) \in FT^+.\) It is then true in general by analytic continuation. \( \square \)

4. Definition of \( \lambda \)

Let \( G = \text{PSL}(2, \mathbb{C}) \) (with the discrete topology). In this section we describe the map \( \lambda : H_3(G; \mathbb{Z}) \to \widehat{\mathbb{P}}(\mathbb{C}). \)

We first describe the combinatorial representation of elements of \( H_3(G; \mathbb{Z}) \) that we will use. As we will describe, a special case is to give a closed oriented triangulated 3-manifold with a flat \( G \)-bundle on it. Any element of \( H_3(G; \mathbb{Z}) \) can be represented this way, but we do not want to restrict to this type of representation.

We need to clarify our terminology for simplicial complexes.

**Definition 4.1.** We use the usual concept of simplicial complex \( K \) except that we do not require that distinct simplices have different vertex sets (but we do require that closed simplices embed in \( |K| \)), i.e., vertices of a simplex are distinct in \( K \).

The (open) star of a 0-simplex \( v \) of \( K \) is the union of \( \tau - \partial \tau \) over simplices that have \( v \) as a vertex. It is an open neighborhood of \( v \) and is the open cone on a simplicial complex \( L_v \) called the link of \( v \). Note that \( L_v \) immerses, but does not necessarily embed in \( K \) as the boundary of the star of \( v \).

We really only need quasi-simplicial complexes — cell complexes whose cells are simplices with simplicial attaching maps, but no requirement that closed simplices embed. We discuss this later (see Proposition 9.2); our current more restrictive hypothesis is to simplify exposition.

4.1. Representing elements of \( H_n(G; \mathbb{Z}) \).

**Definition 4.2.** An ordered \( n \)-cycle will be a compact \( n \)-dimensional simplicial complex \( K \) such that the complement \( |K| - |K^{(n-3)}| \) of its \((n-3)\)-skeleton is an oriented \( n \)-manifold, together with an ordering of the vertices of each \( n \)-simplex of \( K \) so that these orderings agree on common faces. The ordering orients each \( n \)-simplex \( \Delta_i \) of \( K \), and this orientation may or may not agree with the orientation of \( |K| - |K^{(n-3)}| \). Let \( \epsilon_i = +1 \) or \( -1 \) accordingly. Then the \( n \)-cycle \( \sum_i \epsilon_i \Delta_i \) represents a homology class \([K]\) \( \in H_n(|K|; \mathbb{Z}) \) called the fundamental class.

We will also require that the link \( L_v \) of each 0-simplex is connected. This can always be achieved by duplicating 0-simplices if necessary.

One can represent any element of \( H_n(G; \mathbb{Z}) \) by giving an ordered \( n \)-cycle \( K \) and labeling the vertices of each simplex \( \Delta \) of \( K \) by elements of \( G \) so that:

- Two \( G \)-labelings \((g_0, \ldots, g_k)\) and \((g_0', \ldots, g_k')\) of an ordered \( k \)-simplex are considered equivalent if there is a \( g \in G \) with \( gg_i = g_i' \) for each \( i \);
- The \( G \)-labeling of any face of any simplex \( \Delta \) is equivalent to the \( G \)-labeling induced from \( \Delta \).
We will also required that the labels for the vertices of any \( n \)-simplex are distinct (we can do this because \( G \) is infinite).

We describe two ways of seeing how this data describes an element of \( H_n(G, \mathbb{Z}) \). The first is algebraic, and is taken from \cite{16}.

4.1.1. Algebraic description. We recall a standard chain complex for homology of \( G = \text{PSL}(2, \mathbb{C}) \), the chain complex of “homogeneous simplices for \( G \).” We will, however, diverge from the standard by using only non-degenerate simplices, i.e., simplices with distinct vertices — we may do this because \( G \) is infinite.

Let \( C_n(G) \) denote the free \( \mathbb{Z} \)-module on all ordered \((n + 1)\)-tuples \( (g_0, \ldots, g_n) \) of distinct elements of \( G \). Define \( \delta: C_n \to C_{n-1} \) by

\[
\delta(g_0, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \hat{g}_i, \ldots, g_n).
\]

Then each \( C_n \) is a free \( \mathbb{Z}G \)-module under left-multiplication by \( G \). Since \( G \) is infinite the sequence

\[
\cdots \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0
\]

is exact, so it is a \( \mathbb{Z}G \)-free resolution of \( \mathbb{Z} \). Thus the chain complex

\[
\cdots \to C_2 \otimes_{\mathbb{Z}G} \mathbb{Z} \to C_1 \otimes_{\mathbb{Z}G} \mathbb{Z} \to C_0 \otimes_{\mathbb{Z}G} \mathbb{Z} \to 0
\]

computes the homology of \( G \). Note that \( C_n \otimes_{\mathbb{Z}G} \mathbb{Z} \) is the free \( \mathbb{Z} \)-module on symbols \( (g_0 : \ldots : g_n) \), where the \( g_i \) are distinct elements of \( G \) and \( (g_0 : \ldots : g_n) = (g_0' : \ldots : g_n') \) if and only if there is a \( g \in G \) with \( gg_i = g'_i \) for \( i = 0, \ldots, n \) (we call these homogeneous \( n \)-simplices for \( G \)).

Thus an element of \( \alpha \in H_n(G; \mathbb{Z}) \) is represented by a sum

\[
\sum \epsilon_i (g_0^{(i)} : \ldots : g_n^{(i)})
\]

of homogeneous \( n \)-simplices for \( G \) and their negatives (here each \( \epsilon_i \) is \( \pm 1 \)). The fact that this is a cycle means that the \((n-1)\)-faces of these homogeneous simplices cancel in pairs. We choose some specific way of pairing canceling faces and form a geometric quasi-simplicial complex \( K \) by taking a \( n \)-simplex \( \Delta_i \) for each homogeneous \( n \)-simplex of the above sum and gluing together \((n-1)\)-faces of these \( \Delta_i \) that correspond to \((n-1)\)-faces of the homogeneous simplices that have been paired with each other.

As already mentioned, a quasi-simplicial complex actually suffices for our purposes. But we can obtain a simplicial complex by replacing \( K \) by its barycentric subdivision if necessary (label the barycenter of each simplex by an arbitrary element of \( G \) that differs from the \( G \)-labels of the barycenters of all proper faces of that simplex). A standard argument shows that this does not change the homology class represented by \( K \). We thus get a representation of the homology class in the promised form.

4.1.2. Topological description. Given a space \( X \), any element \( \alpha \in H_n(X; \mathbb{Z}) \) can be represented as \( f_*[K] \) for some map \( f: [K] \to X \) of an ordered \( n \)-cycle to \( X \). If \( X = BG \) is a classifying space for \( G \) then this map is determined up to homotopy by the flat \( G \)-bundle \( f^*EG \) over \([K]\), so this flat bundle determines the homology class \( \alpha \in H_n(BG; \mathbb{Z}) = H_n(G; \mathbb{Z}) \).

Flat \( G \)-bundles over \([K]\) are determined by \( G \)-valued 1-cocycles on \( K \) up to the coboundary action of \( G \)-valued 0-cochains (the 1-cocycle is the obstruction to
extending over $|K^{(1)}|$ a chosen section of the flat bundle over $K^{(0)}$). We recall the basic definitions.

Let $S_1(K)$ be the set of ordered $g$-simplices of $K$. A $G$-valued 1-cocycle on $K$ is a map $\sigma : S_1(K) \to G$ with the cocycle property:

$$\sigma(v_0, v_2) = \sigma(v_0, v_1)\sigma(v_1, v_2) \quad \text{for } \langle v_0, v_1, v_2 \rangle \in S_2(K).$$

For convenience we extend the definition of $\sigma$ to reverse-ordered simplices by

$$\sigma(v_1, v_0) = \sigma(v_0, v_1)^{-1} \quad \text{for } \langle v_0, v_1 \rangle \in S_1(K).$$

If $\tau : S_0(K) \to G$ is a 0-cochain, then its coboundary action on 1-cocycles is to replace $\sigma$ by

$$\langle v_0, v_1 \rangle \mapsto \tau(v_0)^{-1}\sigma(v_0, v_1)\tau(v_1).$$

A $G$-valued 1-cocycle $\sigma$ determines a $G$-labeling of simplices: label a simplex $\langle v_0, \ldots, v_k \rangle$ by $(1, \sigma(v_0, v_1), \ldots, \sigma(v_0, v_k))$. Conversely, a $G$-labeling of simplices determines a 1-cocycle: assign to a 1-simplex with label $(g_1, g_2)$ the element $g_1^{-1}g_2 \in G$. These correspondences are clearly mutually inverse. When a 1-cocycle $\sigma$ is changed by the coboundary action of a 0-cochain $\tau$, the corresponding $G$-labeling $(g_0, \ldots, g_k)$ of a $k$-simplex $\langle v_0, \ldots, v_k \rangle$ is replaced by $(g_0\tau(v_0), \ldots, g_k\tau(v_k))$.

Thus, we get our desired representative as in subsection 4.1 for a homology class $\alpha \in H_n(G; \mathbb{Z})$ by representing $\alpha$ by a flat $G$ bundle over an ordered $n$-cycle $K$, representing that by a $G$-valued 1-cycle on $K$, and then taking the corresponding $G$-labeling of $K$. We want labels of the vertices of any $n$-simplex to be distinct, which means that the 1-cocycle should never take the value 1. Since $G$ is infinite, this can be achieved by modifying by a coboundary if necessary.

4.2. Definition of $\lambda : H_3(G; \mathbb{Z}) \to \hat{\mathcal{P}}(\mathbb{C})$.

Let $K$ be an ordered 3-cycle. We call a closed path $\gamma$ in $|K|$ a normal path if it meets no 0- or 1-simplices of $K$ and crosses all 2-faces that it meets transversally. When such a path passes through a 3-simplex $\Delta_i$, entering and departing at different faces, there is a unique edge $E$ of the 3-simplex between these faces. We say the path passes this edge $E$.

(In the following it will often be necessary to distinguish between a 1-simplex of $K$ and the various edges of 3-simplices that are identified with this 1-simplex. To avoid excess notation, we will often use the same symbol for a 1 simplex of $K$ and the edges of 3-simplices that are identified with it, but we will refer “edges” or “1-simplices” to flag which we mean.)

Consider a choice of combinatorial flattening $w_i$ for each simplex $\Delta_i$. Then for each edge $E$ of a simplex $\Delta_i$ of $K$ we have a log-parameter $l_E = l_E(\Delta_i, w_i)$ assigned. Recall that this log-parameter has the form $\log z + s\pi i$ where $z$ is the cross-ratio parameter associated to the edge $E$ of simplex $\Delta_i$ and $s$ is some integer. We call $(s \mod 2)$ the parity parameter at the edge $E$ of $\Delta_i$ and denote it $\delta_E = \delta_E(\Delta_i, w_i)$.

**Definition 4.3.** Suppose $\gamma$ is a normal path in $|K|$. The parity along $\gamma$ is the sum

$$\sum_E \delta_E \pmod{2}$$

of the parity parameters of all the edges $E$ that $\gamma$ passes. Moreover, if $\gamma$ runs in the star of some fixed 0-simplex $v$ of $K$, then the log-parameter along the path is
the sum
\[ \sum_{E} \pm \epsilon_{i(E)} |E|, \]
summed over all edges \( E \) that \( \gamma \) passes, where:

- \( \ell_i(E) \) is the index \( i \) of the simplex \( \Delta_i \) that the edge \( E \) belongs to and \( \epsilon_{i(E)} \) is the coefficient \( \pm 1 \) that encodes whether the ordering of the vertices of \( \Delta_i(E) \) matches the orientation of \( |K| \) or not.
- the extra sign \( \pm \) is + or – according as the edge \( E \) is passed in a counterclockwise or clockwise fashion as viewed from the vertex.

If \( \gamma \) just encircles a 1-simplex (so the extra signs are all + or all –) we speak of the parity or log-parameter “around the 1-simplex.”

We assume we have represented an element \( \alpha \in H_3(G; \mathbb{Z}) \) by a \( G \)-labeled ordered 3-cycle \( K \) as in subsection 4.1. So to each 3-simplex \( \Delta_i \) of \( K \) is associated a 4-tuple \( \langle g_0^{(i)} : \ldots : g_3^{(i)} \rangle \), defined up to left-multiplication by elements of \( G \), and \( \epsilon_i = \pm 1 \) encodes whether the ordering of the vertices of \( \Delta_i \) matches the orientation of \( K \) or not.

In the following theorem we identify \( G = PSL(2, \mathbb{C}) \) with the isometry group \( \text{Isom}^+(\mathbb{H}^3) \).

**Theorem 4.4.** Choose \( z \in \partial \mathbb{H}^3 \) so that \( g_0^{(i)} z, g_1^{(i)} z, g_2^{(i)} z, g_3^{(i)} z \) are distinct points of \( \partial \mathbb{H}^3 \) for each \( i \). This defines an ideal hyperbolic simplex shape for each simplex \( \Delta_i \) of \( K \) and an associated cross ratio \( x_i = [g_0^{(i)} z : g_1^{(i)} z : g_2^{(i)} z : g_3^{(i)} z] \). There exist combinatorial flattenings \( w_i = \ell(x_i; p_i, q_i) \) of the simplices of \( K \) such that the parity along any normal path in \( K \) is zero and the log-parameter along any normal path in the star of any 0-simplex of \( K \) is zero (in particular, parity and log-parameter around each 1-simplex are zero). We call such an assignment a “flattening of \( K \).”

For such a flattening of \( K \) the element \( \sum \epsilon_i x_i [p_i; q_i] \in \widehat{P}(\mathbb{C}) \) only depends on the homology class \( \alpha \in H_3(PSL(2, \mathbb{C}); \mathbb{Z}) \). We denote it \( \lambda(\alpha) \). Moreover, \( \lambda(\alpha) \in \widehat{B}(\mathbb{C}) \) and \( \lambda: H_3(PSL(2, \mathbb{C}); \mathbb{Z}) \rightarrow \widehat{B}(\mathbb{C}) \) is a homomorphism.

In Remark 8.3 we point out that instead of choosing a single \( z \) in the above theorem, one may choose a different \( z \) for each vertex of \( K \).

If \( K \) is a triangulation of a manifold then the condition on log-parameters involved in a flattening of \( K \) is just that the log-parameter around each 1-simplex of \( K \) is zero. In fact this and the parity condition suffice in general to define \( \lambda \):

**Scholium.** To define \( \lambda(\alpha) \) in the above theorem the condition on log-parameters may be weakened to just require that the log-parameter about each 1-simplex is zero. (We then speak of a “weak flattening of \( K \).”)

The proof of theorem 4.4 is rather long so we postpone it to sections 7 through 9. We make one remark however. For the log parameters along normal paths as in the theorem to be zero, it must be true that they are multiples of \( \pi i \) if we use flattenings \( \ell(x; 0, 0) \) instead of the ones of the theorem, since the log parameters for \( \ell(x; 0, 0) \) and \( \ell(x; p_i, q_i) \) differ by multiples of \( \pi i \). We will show, in fact:

**Proposition 4.5.** Notation as in Theorem 4.4. If we use flattenings \( \ell(x; 0, 0) \) for the simplices of \( K \) then the log parameter along any normal path in the star of a 0-simplex is a multiple of \( 2\pi i \).
The proof of this will involve the “developing map,” which also helps to better visualize the meaning of the flattening condition of the theorem. It also uses the following related proposition, which we prove first.

**Proposition 4.6.** Notation as in Theorem 4.4. If we use flattenings ℓ(x; 0, 0) for the simplices of K then the parity along any normal path in K is zero.

**Proof.** As we follow a normal path the contribution to the parity as we pass an edge of a simplex is 0 if the edge is the 01, 03, 12, or 23 edge of the simplex and the contribution is ±1 if it is the 02 or 13 edge. Consider the orientations of the triangular faces we cross as we follow the path, where the orientation is the one induced by the ordering of its vertices. As we pass a 02 or 13 edge this orientation changes while for the other edges it does not. Since K is oriented, we must have an even number of orientation changes as we traverse the normal path, proving the claim. □

The following useful corollary of Propositions 4.5 and 4.6 is immediate from the definition of parity.

**Corollary 4.7.** For a normal path in the star of a vertex, if the flattening condition for log-parameters is satisfied, then so is the parity condition. □

### 4.3. Developing map

Suppose we have a G-labeled 3-cycle K as above. Choose a generic point z ∈ ∂H³ as in Theorem 4.4, so that each simplex Δi of K corresponds to a non-degenerate ideal simplex <g₀(1)z, g₁(1)z, g₂(1)z, g₃(1)z> in H³. This simplex is determined up to isometry, since <g₀(1), . . . , g₃(1)> is well defined up to left multiplication by elements of G = PSL(2, C) = Isom⁺(H³). This associates a geometry as an ideal simplex to each simplex of K, and hence, by lifting, also to each simplex of the universal cover Ō.

We would like to use this geometry to construct a developing map |Ō| → H³. The next lemma says that this is possible. Note that the ideal simplex corresponding to a 3-simplex of K inherits an orientation from H³ that will not in general agree with the orientation of the simplex in the manifold |K − K(10)|, so the developing map may be “folded” in that some simplices may be mapped reversing orientation. In particular, adjacent simplices will map to the same side of a common 2-face whenever the ideal simplex orientation of one of the simplices agrees with its orientation in |K| and for the other simplex it differs.

**Lemma 4.8.** There is a map D: |Ō| → H³ (unique up to isometries of H³) which maps each simplex of the universal cover Ō of K isometrically and preserving orientation with respect to its ideal simplex shape just described. We call it the “developing map.” The map is equivariant with respect to an action of π₁(K) on H³.

**Proof.** Consider the corresponding 1-cocyle σ (subsection 4.1.2) whose value on an unordered 1-simplex <v₁, v₂> with label (g₁, g₂) is σ(<v₁, v₂>) = g₁⁻¹g₂. Any edge path <v₀, v₁><v₁, v₂> . . . <vₖ₋₁, vₖ> determines an element

σ(<v₀, v₁>)σ(<v₁, v₂>) . . . σ(<vₖ₋₁, vₖ>) ∈ G

and this gives a homomorphism φK from the groupoid of edge paths to G. In particular, it restricts to a group homomorphism π₁(K, v₀) → G. We thus get an action of π₁(K, v₀) on H³.
Choose a 0-simplex \( \tilde{v}_0 \) as a basepoint in the universal cover \( \tilde{K} \) of \( K \). We can then \( G \)-label the vertices of \( \tilde{K} \) by labeling vertex \( \tilde{v} \) by the element \( \phi_K(\gamma) \) where \( \gamma \) is the image of any edge path from \( \tilde{v}_0 \) to \( \tilde{v} \) in \( \tilde{K} \). This gives us a lift of the \( G \)-labeling of \( K \) to a labeling of \( \tilde{K} \) in which labels are absolute rather than just well defined up to left multiplication by \( G \). Using this labeling, the desired mapping of \([K] \) to \( PL^1 \) maps a simplex of \( \tilde{K} \) with labels \((g_1, \ldots, g_4)\) to the ideal simplex \((g_1z, \ldots, g_4z)\).

To show the equivariance of this map it helps to describe it more explicitly. For any two 0-simplices \( w_1, w_2 \in \tilde{K} \), let \( \phi[w_1, w_2] \) be \( \phi_K \) of the image in \( K \) of a simplicial path \([w_1, w_2]\) from \( w_1 \) to \( w_2 \) in \( \tilde{K} \). \( \phi[w_1, w_2] \) only depends on \( w_1 \) and \( w_2 \) since any two paths from \( w_1 \) to \( w_2 \) are homotopic. With this notation the developing map is

\[
D(w) = \phi[\tilde{v}_0, w]z \quad \text{for any } w \in \tilde{K}^{(0)}.
\]

The action of \( \pi_1(K, v_0) \) by covering transformations is such that the lift of a closed path representing \( \gamma \in \pi_1(K, v_0) \) is a path \([\tilde{v}_0, \gamma\tilde{v}_0]\).

Thus for any \( w \in \tilde{K}^{(0)} \) and \( \gamma \in \pi_1(K, v_0) \) we have

\[
\phi_K(\gamma)D(w) = \phi_K(\gamma)\phi[\tilde{v}_0, w]z = \phi[\tilde{v}_0, \gamma\tilde{v}_0][\phi[v_0, w]z] = \\
= \phi[\tilde{v}_0, \gamma\tilde{v}_0][\phi[v_0, \gamma w]z] = \phi[v_0, \gamma w]z = D(\gamma w)
\]

This proves the equivariance. \( \square \)

Proof of Proposition 4.5. Consider a normal path \( \gamma \) in the star \( N \) of a 0-simplex \( v \) of \( K \). Since \( N \) is contractible, it lifts homeomorphically to the star \( \tilde{N} \) of a lift \( \tilde{v} \) of \( v \) in \( \tilde{K} \). Use the upper half-space model of \( H^3 \) and choose the developing map \( D: [\tilde{K}] \rightarrow H^3 \) so \( D(\tilde{v}) = \infty \). Thus for each 3-simplex of the closure of \( \tilde{N} \), the other three vertices of the simplex map to points in \( C \subset C \cup \{\infty\} = \partial H^3 \), and thus determine a triangle in \( C \), with edges corresponding to 2-faces of the 3-simplex.

Denote the lift of \( \gamma \) to \( \tilde{N} \) by \( \tilde{\gamma} \). The path \( D \circ \tilde{\gamma} \) has a “shadow” in \( C \). As \( \gamma \) passes an edge of a 3-simplex of \( \tilde{N} \), the shadow passes the corresponding vertex of the corresponding triangle in \( C \), entering at one edge of the triangle and exiting at another. The cross-ratio parameter corresponding to the 3-simplex edge (or its inverse, depending on vertex ordering and how we are passing the edge) is the derivative of the linear map of \( C \) that takes the entering edge of the triangle to the exiting edge, fixing the common vertex. Since the path \( D \circ \tilde{\gamma} \) ends up where it started, the product of these cross-ratio parameters (each raised to an appropriate power \( \pm 1 \)) is 1, so the appropriately signed sum of their logs is a multiple of \( 2\pi i \). This sum is precisely the log-parameter along the path, except for the adjustments by multiples of \( \pi i \) that are involved in forming the log-parameters from the logarithms of cross-ratio parameters. To complete the proof we need that these adjustments add up to a multiple of \( 2\pi i \). In other words, we need the parity along the path \( \gamma \) to be zero. This was proved in Proposition 4.6. \( \square \)

The developing map gives a convenient way to visualize what it means to satisfy the flattening condition in the neighborhood of a 0-simplex \( v \) of \( K \). Suppose we have developed the star of \( v \) into \( H^3 \) as in the proof of Proposition 4.5, so that \( v \) maps to \( \infty \in C \cup \{\infty\} = \partial H^3 \) and the \( k \)-simplices in the link \( L_v \) of \( v \) therefore map to straight \( k \)-simplices in \( C \) for \( 0 \leq k \leq 2 \). Each 2-simplex incident to \( v \) determines a 1-simplex in \( L_v \), and hence a line segment in \( C \). For each such segment \( S \), the angle of this line segment from horizontal is well defined in \( \mathbb{R}/\pi \mathbb{Z} \); choose a specific lift \( a(S) \) of this number in \( \mathbb{R} \) and call it the rotation level for \( S \). Then
Figure 3. Three triangles in developing image of $N$ in $C$ showing rotation levels of edges and resulting adjusted angles. One triangle is from a “folded” simplex.

differences of rotation levels for segments corresponding to adjacent 2-faces of a 3-simplex determine a flattening (see Fig. 3). Specifically:

Suppose we have two adjacent 2-faces incident to $v$ of a 3-simplex $\Delta_i$ of $N$. Let $S_1$ and $S_2$ be the corresponding segments in the plane $C$, taken in the order specified by the orientation $\Delta_i$ inherits from $K$ (so $S_1$ and $S_2$ are in clockwise order around the triangle in $C$ determined by $\Delta$ if the developing map $D$ preserves orientation of $\Delta$ and anticlockwise order otherwise). Specify the adjusted angle between these 2-faces of $\Delta_i$ to be $\epsilon_i(\alpha(S_2) - \alpha(S_1))$ (where, as usual, $\epsilon_i = \pm 1$ is determined by vertex-order of $\Delta_i$).

This clearly specifies a flattening for the 3-simplices of $N$ and the flattening conditions are satisfied at $v$. Conversely, such a flattening for $N$ plus a rotation level $\alpha(S)$ for one of the line segments determines $\alpha(S)$ for every segment, so the flattening determines the rotation level function $\alpha$ up to a multiple of $\pi$.

5. Computation of $\hat{B}(C)$

In this section we work out the relationship of the extended Bloch group with the usual one (Theorem 5.6). For this we will need a general relation (Lemma 5.1 below) in $\hat{P}(C)$ that follows from the lifted five term relation.

Let $K$ be a simplicial complex obtained by gluing 3-simplices $\Delta_1, \ldots, \Delta_n$ together in sequence around a common 1-simplex $E$. Thus, for each index $j$ modulo $n$, $\Delta_j$
is glued to each of $\Delta_{j-1}$ and $\Delta_{j+1}$ along one of the two faces of $\Delta_j$ incident to $E$. Suppose, moreover, that the vertices of each $\Delta_j$ are ordered such that orderings agree on the common 2-faces of adjacent 3-simplices.

There is then a sequence $\epsilon_1 = \pm 1, \ldots, \epsilon_n = \pm 1$ such that the 2-faces used for gluing all cancel in the boundary of the 3-chain $\sum_{j=1}^n \epsilon_j \Delta_j$. (Proof: choose $\epsilon_1 = 1$ and then for $i = 2, \ldots, n$ choose $\epsilon_i$ so the common face of $\Delta_{i-1}$ and $\Delta_i$ cancels. The common face of $\Delta_n$ and $\Delta_1$ must then cancel since otherwise that face occurs with coefficient $\pm 2$ in $\partial \sum_{j=1}^n \epsilon_j \Delta_j$, and $E$ occurs with coefficient $\pm 2$ in $\partial \partial \sum_{j=1}^n \epsilon_j \Delta_j$.)

Suppose now further that a combinatorial flattening $w_j$ has been chosen for each $\Delta_j$ such that the “signed sum” of log parameters around the 1-simplex $E$ vanishes and the same for parity parameters:

\begin{equation}
\sum_{j=1}^n \epsilon_j l_E(\Delta_j, w_j) = 0, \quad \sum_{j=1}^n \epsilon_j \delta_E(\Delta_j, w_j) = 0.
\end{equation}

We think of the 1-simplex $E$ as being vertical, so that we can label the two 1-simplices other than $E$ of the common triangle of $\Delta_j$ and $\Delta_{j+1}$ as $T_j$ and $B_j$ (for “top” and “bottom”). Let $w_j'$ be the flattening obtained from $w_j$ by adding $\epsilon_j \pi i$ to the log parameter at $T_j$ and its opposite edge in $\Delta_j$ and subtracting $\epsilon_j \pi i$ from the log parameter at $B_j$ and its opposite edge in $\Delta_j$. If we do this for each $j$ then the total log parameter and parity parameter around any 1-simplex of the complex $K$ is not changed (we sum log-parameters with the appropriate sign $\epsilon_j$): — at $E$ no log-parameter has changed while at every other 1-simplex $\pi i$ has been added at one of the two adjacent 3-simplices and subtracted at the other.

**Lemma 5.1** (Cycle relation about $E$). With the above notation,

\[ \sum_{j=1}^n \epsilon_j [w_j] = \sum_{j=1}^n \epsilon_j [w_j'] \in \hat{P}(\mathbb{C}), \]

where we are using $[w]$ as a shorthand for $[\ell^{-1}w]$ (i.e., $[w]$ means $[z; p, q]$ where $\ell(z; p, q) = w$; see Lemma 3.2).

**Proof.** Each of the 3-simplices $\Delta_i$ has an associated ideal hyperbolic structure compatible with the combinatorial flattenings $w_j$. This ideal hyperbolic structure is also compatible with the flattening $w_j'$. Choose a realization of $\Delta_1$ as an ideal simplex in $\mathbb{H}^3$. We think of this as a mapping of $\Delta_1$ to $\mathbb{H}^3$. We can extend this
to a mapping of $K$ to $\mathbb{H}^3$ which maps each $\Delta_j$ to an ideal simplex with shape appropriate to its combinatorial flattening. Adjacent simplices will map to the same side of their common face in $\mathbb{H}^3$ if either their orientations or the signs $\epsilon_j$ do not match and will be on opposite sides otherwise. The fact that the signed sums of log and parity parameters around the 1-simplex $E$ are zero guarantees that the identifications match up as we go once around $E$.

Note that $K$ has $n + 2$ vertices. We first consider the special case that $n = 3$ and there is an ordering $v_0, \ldots, v_4$ of the five vertices of $K$ that restricts to the given vertex ordering for each simplex. We also assume the five vertices of $K$ map to distinct points $z_0, \ldots, z_4$ of $\partial \mathbb{H}^3$.

Each 3-simplex $\Delta_j$ for $j = 1, 2, 3$ has vertices obtained by omitting one of the five vertices $v_0, \ldots, v_4$. Denote by $\Delta_4$ and $\Delta_5$ the 3-simplices obtained by omitting each of the other two vertices. The fact that the common 2-faces of the $\Delta_j$ cancel when taking boundary of the chain $\epsilon_1 \Delta_1 + \epsilon_2 \Delta_2 + \epsilon_3 \Delta_3$ means that, up to sign this sum corresponds to three summands of the chain $\partial(v_0, \ldots, v_4) = \sum (-1)^i (v_0, \ldots, \hat{v}_i, \ldots, v_4)$. Choose $\epsilon_4$ and $\epsilon_5$ so that $\sum_{j=1}^5 \epsilon_j \Delta_j$ is $\pm \partial(v_0, \ldots, v_4)$.

We now claim that we can choose unique combinatorial flattenings $w_4$ and $w_5$ of $\Delta_4$ and $\Delta_5$ so that the signed sum of log parameters and parity parameters around any 1-simplex of $K \cup \Delta_4 \cup \Delta_5$ is zero. Indeed, this claim does not depend on the order of the vertices, so by permuting the vertices we can assume the five vertices are ordered so that $\Delta_1$, $\Delta_2$ and $\Delta_3$ are the first three 3-simplices occurring in the five term relation. Then the common 1-simplex $E$ is $v_3v_4$ and the fact that the simplices fit together around this 1-simplex is the condition that their cross-ratio parameters $x_0, x_1$, and $x_2$ satisfy $x_0 x_1^{-1} x_2 = 1$. Writing the flattenings as elements of $\hat{C}$ as $(x_0; p_0, q_0)$, $(x_1; p_1, q_1)$, and $(x_2; p_2, q_2)$, the equation saying signed sum of log parameters around this 1-simplex is zero is $(\log x_0 + p_0 \pi i) - (\log x_1 + p_1 \pi i) + (\log x_2 + p_2 \pi i) = 0$. If $x_0, x_1, \text{and } x_2$ are in the complex upper half plane this implies the equation $p_2 = p_1 - p_0$ of the lifted five term relation, while otherwise it implies the appropriate analytic continuations in $\hat{C}$ of this. The desired choice of flattenings of $\Delta_4$ and $\Delta_5$ is thus determined as in the lifted five term relation by the choice of $p_0, p_1, q_0, q_1, \text{and } q_2$ (namely $p_3 = p_1 - p_0 + q_1 - q_0$, $q_3 = q_2 - q_1$, $p_4 = q_1 - q_0$, and $q_4 = q_2 - q_1 - p_0$ if $x_0, x_1, x_2$ are in the upper half plane and otherwise the appropriate analytic continuation).

Note that $w_4$ and $w_5$ do not change if we replace $w_1, \ldots, w_3$ by $w'_1, \ldots, w'_3$ (with the above reordering of vertices this just subtracts 1 from each of $q_0, q_1$ and $q_2$ in the lifted five term relation, so it does not alter $p_3, q_3, p_4, q_4$). By Lemma 3.4 we then have

$$\epsilon_1 w_{1} + \epsilon_2 w_{2} + \epsilon_3 w_{3} = -(\epsilon_4 w_{4} + \epsilon_5 w_{5})$$

$$\epsilon_1 w'_{1} + \epsilon_2 w'_{2} + \epsilon_3 w'_{3} = -(\epsilon_4 w'_{4} + \epsilon_5 w'_{5}),$$

proving this case.

We next consider the case that for some index $j$ modulo $n$ the images of $\Delta_j$ and $\Delta_{j+1}$ in $\mathbb{H}^3$ do not coincide, so their union has five distinct vertices. By cycling our indices we may assume $j = 1$. Since the orderings of the vertices of $\Delta_1$ and of $\Delta_2$ agree on the three vertices they have in common, there is an ordering of all five vertices compatible with both $\Delta_1$ and $\Delta_2$. Let $\Delta_0$ be the simplex determined by the common 1-simplex $E$ and the two vertices that $\Delta_1$ and $\Delta_2$ do not have in common. Then there is an $\epsilon_0 = \pm 1$ such that the common faces of $\Delta_0$, $\Delta_1$, and
\( \Delta_2 \) cancel in the boundary of the chain \( \epsilon_0 \Delta_0 + \epsilon_1 \Delta_1 + \epsilon_2 \Delta_2 \). Choose a flattening \( w_0 \) of \( \Delta_0 \) such that \( \epsilon_0 l_E(\Delta_0, w_0) + \epsilon_1 l_E(\Delta_1, w_1) + \epsilon_2 l_E(\Delta_2, w_2) = 0 \). Then the relation of the lemma has already been proved for \( w_0, w_1, w_2 \), and by subtracting this relation from the relation to be proved for \( w_1, \ldots, w_n \) we obtain a case of the lemma with one fewer simplices. Thus, if we assume the lemma proved for \( n - 1 \) simplices then this case is also proved.

The above induction argument fails only for the case that there are \( 2m \) simplices that alternately “fold back on each other” so that their images in \( \mathbb{H}^3 \) all have the same four vertices. The above induction eventually reduces us to this case (usually with \( m = 1 \)). We must therefore deal with this situation to complete the proof. We first consider the case that \( m = 1 \) so \( n = 2 \). We then have four vertices \( z_0, \ldots, z_3 \) in \( \partial \mathbb{H}^3 \). We assume the 1-simplex \( E \) is \( z_0 z_1 \). Then the ordering of the vertices of the faces \( z_0 z_1 z_2 \) and \( z_0 z_1 z_3 \) is the same in each of \( \Delta_1 \) and \( \Delta_2 \). Choose a new point \( z_4 \) in \( \partial \mathbb{H}^3 \) distinct from \( z_0, \ldots, z_3 \) and consider the ordered simplex with vertices \( z_0, z_1, z_2 \) ordered as above followed by \( z_4 \). Call this \( \Delta_3 \). Similarly make \( \Delta_4 \) using \( z_0, z_1, z_3 \) ordered as above followed by \( z_4 \). Choose flattenings of \( \Delta_3 \) and \( \Delta_4 \) so that the signed sum of log parameters for \( \Delta_1, \Delta_3, \Delta_4 \) around \( E \) is zero. Then we obtain a three simplex relation of the type already proved for \( \Delta_1, \Delta_3, \Delta_4 \) and another for \( \Delta_2, \Delta_3, \Delta_4 \), and the difference of these two relations gives the desired two-simplex relation.

More generally, if we are in the above “folded” case with \( m > 1 \) we can use an instance of the three-simplex relation to replace one of the \( 2m \) simplices by two. We then use the induction step to replace one of these new simplices together with an adjacent old simplex by one simplex and then repeat for the other new simplex. In this way we reduce to a relation involving \( 2m - 1 \) simplices, completing the proof. \( \square \)

Before we continue, we note a consequence of the first case we considered in the above proof that we will need later. If vertices have been reoredded as in that proof then the relation we proved can be written (with the appropriate relationship among \( p_0, p_1, p_2 \)):

\[
(x; p_0, q_0) - (y; p_1, q_1) + (y; x, p_2, q_2) =
\]

\[
(x; p_0, q_0 - 1) - (y; p_1, q_1 - 1) + (y; x, p_2, q_2 - 1).
\]

This is true for any choice of \( q_0, q_1, q_2 \) so long as \( p_0, p_1, p_2 \) satisfy the appropriate relation. Thus if we just change \( q_0 \) and subtract the resulting equation from the above we get

\[
(x; p_0, q_0) - (x; p_0, q'_0) = (x; p_0, q_0 - 1) - (x; p_0, q'_0 - 1).
\]

From the versions of the above three-simplex case with different orderings of the vertices we can derive three versions of this relation:

\[
(x; p, q) - (x; p, q') = (x; p, q - 1) - (x; p, q' - 1)
\]

\[
(x; p, q) - (x; p', q) = (x; p - 1, q) - (x; p' - 1, q)
\]

\[
(x; p, q) - (x; p + s, q - s) = (x; p + 1, q - 1) - (x; p + s + 1, q - s - 1)
\]

From these we obtain:

**Lemma 5.2.** \((x; p, q) = pq[x; 1, 1] - (pq - p)[x; 1, 0] - (pq - q)[x; 0, 1] + (pq - p - q + 1)[x; 0, 0].\)
Proof. The first of the relations (8) implies \([x; p, q] = [x; p, q - 1] + [x; p, 1] - [x; p, 0]\) and applying this repeatedly shows
\[
[x; p, q] = q[x; p, 1] - (q - 1)[x; p, 0].
\]
The second equation of (8) implies similarly that \([x; p, q] = p[x; 1, q] - (p - 1)[x; 0, q]\) and using this to expand each of the terms on the right of (9) gives the desired equation.

Up to this point we have not used the transfer relation (3). We digress briefly to show that the transfer relation almost follows from the five term relation.

**Proposition 5.3.** If \(\hat{\mathcal{P}}(\mathbb{C})\) and \(\hat{\mathcal{B}}(\mathbb{C})\) are defined like \(\hat{\mathcal{P}}(\mathbb{C})\) and \(\hat{\mathcal{B}}(\mathbb{C})\) but without the transfer relation, then in \(\hat{\mathcal{P}}(\mathbb{C})\) the element \(\kappa := [x; 1, 1] + [x; 0, 0] - [x; 1, 0] - [x; 0, 1]\) is independent of \(x\) and has order 2. Moreover, \(\hat{\mathcal{P}}(\mathbb{C}) = \hat{\mathcal{P}}(\mathbb{C}) \times C_2\) and \(\hat{\mathcal{B}}(\mathbb{C}) = \hat{\mathcal{B}}(\mathbb{C}) \times C_2\), where \(C_2\) is the cyclic group of order 2 generated by \(\kappa\).

**Proof.** If we subtract equation (7) with \(p_0 = p_1 = q_0 = q_1 = q_2 = 1\) from the same equation with \(p_0 = p_1 = 0, q_0 = q_1 = q_2 = 1\) we obtain \([x; 1, 1] - [y; 1, 1] - [x; 0, 1] + [y; 0, 1] = [x; 1, 0] - [y; 1, 0] - [x; 0, 0] + [y; 0, 0]\), which rearranges to show that \(\kappa\) is independent of \(x\). The last of the equations (8) with \(p = q = 0\) and \(s = -1\) gives \(2[x; 0, 0] = [x; 1, -1] + [x; -1, 1]\) and expanding the right side of this using Lemma 5.2 gives \(2[x; 0, 0] = -2[x; 1, 1] + 2[x; 1, 0] + 2[x; 0, 1]\), showing that \(\kappa\) has order dividing 2.

To show \(\kappa\) has order exactly 2 we note that there is a homomorphism \(\epsilon: \hat{\mathcal{P}}(\mathbb{C}) \to \mathbb{Z}/2\) defined on generators by \([z; p, q] \mapsto (pq \text{ mod } 2)\). Indeed, it is easy to check that this vanishes on the lifted five-term relation, and is thus well defined on \(\hat{\mathcal{P}}(\mathbb{C})\). Since \(\epsilon(\kappa) = 1\) we see \(\kappa\) is non-trivial. Finally, Lemma 5.2 implies that the effect of the transfer relation is simply to kill the element \(\kappa\), so the final sentence of the proposition follows. \(\square\)

**Lemma 5.4.** For any \([x; p, q] \in \hat{\mathcal{P}}(\mathbb{C})\) one has \([x; p, q] + [1-x; -q, -p] = 2[1/2; 0, 0]\).

**Proof.** Assume first that \(0 < y < x < 1\). Then, as remarked in the proof of Proposition 2.5,
\[
[x; p_0, q_0] - [y; p_1, q_1] + \left(\frac{1}{1 - y}; p_1 - p_0, q_2\right) - \left(\frac{1}{1 - x}; p_1 - p_0 + q_1 - q_0, q_2 - q_1\right) + \left(\frac{1}{1 - x}; q_1 - q_0, q_2 - q_1 - p_0\right) = 0
\]
is an instance of the lifted five term relation. Replacing \(y\) by \(1 - x\), \(x\) by \(1 - y\), \(p_0\) by \(-q_1\), \(q_1\) by \(-q_0\), \(q_0\) by \(-p_1\), \(p_1\) by \(-p_0\), and \(q_2\) by \(q_2 - q_1 - p_0\) replaces this relation by exactly the same relation except that the first two terms are replaced by \([1 - y; -q_1, -p_1] - [1 - x; -q_0, -p_0]\). Thus subtracting the two relations gives:
\[
[x; p_0, q_0] - [y; p_1, q_1] - [1 - y; -q_1, -p_1] + [1 - x; -q_0, -p_0] = 0.
\]
Putting \([y; p_1, q_1] = [1/2; 0, 0]\) now proves the lemma for \(1/2 < x < 1\). But since we have shown this as a consequence of the lifted five term relation, we can analytically continue it over the whole of \(\hat{\mathbb{C}}\). \(\square\)

**Proposition 5.5.** The following sequence is exact:
\[
0 \rightarrow \mathbb{C}^* \xrightarrow{x} \hat{\mathcal{P}}(\mathbb{C}) \rightarrow \mathcal{P}(\mathbb{C}) \rightarrow 0
\]
where \( \hat{\mathcal{P}}(\mathbb{C}) \to \mathcal{P}(\mathbb{C}) \) is the natural map and \( \chi(z) := [z;0,1] - [z;0,0] \) for \( z \in \mathbb{C}^* \).

**Proof.** Denote \( \{z,p\} := [z;p,q] - [z;p,q-1] \) which is independent of \( q \) by the first equation of (8). By Lemma 5.4 we have \( [z;p,q] - [z;p-1,q] = -\{1 - z, -q\} \). It follows that elements of the form \( \{z,p\} \) generate \( \text{Ker}(\hat{\mathcal{P}}(\mathbb{C}) \to \mathcal{P}(\mathbb{C})) \). Computing \( \{z,p\} \) using Lemma 5.2 and the transfer relation, one finds \( \{z,0\} = \{z,0\} \) which only depends on \( z \). Thus the elements \( \{z,0\} = \chi(z) \) generate \( \text{Ker}(\hat{\mathcal{P}}(\mathbb{C}) \to \mathcal{P}(\mathbb{C})) \). If we take equation (7) with even \( p_i \) and subtract the same equation with the \( q_i \) reduced by 1 we get an equation that says that \( \chi: \mathbb{C}^* \to \text{Ker}(\hat{\mathcal{P}}(\mathbb{C}) \to \mathcal{P}(\mathbb{C})) \) is a homomorphism. We have just shown that it is surjective, and it is injective because \( R \circ \chi \) is the map \( \mathbb{C}^* \to \mathbb{C}/\pi^2 \) defined by \( z \mapsto \frac{z}{2} \log z \). \( \square \)

We can now describe the relationship of our extended groups with the “classical” ones.

**Theorem 5.6.** There is a commutative diagram with exact rows and columns

\[
\begin{array}{cccccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \mu^* & \mathcal{C}^* & \mathcal{C}^*/\mu^* & 0 \\
\chi|\mu^* & \chi & \beta & \\
0 & \hat{\mathcal{B}}(\mathbb{C}) & \hat{\mathcal{P}}(\mathbb{C}) & \mathcal{C} \wedge \mathcal{C} & K_2(\mathbb{C}) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{B}(\mathbb{C}) & \mathcal{P}(\mathbb{C}) & \mathcal{C}^* \wedge \mathcal{C}^* & K_2(\mathbb{C}) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0
\end{array}
\]

Here \( \mu^* \) is the group of roots of unity and the labeled maps defined as follows:

\[
\begin{align*}
\chi(z) &= [z;0,1] - [z;0,0] \\
\nu[z;p,q] &= (\log z + p\pi i) \wedge (-\log(1 - z) + q\pi i); \\
\nu'[z] &= 2(z \wedge (1 - z)); \\
\beta[z] &= \log z \wedge \pi i; \\
\epsilon(w_1 \wedge w_2) &= -2(\epsilon^{w_1} \wedge \epsilon^{w_2});
\end{align*}
\]

and the unlabeled maps are the obvious ones.

**Proof.** The top horizontal sequence is trivially exact while the other two are exact at their first two non-trivial groups by definition of \( \hat{\mathcal{B}} \) and \( \mathcal{B} \). The bottom row is exact also at its other two places by Milnor’s definition of \( K_2 \). The exactness of the third vertical sequence is elementary and the second one has just been proved. The commutativity of all but the top left square is elementary. A diagram chase confirms that \( \chi \) maps \( \mu^* \) to \( \hat{\mathcal{B}}(\mathbb{C}) \) and that the left vertical sequence is also exact. Another confirms exactness of the middle row. \( \square \)
6. The more extended Bloch group

This section is a digression. We describe a slightly more natural looking variant of the extended Bloch group, based on the universal abelian cover $X_{00}$ of $\mathbb{C} - \{0, 1\}$. It turns out to be a $\mathbb{Z}/2$ extension of $\hat{B}(\mathbb{C})$. We are not sure of its significance, so we describe it briefly.

Recall that $\hat{C}$ consists of four components $X_{00}, X_{01}, X_{10},$ and $X_{11},$ of which $X_{00}$ is naturally the universal abelian cover of $\mathbb{C} - \{0, 1\}$.

Let $\hat{FT}_{00}$ be $\hat{FT} \cap (X_{00})^5$, so $\hat{FT}_{00} = \hat{FT}_0 + 2V$ in the notation of Definition 2.2. As mentioned earlier, $\hat{FT}_{00}$ is, in fact, equal to $\hat{FT}_0$, but we do not need this.

Define $EP(C)$ to be the free $\mathbb{Z}$-module on $X_{00}$ factored by all instances of the relation

$$\sum_{i=0}^{4} (-1)^{i}(x_i; 2p_i, 2q_i) = 0 \quad \text{with} \quad (x_0; 2p_0, 2q_0), \ldots, (x_4; 2p_4, 2q_4) \in \hat{FT}_{00}.$$  

As before, we have a well-defined map

$$\nu: EP(C) \rightarrow \mathbb{C} \wedge \mathbb{C},$$

given by $\nu[z; 2p, 2q] = (\log z + 2p\pi i) \wedge (-\log(1 - z) + 2q\pi i)$, and we define

$$\mathcal{EB}(\mathbb{C}) := \text{Ker} \nu.$$  

The proof of Proposition 2.5 shows:

**Proposition 6.1.** The function $R(z; 2p, 2q) := R(z) + \pi i(p \log(1 - z) + q \log z) - \frac{\pi^2}{4}$ gives a well defined map $X_{00} \rightarrow \mathbb{C}/2\pi^2 \mathbb{Z}$ and induces a homomorphism $R: EP(\mathbb{C}) \rightarrow \mathbb{C}/2\pi^2 \mathbb{Z}$.  

We can repeat the computations in Sect. 5 word-for-word, replacing anything of the form $[x; p, q]$ by $[x; 2p, 2q]$, to show that $\text{Ker}(EP(\mathbb{C}) \rightarrow P(\mathbb{C}))$ is generated by elements of the form $\hat{\chi}(z) := [z; 0, 2] - [z; 0, 0]$. We thus get:

**Proposition 6.2.** The following sequence is exact:

$$0 \rightarrow \mathbb{C}^* \xrightarrow{\hat{\chi}} EP(\mathbb{C}) \rightarrow P(\mathbb{C}) \rightarrow 0$$

where $\hat{P}(\mathbb{C}) \rightarrow P(\mathbb{C})$ is the natural map and $\hat{\chi}(z) := [z; 0, 2] - [z; 0, 0]$ for $z \in \mathbb{C}^*$.

**Proof.** The only thing to prove is the injectivity of $\hat{\chi}$ which follows by noting that $R(\hat{\chi}(z)) = \pi i \log z \in \mathbb{C}/2\pi^2$.
Corollary 6.3. We have a commutative diagram with exact rows and columns:

\[
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
\mathbb{Z}/2 & \mathbb{Z}/2 \\
\downarrow & \downarrow \\
0 & \mathbb{C}^* & \mathcal{E}\mathcal{P}(\mathbb{C}) & \mathcal{P}(\mathbb{C}) & 0 \\
\downarrow_{z\mapsto z^2} & \downarrow & \downarrow \\
0 & \mathbb{C}^* & \hat{\mathcal{P}}(\mathbb{C}) & \mathcal{P}(\mathbb{C}) & 0 \\
\downarrow & \downarrow \\
0 & 0
\end{array}
\]

and analogously for the Bloch group:

\[
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
\mathbb{Z}/2 & \mathbb{Z}/2 \\
\downarrow & \downarrow \\
0 & \mu^* & \mathcal{E}\mathcal{B}(\mathbb{C}) & \mathcal{B}(\mathbb{C}) & 0 \\
\downarrow_{z\mapsto z^2} & \downarrow & \downarrow \\
0 & \mu^* & \hat{\mathcal{B}}(\mathbb{C}) & \mathcal{B}(\mathbb{C}) & 0 \\
\downarrow & \downarrow \\
0 & 0
\end{array}
\]

7. Start of proof of Theorem 4.4

In this section we start the proof of Theorem 4.4. We shall show that the flattening of $K$ promised in that theorem exists, and that for such a flattening the resulting element of $\hat{\mathcal{P}}(\mathbb{C})$ is in $\hat{\mathcal{B}}(\mathbb{C})$ and independent of the choice of flattening. In Section 8 we then show that it only depends on the complex $K$ and the flat bundle over it, and finally in Section 9 that it only depends on the homology class in $H_3(G;\mathbb{Z})$. This also leads to a proof of the Scholium to Theorem 4.4.

Recall that $K$ is an ordered 3-cycle: the vertices of each 3-simplex are ordered with orderings agreeing on common faces. We shall initially prove the theorem under the stronger condition that the vertex-orderings of simplices are induced from a “global” ordering of the set of all 0-simplices of $K$. This stronger condition is not needed in this section; it is first needed in Lemmas 8.4 and 8.5. In Proposition 9.2 we then show that this stronger condition can be eliminated; also that it is sufficient that $K$ be quasi-simplicial rather than simplicial.
The underlying space $|K|$ is an oriented 3-manifold except possibly at 0-simplices, where it is topologically the cone on a connected oriented surface, $L_v$, the link of $v$ (see Definitions 4.1 and 4.2).

We must first recall some notation and results from [12]. (The results of [12] do not require as strong conditions on $K$ as we have here: there, the ordering is not needed and $K$ is only required to be a quasi-simplicial complex.)

To an oriented 3-simplex $\Delta$ of $K$ we associate a 2-dimensional bilinear space $J_\Delta$ over $\mathbb{Z}$ as follows. As a $\mathbb{Z}$-module $J_\Delta$ is generated by the six edges $e_0, \ldots, e_5$ of $\Delta$ (see Fig. 5) with the relations:

$$e_i - e_{i+3} = 0 \quad \text{for } i = 0, 1, 2.$$  
$$e_0 + e_1 + e_2 = 0.$$

Thus, opposite edges of $\Delta$ represent the same element of $J_\Delta$, so $J_\Delta$ has three “geometric” generators, and the sum of these three generators is zero. The bilinear form on $J_\Delta$ is the non-singular skew-symmetric form given by

$$\langle e_0, e_1 \rangle = \langle e_1, e_2 \rangle = \langle e_2, e_0 \rangle = -\langle e_1, e_0 \rangle = -\langle e_3, e_1 \rangle = -\langle e_0, e_2 \rangle = 1.$$

Let $J$ be the direct sum $\coprod J_\Delta$, summed over the oriented 3-simplices of $K$. For $i = 0, 1$ let $C_i$ be the free $\mathbb{Z}$-module on the unoriented $i$-simplices of $K$. Define homomorphisms

$$\alpha: C_0 \to C_1 \quad \text{and} \quad \beta: C_1 \to J$$

as follows. $\alpha$ takes a vertex to the sum of the incident 1-simplices (with a 1-simplex counted twice if both endpoints are at the given vertex). The $J_\Delta$ component of $\beta$ takes a 1-simplex $E$ of $K$ to the sum of those edges $e_i$ in the edge set $\{e_0, e_1, \ldots, e_5\}$ of $\Delta$ which are identified with $E$ in $K$.

The natural basis of $C_i$ gives an identification of $C_i$ with its dual space and the bilinear form on $J$ gives an identification of $J$ with its dual space. With respect to these identifications, the dual map

$$\alpha^*: C_1 \to C_0$$

is easily seen to map a 1-simplex $E$ of $K$ to the sum of its endpoints, and the dual map

$$\beta^*: J \to C_1$$
can be described as follows. To each 3-simplex $\Delta$ of $K$ we have a map $j = j_\Delta$ of the edge set $\{e_0, e_1, \ldots, e_5\}$ of $\Delta$ to the set of 1-simplices of $K$: put $j(e_i)$ equal to the 1-simplex that $e_i$ is identified with in $K$. For $e_i$ in $J_{\Delta}$ we have

$$\beta^*(e_i) = j(e_{i+1}) - j(e_{i+2}) + j(e_{i+4}) - j(e_{i+5}) \quad (\text{indices mod 6}).$$

Let $K_0$ be the result of removing a small open cone neighborhood of each 0-simplex $v$ of $K$, so $\partial K_0$ is the disjoint union of the links $L_v$ of the vertices of $K$.

**Theorem 7.1** ([12], Theorem 4.2). The sequence

$$J : 0 \longrightarrow C_0 \xrightarrow{\alpha} C_1 \xrightarrow{\beta} J \xrightarrow{\beta^*} C_1 \xrightarrow{\alpha^*} C_0 \longrightarrow 0$$

is a chain complex. Its homology groups $H_i(J)$ (indexing the non-zero groups of $J$ from left to right with indices 5, 4, 3, 2, 1) are

$$H_5(J) = 0, \quad H_4(J) = \mathbb{Z}/2, \quad H_3(J) = \mathbb{Z}/2,$$

where $H = \text{Ker}(H_1(\partial K_0; \mathbb{Z}) \longrightarrow H_1(K_0; \mathbb{Z}/2))$. Moreover, the isomorphism $H_2(J) \xrightarrow{\text{iso}} H_1(K; \mathbb{Z}/2)$ results by interpreting an element of $\text{Ker}(\alpha^*) \subset C_1$ as an unoriented 1-cycle in $K$. \hfill $\square$

We now return to the proof of the existence of the flattenings promised in Theorem 4.4. Give each simplex $\Delta_i$ of the complex $K$ of that theorem the flattening $w^{(0)}_i := \ell(x_i; 0, 0)$. Recall that if this choice of flattenings satisfies the conditions of the theorem it is called a “flattening of $K$.” This will not hold in general, so we need to describe how to modify it so that the conditions of the theorem are satisfied.

Recall that to each 3-simplex $\Delta_i$ of $K$ is associated a sign $\epsilon_i = \pm 1$ that says whether the vertex-ordering of $\Delta_i$ agrees or disagrees with the orientation $\Delta_i$ inherits from the orientation of $K$.

If $\Delta$ is an ideal simplex and $w = (w_0, w_1, w_2)$ is a flattening of it, then denote

$$\xi(w) := w_1e_0 - w_0e_1 \in J_\Delta \otimes \mathbb{C}.$$ 

This definition is only apparently unsymmetrical since $w_1e_0 - w_0e_1 = w_2e_1 - w_1e_2 = w_0e_2 - w_2e_0$. Denote by $\omega$ the element of $J \otimes \mathbb{C}$ whose $\Delta_i$-component is $\epsilon_i\xi(w^{(0)}_i)$ for each $i$. That is, the $\Delta_i$-component of $\omega$ is $-\epsilon_i(\log(1 - x_i)e_0 + \log(x_i)e_1)$.

**Lemma 7.2.** $\frac{1}{\lambda_i} \beta^*(\omega)$ is an integer class in the kernel of $\alpha^*$, so it represents an element of the homology group $H_2(J)$. Moreover this element in $H_2(J)$ vanishes, so $\frac{1}{\lambda_i} \beta^*(\omega) = \beta^*(\delta)$ for some $\delta \in J$.

**Proof.** Let $J_\Delta$ be defined like $J_\Delta$ but without the relation $e_0 + e_1 + e_2 = 0$, so it is generated by the six edges $e_0, \ldots, e_5$ of $\Delta$ with relations $e_i = e_{i+3}$ for $i = 0, 1, 2$. Let $J$ be the direct sum $\bigoplus J_\Delta$ over 3-simplices $\Delta$ of $K$.

The map $\beta^* : J \rightarrow C_2$ factors as

$$\beta^* : J \xrightarrow{\beta_1} J \xrightarrow{\beta_2} C_2,$$

with $\beta_1$ and $\beta_2$ defined on each component by:

$$\begin{align*}
\beta_1(e_i) &= e_{i+1} - e_{i+2} \\
\beta_2(e_i) &= j(e_i) + j(e_{i+3})
\end{align*}$$

for $i = 0, 1, 2$. 

Note that \( \beta_1(\xi(w)) = w_0e_0 + w_1e_1 + w_2e_2 \in J_\Delta \otimes \mathbb{C} \). Thus if \( E \) is a 1-simplex of \( K \) then the \( E \)-component of \( \beta^*(\omega) = \beta_2\beta_1(\omega) \) is the signed sum of the log-parameters for \( E \) in the ideal simplices of \( K \) around \( E \) and is hence a multiple of \( 2\pi i \) by Proposition 4.5. That is,

\[
(12) \quad \frac{1}{\pi i} \beta^*(\omega) \in 2C_2.
\]

The lemma follows, since the isomorphism \( H_2(J) \rightarrow H_1(K;\mathbb{Z}/2) \) is the map which interprets an element of \( \text{Ker}(\alpha^*) \) as an unoriented 1-cycle in \( K \), and equation (12) says this 1-cycle is zero modulo 2. \( \square \)

Let \( \omega' := \omega - \pi i \delta \in J \otimes \mathbb{C} \) with \( \delta \) as in the lemma, so \( \beta^*(\omega') = 0 \). The \( \Delta_i \)-component of \( \omega' \) is \( \epsilon_i \xi(\omega_i) \), where \( \omega_i = \ell(x_i;p_i,q_i) \) with the integers \( p_i,q_i \) determined by the coefficients occurring in the element \( \delta \in J \). The element \( \delta \in J \) is only determined by the lemma up to elements of \( \text{Ker}(\beta^*) \). We want to show that for suitable choice of \( \delta \), the \( \omega_i \) satisfy the parity and log-parameter conditions of Theorem 4.4. We will need to review a computation of \( H_3(J) \) from [12].

We define a map \( \gamma' : H_3(J) \rightarrow H^1(\partial K_0;\mathbb{Z}) = \text{Hom}(H_1(\partial K_0),\mathbb{Z}) \) as follows. Given elements \( a \in H_3(J) \) and \( c \in H_1(\partial K_0) \) we wish to define \( \gamma'(a)(c) \). It is enough to do this for a class \( c \) which is represented by a normal path \( C \) in the link of some vertex of \( K \). Represent \( a \) by an element \( A \in J \) with \( \beta^*(A) = 0 \) and consider the element \( \beta_1(A) \in J \). This element has a coefficient for each edge of each simplex of \( K \). To define \( \gamma'(a)(c) \) we consider the coefficients of \( \beta_1(A) \) corresponding to edges of simplices that \( C \) passes and sum these using the orientation conventions of Definition 4.3. It is easy to see that the result only depends on the homology class of \( C \).

We can similarly define a map \( \gamma_2' : H_3(J) \rightarrow H^1(K_0;\mathbb{Z}/2) = \text{Hom}(H_1(K_0),\mathbb{Z}/2) \) by using normal paths in \( K_0 \) and taking modulo 2 sum of coefficients of \( \beta_1(A) \).

**Lemma 7.3** ([12], Theorem 5.1). The sequence

\[
0 \rightarrow H_3(J) \xrightarrow{(\gamma',\gamma_2')} H^1(\partial K_0;\mathbb{Z}) \oplus H^1(K_0;\mathbb{Z}/2) \xrightarrow{r-i^*} H^1(\partial K_0;\mathbb{Z}/2) \rightarrow 0
\]

is exact, where \( r : H^1(\partial K_0;\mathbb{Z}) \rightarrow H^1(\partial K_0;\mathbb{Z}/2) \) is the coefficient map and the map \( i^* : H^1(K_0;\mathbb{Z}/2) \rightarrow H^1(\partial K_0;\mathbb{Z}/2) \) is induced by the inclusion \( \partial K_0 \rightarrow K_0 \). \( \square \)

Returning to the choice of \( \delta \) above, assume we have made a choice so that the resulting flattenings \( \omega_i \) do not lead to zero log-parameters and parities for normal paths. Taking \( \frac{1}{3} \) times the log-parameters of normal paths leads as above to an element \( c \in H^1(\partial K_0;\mathbb{Z}) \). Similarly, parities of normal paths leads to an element of \( c_2 \in H^1(K_0;\mathbb{Z}/2) \). These elements satisfy \( r(c) = i^*(c_2) \). The lemma thus gives an element of \( H_3(J) \) that maps to \( (c,c_2) \). Subtracting a representative for this element from \( \delta \) gives the desired correction of \( \delta \) so the log-parameters and parities of normal paths with respect to the corresponding changed \( \omega_i \)’s are zero.

This completes the proof of the existence of the desired flattening of \( K \) in Theorem 4.4. We also see that the choice of flattening does not affect the resulting element \( \sum \epsilon_i[x_i;p_i,q_i] \in \overline{\mathcal{P}}(\mathbb{C}) \). For if we have a different choice of flattenings \( \omega_i \) satisfying the parity and log-parameter conditions of the theorem then the above lemma implies that the difference between the corresponding elements \( \delta \) represents 0 in \( H_3(J) \) and is thus in the image of \( \beta \). For \( E \in C_2 \) the effect of changing \( \delta \) by \( \beta(E) \) is to change the element \( \sum \epsilon_i[x_i;p_i,q_i] \in \overline{\mathcal{P}}(\mathbb{C}) \) by the cycle relation about \( E \).
of Lemma 5.1. Since this is a consequence of the lifted five term relations, the element in \( \tilde{P}(\mathbb{C}) \) is unchanged. (The stronger result that \( \sum \epsilon_i[x_i; p_i, q_i] \) is independent of weak flattening, as in the Scholium to 4.4, will follow in Section 9.)

**Lemma 7.4.** The element \( \sum \epsilon_i[x_i; p_i, q_i] \in \tilde{P}(\mathbb{C}) \) lies in \( \tilde{B}(\mathbb{C}) \) (even under the weaker assumption of the Scholium to Theorem 4.4).

**Proof.** For any \( \mathbb{Q} \)-vector-space \( V \), the skew-symmetric bilinear form \( \langle \cdot \rangle \) on \( J \) induces a symmetric bilinear map

\[
B: (J \otimes V) \otimes (J \otimes V) \to V \land V, \quad (a \otimes v) \otimes (b \otimes w) \mapsto \langle a, b \rangle v \land w.
\]

Theorem 4.1 of [12] says that after tensoring with \( \mathbb{Q} \), the bilinear form \( \langle \cdot \rangle \) on \( \text{Ker}(\beta^*): J \otimes \mathbb{Q} \to C_1 \otimes \mathbb{Q} \) induces two times the intersection form on \( H_1(\partial K_0; \mathbb{Q}) = \text{Ker} \beta^*/\text{Im} \beta \). Hence, on \( (\text{Ker} \beta^*/\text{Im} \beta) \otimes V = H_1(\partial K_0) \otimes V = H_1(\partial K_0; V) \), the above bilinear map \( B \) induces the map

\[
B': (H_1(\partial K_0) \otimes V) \otimes (H_1(\partial K_0) \otimes V) \to V \land V
\]

given by

\[
([a] \otimes v) \otimes ([b] \otimes w) \mapsto 2([a] \cdot [b])v \land w
\]

where \([a] \cdot [b]\) is intersection form.

We take \( V = \mathbb{C} \). For our element \( w' = w - \pi i \delta \in J \otimes \mathbb{C} \) we have

\[
B(w', w') = 2 \sum \epsilon_i(\log x_i + p_i \pi i) \land (-\log(1 - x_i) + q_i \pi i) \in \mathbb{C} \land \mathbb{C}.
\]

Thus, we want to show that \( B(w', w') = 0 \). Under the assumption of Theorem 4.4, \( w' \) is in \( \text{Ker}(\beta^*) \otimes \mathbb{C} \) and represents zero in \( H_1(\partial K_0; \mathbb{C}) \). Thus \( B(w', w') = B'(w', [w']) = B'(0, 0) = 0 \), as desired. Under the weaker assumption of the Scholium, rather than \( w' \) representing zero in \( H_1(\partial K_0; \mathbb{C}) \), it represents \( \pi i \) times an integral homology class \( \alpha \) say, and we still have \( B(w', w') = B'([w'], [w']) = (\alpha \cdot \alpha) \pi i \land \pi i = 0 \).

\[
\square
\]

8. Second part of the proof of Theorem 4.4

At this point we have shown that the element \( \sum \epsilon_i[x_i; p_i, q_i] \) lies in \( \tilde{B}(\mathbb{C}) \) and is independent of choice of flattening of \( K \). In the remainder of this section we show that it depends only on \( |K| \) and the flat \( G \)-bundle over it.

**Lemma 8.1.** Changing the choice of the point \( z \in \partial \mathbb{H}^3 \) in Theorem 4.4 does not change the element \( \sum \epsilon_i[x_i; p_i, q_i] \in \tilde{B}(\mathbb{C}) \).

**Proof.** This is a special case of the following lemma, since if we change the 1-cocycle corresponding to a \( G \)-labeling by the coboundary of the constant 0-cochain with value \( g \), the effect is the same as replacing \( z \) by \( gz \) (see subsection 4.1.2).

\[
\square
\]

**Lemma 8.2.** If we change the \( G \)-labeling of \( K \) by changing the corresponding 1-cocycle by a coboundary (subsection 4.1.2) then the element \( \sum \epsilon_i[x_i; p_i, q_i] \in \tilde{B}(\mathbb{C}) \) does not change.

**Proof.** The corresponding result is known for the element in \( B(\mathbb{C}) \), so the change is in \( \text{Ker}(\tilde{B}(\mathbb{C}) \to B(\mathbb{C})) = \mathbb{Q}/\mathbb{Z} \). Thus, if we knew that the coboundary action of 0-cochains was continuous using the standard topology of \( G = \text{PSL}(2, \mathbb{C}) \) the Lemma would follow. This continuity seems “self-evident” but we do not know
an easier proof than what follows, which directly proves the local constancy of the
element of $\hat{B}(C)$ under this action.

It suffices to prove the lemma for the change given by the coboundary action
of a 0-cochain $\tau$ that takes the value 1 on all 0-simplices except one. Denote
that one 0-simplex by $v$. The effect is that the ideal simplex corresponding to a
simplex $\Delta$ of $K$ is unchanged if $v$ is not one of its vertices, while if it has vertices
$v, v_1, v_2, v_3$ with $G$-weights $g, g_1, g_2, g_3$ before the change, then the corresponding
ideal simplex has vertices $g z, g_1 z, g_2 z, g_3 z$ before the change and $g \tau(v) z, g_1 z, g_2 z, g_3 z$
after. We assume $\tau(v)$ is sufficiently close to 1 that none of these latter simplices
are degenerate (two vertices equal).

Since the flattening condition on $K$ of Theorem 4.4 is a discrete condition (it
can only fail by multiples of $\pi$), it will stay valid if we vary flattenings of simplices
continuously as $\tau(v) \in G$ varies (considering $G = \text{PSL}(2, \mathbb{C})$ with its usual topology,
rather than as a discrete group). Thus we get a continuous family of flattenings
of $K$ as $\tau(v)$ varies in a neighborhood of 1 $\in G$. We must show that any two of
them lead to equal elements of $\hat{B}(C)$, i.e., they can be related using lifted five-term
relations. Since only simplices in the star $N$ of $v$ are affected, it suffices to show that

$$\sum_{\Delta \in N} \epsilon_i([x_i; p_i, q_i] - [x'_i; p'_i, q'_i]) = 0 \in \hat{P}(C),$$

where $\epsilon_i([x_i; p_i, q_i])$ and $\epsilon_i([x'_i; p'_i, q'_i])$ are the flattenings of $\Delta_i$ before and after chang-
ing the $G$-label of $v$ by $\tau(v)$.

Denote the 0-simplices in the simplicial link of $v$ by $v_1, \ldots, v_m$, and let their
$G$-labels be $g_1, \ldots, g_m$. For each ordered 2-simplex $\langle v_i, v_j, v_k \rangle$ in the simplicial link
of $v$ we wish to give a lifted five-term relation based on the five points $g z, g \tau(v) z,$
g_1 z, g_2 z, g_3 z$ so that when we sum these relations (with appropriate signs) we get
the left side of equation (13).

We already have flattenings of the two simplices

$$\langle g z, g_1 z, g_2 z, g_3 z \rangle \quad \text{and} \quad \langle g \tau(v) z, g_1 z, g_2 z, g_3 z \rangle.$$

We wish to find flattenings of the other three simplices

$$\langle g z, g \tau(v) z, g_j z, g_k z \rangle, \quad \langle g z, g \tau(v) z, g_i z, g_k z \rangle, \quad \langle g z, g \tau(v) z, g_i z, g_j z \rangle$$

so that we have an instance of the five-term relation.

Consider the appropriate signed sum of the three adjusted angles about the 1-
simplex $\langle g_j z, g_k z \rangle$ in the the flattening condition of the five-term relation. This
sum is a multiple of $\pi$ which should be zero. The two adjusted angles at this
edge in $\langle g z, g_i z, g_j z, g_k z \rangle$ and $\langle g \tau(v) z, g_i z, g_j z, g_k z \rangle$ almost cancel in the sum. Since
we may assume $\tau(v)$ is close to 1, the angle at the edge $\langle g_j z, g_k z \rangle$ in the simplex
$\langle g z, g \tau(v) z, g_j z, g_k z \rangle$ will be small. Thus the adjustment to this angle must be
zero. Similarly for the $\langle g_i z, g_k z \rangle$ edge of $\langle z, \tau(v) z, g_i z, g_k z \rangle$ and the $\langle g_i z, g_j z \rangle$ edge
of $\langle z, \tau(v) z, g_i z, g_j z \rangle$. In each of these three simplices the adjusted angle $\langle z, \tau(v) z \rangle$
equals the adjusted angle at the opposite edge and is hence small, so the sum of these
adjusted angles must be 0 (since it is a multiple of $\pi$) so the flattening
condition holds also at the edge $\langle z, \tau(v) z \rangle$.

It remains to choose the angle adjustments at edges of the form $\langle \tau(v) z, g_i z \rangle$ of
these simplices. We use the description of flattenings in terms of rotation levels on
edges described at the end of subsection 4.3. So we position the developing map
restricted to $\mathcal{N}$ so that vertex $v$ goes to $gz = \infty$. Then $g\tau(v)z$ is close to $\infty$, i.e., very far from the points $g\nu z$ for $\nu = 1, \ldots, m$. Thus the segments joining $g\tau(v)z$ to the points $g\nu z$ will be almost mutually parallel, so we can choose the rotation levels of these segments to be almost equal to each other. With such a choice it is clear by inspection that the remaining flattening conditions for the desired five-term relations hold. When we sum these five-term relations with appropriate signs, each flattened simplex with vertices of the form $gz = \infty, g\tau(v)z, g\nu z, g\nu z$ occurs in two of the relations and cancels and we are left with the desired relation (13).

This proves the existence of the desired lifted five-term relations when $\tau(v)$ is close to 1. This shows that the element of $\hat{\mathcal{B}}(\mathbb{C})$ is locally constant under the coboundary action of 0-cochains, and since the space $G^{K(0)}$ of 0-cochains is connected, the Lemma follows. \hfill \Box

**Remark 8.3.** The above lemma can be interpreted to say that, instead of choosing a single $z \in \partial\mathbb{H}^3$ in Theorem 4.4 to obtain the ideal simplex shapes for the simplices of $K$, we can choose a different $z$ for each 0-simplex of $K$. This will be important when discussing the applications to 3-manifolds in Section 12.

Recall that we are assuming for the moment that the vertex-ordering of the simplices of $K$ comes from a global ordering of the set of all 0-simplices of $K$. The next two lemmas are where we use this.

**Lemma 8.4.** Changing the ordering of the 0-simplices of $K$ does not change the element $\sum_i \epsilon_i [x_i : p_i, q_i] \in \hat{\mathcal{B}}(\mathbb{C})$.

*Proof.* The full permutation group on $K^{(0)}$ is generated by transpositions on adjacent elements with respect to the ordering, so we need only consider such transpositions. If the two 0-simplices are not joined by a 1-simplex then the the transposition has no effect on the vertex-ordering of any 3-simplex so there is nothing to prove. Assume therefore that $v$ and $w$ are two 0-simplices that are adjacent in the ordering of $K^{(0)}$ and are joined by a 1-simplex. Only the 3-simplices that have $(v, w)$ as an edge have their vertex-ordering changed by the transposition, so we just consider these simplices. The configuration of these simplices is as in Lemma 5.1 and the same inductive argument used in the proof of that lemma lets us deduce the lemma from the case that there are just three 3-simplices about this 1-simplex and their vertex orderings are induced from an ordering of the five 0-simplices involved. In the proof of Lemma 5.1 it is shown that there is then a lifted five-term relation which replaces these three simplices by two. After doing so, $v$ and $w$ no longer joined by an edge, so the transposition no longer has an effect on the vertex-order of any three-simplex, so the lemma follows. \hfill \Box

Finally for this section, we show that we can change the triangulation of $K$ by Pachner moves without changing the element of $\hat{\mathcal{B}}(\mathbb{C})$. These are the moves ($1 \to 4, 2 \to 3, 3 \to 2$ and $4 \to 1$ on numbers of simplices) that replace a union of 3-simplices of $K$ that is combinatorially equivalent to part of the boundary of a 4-simplex by the union of the complementary set of 3-simplices in the boundary of the 4-simplex. Only the $1 \to 4$ Pachner move adds a new 0-simplex. In this case, we insert the new 0-simplex anywhere in the ordering of 0-simplices and, if the old simplex had $G$-labels $(g_0, g_1, g_2, g_3)$, we give the four new simplices $G$-labels $(g_1, g_2, g_3, g)$, $(g_0, g_2, g_3, g)$, $(g_0, g_1, g_3, g)$, and $(g_0, g_1, g_2, g)$, for some $g \neq g_0, g_1, g_2, g_3$. We are again using the assumption of a global ordering of the 0-simplices of $K$: if we
only had local vertex-orderings of simplices, we could not guarantee that the two 3-simplices produced by a $3 \to 2$ Pachner move have vertex-orderings compatible with adjacent simplices.

**Lemma 8.5.** A change of triangulation of $K$ by Pachner moves does not change the represented element $\sum \epsilon_i [x_i; p_i, q_i] \in \hat{B}(C)$.

*Proof.* We want to leave unchanged the flattenings on the unaltered simplices of $K$ and put flattenings on the new simplices to get a flattening of the new complex $K'$. We can do this if and only if the flattenings of the changed simplices (the ones in $K$ that have been replaced and the ones they have been replaced by) give a flattening of the boundary of the 4-simplex (in the sense that log and parity parameters around edges are zero). In particular, the change in the element of $\hat{B}(C)$ is then an instance of the lifted five-term relation, and hence zero. The following lemma thus completes the proof. □

**Lemma 8.6.** Any flattening defined on part of the boundary of a 4-simplex can be extended over the whole 4-simplex.

*Proof.* Denote the vertices of the 4-simplex by $v_1, \ldots, v_5 \in \partial \mathbb{H}^3$. Denote the 3-simplex obtained by omitting vertex $v_i$ by $\Delta_i$, so the boundary of the 4-simplex is the union of the simplices $\Delta_1, \ldots, \Delta_5$. It is easy to show that we can extend a flattening defined on just $\Delta_1$ or $\Delta_1 \cup \Delta_2$, while the case that the flattening is already defined on $\Delta_1 \cup \Delta_2 \cup \Delta_3$ was dealt with in the proof of Lemma 5.1. Thus we need only prove the case that the flattening is defined on $\Delta_1 \cup \cdots \cup \Delta_4$. Now, by the case just mentioned, the flattening on $\Delta_1 \cup \Delta_2 \cup \Delta_3$ extends to the union of all five 3-simplices. We thus have two flattenings of $\Delta_4$: the given one and the one resulting from this extension. We claim they are equal, which completes the proof of the lemma. Indeed, the flattening parameter of $\Delta_4$ at the edge $v_1v_5$ of $\Delta_4$ is determined by the flattenings of $\Delta_2$ and $\Delta_3$ since only $\Delta_2, \Delta_3$ and $\Delta_4$ meet at this edge. Similarly the flattening parameter of $\Delta_4$ at the edge $v_2v_3$ is determined by $\Delta_1$ and $\Delta_3$. Since the flattening of $\Delta_4$ is determined by two edges, the flattening of $\Delta_4$ is determined by the flattenings of $\Delta_1, \ldots, \Delta_3$. □

It follows from Pachner [18] that any two simplicial triangulations of $|K|$ are related by Pachner moves, so at this point we know that the element of $\hat{B}(C)$ is determined just by $|K|$ and the flat $G$-bundle over it. We will actually only need this in the case that $|K|$ is a manifold, which is explicit in Pachner’s work. In the next section we reduce to the case that $|K|$ is a manifold and use this to complete the proof of Theorem 4.4.

9. Completion of proof of Theorem 4.4

In this section we show that we can always resolve the singularities of $|K|$ to replace it by a triangulated 3-manifold and then discuss surgery on this 3-manifold to complete the proof of Theorem 4.4.

The underlying space of $K$ is a 3-manifold except at 0-simplices, where it may look locally like the cone on a closed surface of genus $\geq 1$. Suppose this occurs at some 0-simplex $v$. Choose a non-separating simple closed simplicial curve $C$ in the link $L_v$. This curve is determined by an open disk $D$ in the star of $v$. We slice $|K|$ open along $D$ and splice in a complex as in Fig. 6, glueing the top and the bottom
of that complex to the two copies of $D$ resulting from slicing $|K|$. This introduces a new 0-simplex $v'$ in $K$. We give $v'$ a $G$-label distinct from that of each adjacent 0-simplex. We order 0-simplices by inserting $v'$ anywhere (e.g., as a new maximal element). We must check that we can put flattenings on the inserted 3-simplices so that the flattening conditions hold.

We first show that it suffices to satisfy the log-parameter flattening conditions. Indeed, by Corollary 4.7, if we satisfy the logarithmic flattening conditions, the parity conditions hold around edges. Parity along normal paths then gives a homomorphism from $H_1(|K - K^{(0)}|)$ to $\mathbb{Z}/2$. If we denote the new complex by $K'$ and the curve in $K'$ given by the two vertical 1-simplices by $C$, then $H_1(|K - K^{(0)}|) = H_1(|K' - K^{(0)}| - C)$ which surjects to $H_1(|K' - (K')^{(0)}|)$ by the inclusion, so the parity condition will hold for all normal paths.

We satisfy the log-parameter flattening condition as follows. The added 3-simplices come in pairs with identical $G$-labellings, adjacent to each other above and below the central disk in the picture. We will give the two simplices of a pair identical flattenings so they cancel in the sum $\sum i \epsilon i[x_i, p_i, q_i] \in \hat{B}(C)$ (they appear with opposite signs in this sum). Flatten all the added simplices in the top layer in order to satisfy the flattening conditions at the top edges in the picture (there are $n$ degrees of freedom in doing this, where $n$ is the number of 1-simplices around the curve $C$). We then need only verify the flattening condition at the central vertical 1-simplex. Since the sum of the three log-parameters of a flattened simplex is zero, the appropriately signed sum of log parameters around the central edge is the negative of the signed sum of the $2n$ log-parameters of these simplices at the top edges in the picture. This, in turn, is equal to the log-parameter along a normal path in the star of $v$ determined by a path in the link $L_v$ that runs parallel to $C$ on one side. It is thus zero by the flattening condition for such normal paths.

(An alternate argument is to look at the developing image centered at $v$ as in subsection 4.3. The flattenings before we change $K$ give us rotation levels on the edges of the image in $\mathbb{C}$. After changing $K$ we have a new point in $\mathbb{C}$ at the image of $v'$ and $n$ segments incident on it, corresponding to the 2-simplices of the central horizontal disk in the picture. Assign any rotation levels to these $n$ segments. As in subsection 4.3, this determines flattenings of the 3-simplices so that the flattening conditions are satisfied.)

We call the above procedure “blowing up” at the 0-simplex $v$. Note that it reduces the genus of the link of $v$. By repeated blowing up we can thus make $|K|$ into a 3-manifold. The following lemma improves on this.

**Lemma 9.1.** If $K$ is weakly flattened as in the Scholium to Theorem 4.4 then we can blow up to make $|K|$ into a 3-manifold without changing the represented element in $\hat{B}(C)$. In particular, the Scholium follows from Theorem 4.4.
Proof. To apply the above argument to blow up and simplify the link of a vertex we need a separating curve \( C \) in the link so that the log-parameter along a parallel normal curve in the link \( L_v \) is zero. Suppose the log-parameter flattening conditions are satisfied around edges, but not necessarily along normal curves. Log-parameters along normal curves then give a homomorphism \( H_1(L_v;\mathbb{Z}) \rightarrow \mathbb{Z}[\pi] \subset \mathbb{C} \). It is well known that for any closed oriented surface \( S \) and non-trivial homomorphism \( H_1(S;\mathbb{Z}) \rightarrow \mathbb{Z} \) there exist elements in the kernel represented by non-separating simple closed curves (in fact, the mapping class group acts transitively on the set of surjective homomorphisms \( H_1(S;\mathbb{Z}) \rightarrow \mathbb{Z} \)). Thus, there exists a curve to use for blowing up, so the lemma follows.

At this point we may assume our element in \( H_3(G;\mathbb{Z}) \) is represented by a triangulated 3-manifold with flat \( G \)-bundle over it, or equivalently, a 3-manifold \( |K| \) with a homotopy class of mappings \( |K| \rightarrow BG \). Since \( H_3(BG;\mathbb{Z}) = \Omega_1(BG) \) (oriented bordism), and bordism is generated by homotopy (i.e., maps of \( |K| \times I \)) and handle addition, it suffices to show that when we modify \( K \) by surgery resulting from a handle-addition to \( |K| \times I \) the represented element in \( \widehat{B}(\mathbb{C}) \) is not changed.

It suffices to consider adding 1-handles and 2-handles, since any bordism of connected 3-manifolds is the composition of bordisms that add 1, 2, and 3 handles in that order, and the latter are inverses of bordisms that add 1-handles.

We first consider adding a 1-handle. Combinatorially, homotoping \( |K| \) in \( BG \) means changing the \( G \)-valued 1-cocycle by the coboundary action. We make such a change so that the handle addition can be realized by gluing a 4-simplex \( \Delta^4 \) to \( K \) by gluing two of the 3-faces of \( \Delta^4 \) to disjoint 3-simplices \( \Delta^3_1 \) and \( \Delta^3_2 \) in \( K \). The common 2-simplex of the two 3-simplices in \( \partial \Delta^4 \) determines 2-faces of \( \Delta^3_1 \) and \( \Delta^3_2 \) that will be glued together in this construction. Our initial change of the \( G \)-valued 1-cocycle arranges for the cocycle to match on these two simplices.

The resulting surgery of \( K \) is the result of first removing the interiors of \( \Delta^3_1 \) and \( \Delta^3_2 \), then gluing the resulting boundary components along the 2-simplices mentioned above to give a manifold with a single 2-sphere boundary triangulated as the suspension of a triangle, and finally gluing in a 3-ball triangulated as three 3-simplices meeting along a common 1-simplex. Call the resulting triangulated manifold \( K' \).

We must make the construction compatible with orderings and provide suitable flattenings. By Lemma 8.4 we may order the vertices of \( K \) as \( v_1, v_2, \ldots \) so that the vertices of \( \Delta^3_1 \) are \( v_1, v_3, v_5, v_7 \) and of \( \Delta^3_2 \) are \( v_2, v_4, v_6, v_8 \) and the 2-simplices that are identified are given by the first three vertices of each in the given order. Then the vertices of \( K' \) are \( v_1 = v_2, v_3 = v_4, v_5 = v_6, v_7, v_8, \ldots \) and the common 3-simplices of \( K \) and \( K' \) have not changed their vertex orderings.

Each of the two 3-simplices \( \Delta^3_1 \) and \( \Delta^3_2 \) had a flattening before the surgery, and we use lemma 8.6 to define the flattening on the three new 3-simplices. The log-parameter part of the flattening condition is then satisfied in \( K' \), but the parity condition may not be. The surgery has added a generator to \( H_1(K,\mathbb{Z}) \), represented by any normal path that passes once through the 2-simplex that we glued. We must check the parity condition along one such path. If it fails, then before performing the surgery we change flattenings of \( K \) by the cycle relation of Lemma 5.1 about the 1-simplex \( v_1v_3 \). After doing so the parity part of the flattening condition will be satisfied for \( K' \).

Finally, we must consider adding a 2-handle along a simplicial curve in \( K \). We can realize this by doing the reverse of a blow-up (a “blow-down”) to collapse the
curve to a single vertex $v$ with link a torus, and then blowing up again using a longitude of the curve we collapsed. We need to show, therefore, that if $K$ contains a curve that maps to the trivial element in $G$ under the homomorphism determined by the $G$-valued 1-cocycle, then we can (after adjusting the triangulation) find a flattening that lets us perform a blow-down to collapse the curve, followed by a blow-up using a longitude of the curve.

By retriangulating we may assume the curve $C$ in question is length 2 and the 3-simplices that have a 1-simplex in common with it are configured as in Fig. 6 (with, for concreteness, 6 simplices around each edge of $C$, as in the figure). In particular, the bottom and top vertices in the figure are the same 0-simplex $v$, and, since the 1-cocycle along the curve gives the trivial element of $G$, if we $G$-label the simplices of the figure all with the same label for $v'$ then each pair of vertically adjacent 3-simplices has the same labeling. The rest of the star of $v$ will be the cone on an annulus. In particular, the 1-simplices radiating from $v$ on the top surface of the figure are distinct from the 1-simplices radiating from $v$ on the bottom surface.

We must verify that we can modify the flattenings so that vertically adjacent 3-simplices have the same flattening, in which case we can perform the desired blow-down.

Denote the 1-simplices radiating from $v$ in the bottom surface of the figure $e_1, \ldots, e_6$ in order, and the 1-simplices above these in the central disk $e_1', \ldots, e_6'$ (with $e_i'$ above $e_i$). Denote by $\Delta_i$ the 3-simplex in the bottom layer with edges $e_i$ and $e_{i+1}$ (indices mod 6). By applying suitable multiples of the cycle relation of Lemma 5.1 around 1-simplices $e_1$ and $e_2$ we can adjust the flattening of $\Delta_1$ arbitrarily, so we can make it match the 3-simplex above it. The flattening condition at 1-simplex $e_2'$ now implies that the log-parameter of $\Delta_2$ at edge $e_2'$ matches the simplex above it, and by applying the cycle relation around $e_3$ we can fix up the flattening of $\Delta_2$ at edge $e_3'$. Continuing this way, we fix $\Delta_3, \ldots, \Delta_5$, at which point the flattening conditions about $e_5'$ and $e_6'$ imply that $\Delta_6$ also has its desired flattening.

We can now do the blow-down by removing the interior of the complex shown in the figure from $K$ and then gluing the top and bottom surfaces. The result is a complex $K'$ in which the link of 0-simplex $v$ is a torus, and if we can blow up using a complementary curve in this torus then our handle addition is complete. However, to blow up we need the log-parameter along the appropriate normal curve to be zero, which need not be the case. We describe how to remedy this.

Suppose we are in the situation just before we do the blow-down, so the flattenings have been matched on pairs of 3-simplices in the figure. If we apply the cycle relation about the lower vertical 1-simplex joining $v$ to $v'$ we destroy this matching. The procedure described above to make flattenings match again then applies the cycle relation once about each of the 1-simplices $e_1, \ldots, e_6$. We claim that the end result is to change the log-parameter along the normal curve we are interested in by $2\pi i$.

Indeed, recall that the full star neighborhood of the curve $C$ consists of the complex of Fig. 6 glued to the cone on an annulus. In Fig. 7 we show a possible triangulation of the annulus (we may assume it is triangulated this way, since we may choose the triangulation). The figure also shows how log-parameters are changed by the above procedure. The normal curve that interests us is the vertical curve, and the log-parameter along it has been changed by $2\pi i$.

Since the log-parameter along the normal curve is a multiple of $2\pi i$ (Corollary
Figure 7. Left and right sides are identified to form an annulus. Signs indicate change of flattening by $\pm \pi i$ and 0 signifies that flattening has been changed by both $\pi i$ and $-\pi i$ so total change is zero. The log-parameters affected by applying the cycle relation about a single $e_i$ are bold.

4.7), we can apply a multiple of the above procedure to make sure that desired log-parameter along the curve is zero before we perform the blow-down.

This would complete the proof of Theorem 4.4 except that we have carried out the proof only for ordered 3-cycles whose vertex-orderings come from a global ordering of the 0-simplices of $K$. We now show that this stronger condition is unnecessary. We will also show, as promised earlier, that $K$ need only be a quasi-simplicial complex (closed simplices do not necessarily embed in $|K|$). An example of such a quasi-simplicial ordered 3-cycle is given by gluing the two 3-simplices in Fig. 8 by matching each face of the left 3-simplex with the correspondingly decorated face of the right 3-simplex (the decorations in question are the single and double arrows on the edges that bound the face). The arrows on the edges order the vertices of each 3-simplex as shown, and, by construction, these orderings match on common faces. The resulting complex has one 0-simplex, two 1-simplices, four 2-simplices, and two 3-simplices. (This 3-cycle is simply connected, so it cannot give a non-trivial element of $H_3(G;\mathbb{Z})$ in Theorem 4.4, but we will return to it when discussing invariants of 3-manifolds, since $|K| - K^{(0)}$ is Thurston’s ideal triangulation of the figure eight knot complement.)

Proposition 9.2. Any quasi-simplicial ordered 3-cycle $K$ is related by Pachner moves that respect vertex-orderings of 3-simplices to a simplicial 3-cycle $K'$ whose vertex-orderings are induced by a global ordering of the 0-simplices of $K'$.

Proof. We will start with an arbitrary ordering of the 0-simplices of $K$. At first this ordering will not induce the given vertex-orderings of simplices. We will perform various Pachner moves on $K$. Each time we perform a $1 \rightarrow 4$ move we will add the new 0-simplex as a new maximal element both in our ordering of 0-simplices and also in the vertex-orderings of the new 3-simplices created by the move. At each
stage in the process, the ordering of the two vertices of a 1-simplex will be induced
by the global ordering of the 0-simplices except maybe if both vertices are “old”
0-simplices (i.e., they occurred in \( K \)). We will therefore be done once no two old
0-simplices are joined by a 1-simplex.

We will assume no 3-simplex of \( K \) meets itself across a 2-face (do a \( 1 \to 4 \) Pachner
move on any such simplex if it occurs). Suppose \( K \) has \( n \) 3-simplices. We first do
\( 1 \to 4 \) Pachner moves on each of the 3-simplices of \( K \), creating \( n \) new 0-simplices.
Next, for each old 2-simplex (i.e., a 2-simplex that was already in \( K \)) we do a \( 2 \to 3 \)
Pachner move on the two 3-simplices that meet across it. Each 3-simplex of the
resulting complex is in the star of a unique “old” 1-simplex. So if we operate in the
star of an old 1-simplex it will not affect the star of any other old simplex.

If there are three or more three 3-simplices in the star of an old 1-simplex, we
reduce the number to three by repeatedly doing \( 2 \to 3 \) moves on pairs of adjacent
simplices in its star, and then finally do a \( 3 \to 2 \) move on the three 3-simplices in
its star to remove the old 1-simplex. If an old 1-simplex has just two 3-simplices in
its star we do a \( 1 \to 4 \) move on one of them and then do a \( 3 \to 2 \) move to remove
the old 1-simplex.

In this way we alter the complex by Pachner moves until all the old 1-simplices
have been removed. Then no two old 0-simplices are still joined by a 1-simplex, so
the proof is complete. Since Pachner moves do not affect the represented element
in \( \hat{\mathcal{B}}(\mathbb{C}) \), this also completes the proof of Theorem 4.4. □

The parity condition in the definition of flattening is probably essential to The-
orem 4.4, but it’s failure can at most change the resulting element of \( \hat{\mathcal{B}}(\mathbb{C}) \) by
2-torsion:

**Lemma 9.3.** If we represent an element \( \alpha \in \mathbb{H}_3(G;\mathbb{Z}) \) as in Theorem 4.4 but
without requiring the condition on parity then the resulting element differs from
\( \lambda(\alpha) \) at worst by the element of order 2 in \( \hat{\mathcal{B}}(\mathbb{C}) \).

**Proof.** Suppose the log-parameter condition is satisfied but the parity condition
is not satisfied for the flattening of \( K \) and let \( \mu \) be the resulting element in \( \hat{\mathcal{B}}(\mathbb{C}) \).
Parity along paths determines an element of \( H^1(|K| - K^{(0)}; \mathbb{Z}/2) \) and hence a 2-fold
cover of \( K \) (possibly branched at some 0-simplices). The lift of the flattening to
this cover will satisfy the parity condition and hence represent \( \lambda(2\alpha) = 2\lambda(\alpha) \). But
it clearly also represents \( 2\mu \), so \( 2(\mu - \lambda(\alpha)) = 0 \). □

10. Cheeger-Chern-Simons class

The following is simply a restatement of Theorem 2.6.

**Theorem 10.1.** The homomorphism \( \lambda: H_3(G;\mathbb{Z}) \to \hat{\mathcal{B}}(\mathbb{C}) \) is an isomorphism and
it fits in a commutative square:

\[
\begin{array}{ccc}
H_3(G;\mathbb{Z}) & \xrightarrow{\lambda} & \hat{\mathcal{B}}(\mathbb{C}) \\
\downarrow c & & \downarrow R \\
\mathbb{C}/\pi^2\mathbb{Z} & \to & \mathbb{C}/\pi^2\mathbb{Z}
\end{array}
\]

Here \( G \) is, as usual, \( \text{PSL}(2,\mathbb{C}) \) with discrete topology, \( c \) is the Cheeger-Chern-
Simons class, and \( R \) is the Rogers dilogarithm map of Proposition 2.5.
Proof. We first prove commutativity of the diagram. The imaginary part of both $c$ and $R$ gives volume (this was proved for $R$ in slightly different language in [12]; see also [5]) so the only issue is the real part. As pointed out in [5], to prove that the real part of the above diagram commutes, we may replace PSL(2, $\mathbb{C}$) by PSL(2, $\mathbb{R}$).

For an element of $H_3(\text{PSL}(2, \mathbb{R}))$, if we represent it by an ordered 3-cycle $K$ labeled in PSL(2, $\mathbb{R}$) and then choose the point $z$ in Theorem 4.4 in $\partial \mathbb{H}^3 \subset \partial \mathbb{H}^3$ then all the ideal simplices are flat. A flat ideal simplex has a natural flattening in our sense, since two of its dihedral angles are already 0 and the third is $\pm \pi$, so we adjust only the latter angle to zero. If we flatten $K$ this way, we clearly satisfy the logarithmic flattening conditions, but the parity conditions are less clear. By Lemma 9.3 this affects the element of $\hat{B}(\mathbb{C})$, and hence also the value of $R$ on it, at most by 2-torsion.

With this flattening, the function $R$ for a simplex with real cross-ratio parameter $x$ becomes the real Rogers dilogarithm of $x$ used by Dupont in [5]. Thus $R \circ \lambda$ is (up to the 2-torsion mentioned above) the cocycle he considers. He shows there that $R \circ \lambda$ equals $c$ if we map to $\mathbb{C}/\pi^2 \mathbb{Z}$ rather than $\mathbb{C}/\pi^2 \mathbb{Z}$ (since $\mathbb{C}/\pi^2 \mathbb{Z} = (\mathbb{C}/\pi^2 \mathbb{Z})/(\mathbb{Z}/6)$, the possible 2-torsion discrepancy maps to zero). Thus $(R \circ \lambda - c) : H_3(G; \mathbb{Z}) \to \mathbb{C}/\pi^2 \mathbb{Z}$ has image in $\frac{\pi}{6} \mathbb{Z}/\pi^2 \mathbb{Z} \cong \mathbb{Z}/6$. On the other hand, $H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z})$ is known to be a divisible group by [7], so it has no non-trivial finite quotient, so $(R \circ \lambda - c) = 0$. Thus $R \circ \lambda = c$, as was to be proved.

It remains to show that $\lambda$ is an isomorphism. If we compose $\lambda$ with the map to $B(\mathbb{C})$ then the kernel is the torsion subgroup of $H_3(G; \mathbb{Z})$, so the kernel of $\lambda$ is a subgroup of the torsion subgroup $\mathbb{Q}/\mathbb{Z}$ of $H_3(G; \mathbb{Z})$. But it was pointed out in [5] and [7] that the Cheeger-Chern-Simons map $c$ is injective on the torsion subgroup of $H_3(G; \mathbb{Z})$, so it follows that $\lambda$ must also be injective on this subgroup. Thus $\lambda$ is injective. Surjectivity follows similarly: we need only prove it on the torsion subgroup, and for this it suffices to see that $R$ is an isomorphism on the torsion subgroups, which was shown in the proof of Proposition 5.5. □

One can give an explicit computation of $R \circ \lambda$ on the torsion of $H_3(G; \mathbb{Z})$. By [7], the torsion subgroup $\mathbb{Q}/\mathbb{Z} \subset H_3(G; \mathbb{Z})$ is given by the image of $\mu/\{\pm 1\} \to G$, where $\mu$ is the group of roots of unity mapping to diagonal matrices in $G$. Thus the torsion is generated by the images of generators of $H_3(\mathbb{Z}/n; \mathbb{Z}) \cong \mathbb{Z}/n$ in $H_3(G; \mathbb{Z})$. We can represent a generator of this $\mathbb{Z}/n \subset H_3(G; \mathbb{Z})$ by the lens space $L(n, 1)$ with the flat $G$-bundle given by mapping a generator of $\pi_2(L(n, 1)) = \mathbb{Z}/n$ to the $2\pi/n$-rotation in $\text{SO}(2) \subset \text{PSL}(2, \mathbb{R}) \subset G$. We can triangulate $L(n, 1)$ by triangulating its standard fundamental domain as in Fig. 4. If we order the vertices of each simplex in that figure in the order south, north, west, east, then the resulting representative for this class $\alpha_n \in H_3(G; \mathbb{Z})$ is the sum of homogeneous simplices

$$\sum_{0}^{n} \langle h_1, gh_1, g^j h_2, g^{j+1} h_2 \rangle,$$

where $g$ is the $2\pi/n$-rotation and $h_1$ and $h_2$ are any elements of $\text{SO}(2)$ chosen to fulfill our requirement that the $G$-labels of each simplex be distinct. Explicit computation then gives $R(\lambda(\alpha_n)) = \pi^2/n$ modulo $\pi^2$ (we omit the details).
11. Unordered simplices

As we have shown in Proposition 9.2, our complexes can be quasi-simplicial (closed simplices not required to embed). The triangulations that arise in practice (e.g., ideal triangulations of complete finite volume 3-manifolds, see section 12) are often not simplicial. However, a quasi-simplicial triangulation does not always admit vertex-orderings of its 3-simplices that agree on common faces. Thus, to use such a triangulation one may either have to subdivide, or do without vertex-orderings.

In this section we describe how the theory changes if we discard the orderings of vertices of 3-simplices in the definition of $\hat{B}(\mathbb{C})$ in terms of flattened simplices. We show that the result is to quotient $\hat{B}(\mathbb{C})$ by its cyclic subgroup of order 6.

Suppose we have a flattened simplex with parameter $(z; p, q)$ with $z$ in the upper half plane. If we retain its geometry but reorder its vertices by an even permutation (which preserves orientation) then the flattening is replaced by one of

$$(1/(1-z); q, -1 - p - q) \text{ or } (1 - 1/z; -1 - p - q, q),$$

so the element in $\hat{B}(\mathbb{C})$ is changed by subtracting one of

$$[z; p, q] - [1/(1-z); q, -1 - p - q] \text{ or } [z; p, q] - [1 - 1/z; -1 - p - q, p].$$

Similarly, if we reorder its vertices by an odd permutation then the element in $\hat{B}(\mathbb{C})$ is changed by subtracting one of

$$[z; p, q] + [1/z; -p, 1 + p + q], \quad [z; p, q] + [1 - z; -q, -p],$$

or

$$[z; p, q] + [z/(z - 1); 1 + p + q, -q],$$

Proposition 11.1. If $z$ is in the upper half plane then, with $\chi$ as in Proposition 5.5 and Theorem 5.6

$$\begin{align*}
- [1/(1-z); q, -1 - p - q] &= \chi(e^{\pi i/3 + q \pi i}) \\
[z; p, q] - [1 - 1/z; -1 - p - q, p] &= \chi(e^{-\pi i/3 + p \pi i}) \\
[z; p, q] + [1/z; -p, 1 + p + q] &= \chi(e^{p \pi i}) \\
[z; p, q] + [1 - z; -q, -p] &= \chi(e^{\pi i/3}) \\
[z; p, q] + [z/(z - 1); 1 + p + q, -q] &= \chi(e^{2\pi i/3 + q \pi i})
\end{align*}$$

Proof. The last three of the above equations correspond to permutations which exchange two vertices. Since two of these involutions, together with the Klein four-group (which leaves flattenings unchanged) generate the full symmetric group, it suffices to prove the third and fourth equations; the others are then easy consequences. We start with the third equation. Recall that

$$[x; p_0, q_0] - [y; p_1, q_1] + [y; p_1 - p_0, q_2] - [1 - x^{-1}; p_1 - p_0 + q_1 - q_0, q_2 - q_1] + [1 - x^{-1}; q_1 - q_0, q_2 - q_1 - p_0] = 0$$

is an instance of the five-term relation whenever all five of $x, y, \frac{y}{x}, \frac{1-x^{-1}}{1-y^{-1}}$, and $\frac{1-x^{-1}}{1-y^{-1}}$ are in the upper half plane. Similarly one checks that it is an instance of the five-term relation if the first three are in the lower half plane and the last two in the upper half plane. In particular, if all five are in the upper half plane, then replacing
We have $x$ by $x^{-1}$ and $y$ by $y^{-1}$ we get a relation

$$(15) \quad [x^{-1}; p_0, q_0] - [y^{-1}; p_1, q_1] + \left[ \frac{x}{y}; p_1 - p_0, q_2 \right] - \left[ \frac{-p_1}{y}; p_1 - p_0 + q_1 - q_0, q_2 - q_1 \right] + \left[ \frac{-p_1}{y}; q_1 - q_0, q_2 - q_1 - p_0 \right] = 0.$$ 

Putting $p_0 = p_1 = q_0 = q_1 = 0$ and $q_2 = q$ in equation (14) and $p_0 = p_1 = 0$, $q_0 = q_1 = 1$, $q_2 = q + 1$ in (15) and then adding, we get:

$$(16) \quad ([x; 0, 0] + [x^{-1}; 0, 1]) - ([y; 0, 0] + [y^{-1}; 0, 1]) + ([y/x; 0, q] + [x/y; 0, q + 1]) = 0.$$ 

This equation holds if $x$, $y$ and $\frac{y}{x}$ are suitably positioned in the upper half plane. It will continue to hold by analytic continuation if we vary $x$ and $y$ so long as none of $x$, $y$, or $\frac{y}{x}$ stray from the upper half plane. If we vary $x$ and $y$ in the upper half plane so that $\frac{y}{x}$ crosses interval $(0, 1)$ and $\frac{x}{y}$ therefore crosses $(1, \infty)$ we get the equation

$$(17) \quad ([y; 0, 0] + [y^{-1}; 0, 1]) - ([x; 0, 0] + [x^{-1}; 0, 1]) + ([y/x; 0, q - 1] + [x/y; 0, q]) = 0.$$ 

Replacing $q$ by $q + 1$ in equation (17) and then adding to equation (16) gives, with $z := y/x$,

$$2([z; 0, q] + [z^{-1}; 0, q + 1]) = 0.$$ 

Here $z$ is now arbitrary in the upper half plane. Thus $[z; 0, q] + [z^{-1}; 0, q + 1]$ is in the cyclic subgroup of order 2 in $\hat{B}(\mathbb{C})$. On the other hand, we know by the proof of Proposition 5.5 that $R$ is an isomorphism on the torsion subgroup of $\hat{B}(\mathbb{C})$ and one checks easily that $R([z; 0, q] + [z^{-1}; 0, q + 1]) = 0$. Thus $[z; 0, q] + [z^{-1}; 0, q + 1] = 0$. 

As we analytically continue this relation by letting $z$ go repeatedly around the origin crossing the intervals $(-\infty, 0)$ and $(0, 1)$ (so $z^{-1}$ crosses $(-\infty, 0)$ and $(1, \infty)$) we get the relations

$$[z; p, q] + [z^{-1}; -p, p + q + 1] = 0$$

for all even $p$.

Now, using the notation of the proof of Proposition 5.5 and Lemma 5.2 we find

$$+[z^{-1}; 1, q] = ([z; 0, q] + [z^{-1}; 0, q + 1]) + ([z; -1, q] - [z; 0, q]) + ([z^{-1}; 1, q] - [z^{-1}; 0, q]) + ([z^{-1}; 0, q] - [z^{-1}; 0, q + 1])$$

$$= 0 + \{1 - z, 0\} - \{1 - z^{-1}, 0\} - \{z, 0\}$$

$$= \chi(1 - z) - \chi(1 - z^{-1}) - \chi(z) = \chi \left( \frac{1 - z}{(1 - z^{-1})z} \right) = \chi(-1).$$

Analytically continuing this as $z$ circles the origin as before gives us the equations

$$[z; p, q] + [z^{-1}; -p, p + q + 1] = \chi(-1)$$

for all odd $p$. Thus the third equation of the proposition is proved.

For the fourth equation of the proposition, we already know by Lemma 5.4 that $[z; p, q] + [1 - z; -q, -p] = 2[1/2; 0, 0]$, so we need to show that $2[1/2; 0, 0] = \chi(e^{\pi i/3})$. Denote $\omega = e^{\pi i/3}$. Then $1 - \omega = \omega^{-1}$ so $[\omega; 0, 0] + [\omega^{-1}; 0, 0] = 2[1/2; 0, 0]$. We have just shown that $[\omega; 0, 0] + [\omega^{-1}; 0, 1] = 0$. And by definition of $\chi$, we have $[\omega^{-1}; 0, 0] - [\omega^{-1}; 0, 1] = -\chi(\omega^{-1}) = \chi(\omega)$. Adding the last two equations
gives \([\omega; 0, 0] + [\omega^{-1}; 0, 0] = \chi(\omega)\), whence \(2[1/2; 0, 0] = \chi(\omega)\), as desired (in fact, \(1/2; 0, 0] = \chi(e^{\pi i/6})\), as can be seen now by applying \(R\)). □

**Corollary 11.2.** If we define groups analogous to \(\hat{P}(C)\) and \(\hat{B}(C)\) using flattened simplices modulo the lifted five term relation, but ignoring orderings of vertices, we obtain the quotients \(\hat{P}(C)/C_6\) and \(\hat{B}(C)/C_6\), where \(C_6\) is the unique cyclic subgroup of order 6.

**12. Invariants of 3-manifolds**

Suppose we have a compact oriented hyperbolic 3-manifold \(M\), or, more generally, a compact oriented manifold 3-manifold with a flat \(G\)-bundle (equivalently, a homomorphism \(\pi_1(M) \to G\); as usual \(G\) denotes \(\text{PSL}(2,\mathbb{C})\)). Then Theorems 4.4 and 2.6 give a computation of the corresponding homology class in \(H_3(G;\mathbb{Z})\) and its Cheeger-Chern-Simons class from a triangulation of \(M\).

In this section we extend this to the case that \(M\) is an oriented hyperbolic manifold which is complete of finite volume but not necessarily compact. We will also extend to the sort of ideal triangulations that the programs Snap [9, 4] and Snappea [21] use for compact hyperbolic manifolds. The underlying complex of such a “Dehn surgery triangulation” is homeomorphic not to \(M\) itself, but to the result of collapsing to a point a simple closed curve of \(M\), so Theorem 4.4 does not apply directly.

Suppose \(M\) is a non-compact oriented complete hyperbolic manifold of finite volume. It is known by Epstein and Penner [8] that \(M\) has a triangulation by ideal hyperbolic polytopes, and by subdividing these polytopes we may obtain a triangulation by ideal simplices. As has often been pointed out (e.g., [14]), after subdividing the polytopes into ideal simplices, the subdivisions may not match across faces of the polytopes. One can mediate between non-matching subdivisions by including flat ideal simplices in the triangulation. It is still unknown if this is actually necessary in any example. It has been shown by Petronio and Weeks [17] that flat ideal simplices are usually not a serious issue. This is so for our arguments, in fact for us even “folded back tetrahedra” are allowed.

In [12] it is shown that any ideal triangulation of a hyperbolic 3-manifold has a flattening in the sense of Theorem 4.4. If we can appropriately order vertices then the arguments of this paper imply that we get an element of \(\hat{B}(C)\). In general we cannot do this, so, by the previous section, we only get an element of \(\hat{B}(C)/C_6\). We can refine our triangulation to make sure that we can order the 0-simplices. The triangulations we must consider for this are hybrids of usual triangulations and ideal ones.

Suppose \(M\) has \(h\) cusps, and consider the end compactification of \(M\), obtained by adding points \(p_i, i = 1, \ldots, h\), to compactify each cusp. If \(K\) is a complex that triangulates this compactification of \(M\) then \(p_1, \ldots, p_h\) will be 0-simplices of \(K\). We consider \(K_0 := K - \{p_1, \ldots, p_h\}\) to be a “triangulation” of the non-compact manifold \(M\). Some of the simplices of \(K\) have one or more vertices among \(\{p_1, \ldots, p_h\}\); we call these “ideal vertices.”

We wish to assign ideal simplex shapes to all the simplices of \(K\) in an appropriate fashion. This is maybe most easily visualized as follows. We position our triangulation of \(M = \mathbb{H}^3/\Gamma\) so that it triangulates \(M\) by geodesic hyperbolic simplices, with vertices \(p_i\) ideal. We then lift to a triangulation of \(\tilde{M} = \mathbb{H}^3\). Finally, we move each
non-ideal vertex to $\partial \mathbb{H}^3$ in a $\Gamma$-equivariant fashion, making sure that the resulting ideal simplices are non-degenerate (all four vertices distinct).

**Remark 12.1.** We can also describe this in terms of a $G$-valued 1-cocycle “relative to the cusps.” We choose a lift $\hat{p}_i \in \mathbb{H}^3$ for each $p_i$ and denote by $P_i$ the parabolic subgroup of $G$ that fixes $\hat{p}_i$. The $G$-valued 1-cocycle relative to the cusps is a 1-cocycle in the usual sense except that its value on an edge that ends (resp. starts) in an ideal vertex $p_i$ is only well defined modulo right (resp. left) multiplication by elements of $P_i$. This 1-cocycle is well defined up to the usual action of 0-cocycles, but only 0-cocycles that take value 1 in $G$ on ideal vertices are permitted. The 1-cocycle depends on the choice of the lifts $\hat{p}_i$; a change of lift changes the cocycle on edges to $p_i$ by right multiplication by the element of $G$ that moves the new lift to the old.

There is then an equivalent $G$-labeling of the 3-simplices as in Section 4.1, except that the label of an ideal vertex is only defined up to right multiplication by elements of the corresponding parabolic subgroup. We then assign ideal simplex shapes to the simplices of $K$ by choosing a point $z \in \partial \mathbb{H}^3$ for each 0-simplex of $K$, as in Remark 8.3, but with the restriction that the point $z$ chosen for an ideal vertex is the fixed point $\hat{p}_i$ of the corresponding parabolic subgroup. A simplex with $G$-labels $g_1, g_2, g_3, g_4$ and whose vertices have been assigned points $z_1, \ldots, z_4$ in $\partial \mathbb{H}^3$ receives the ideal simplex shape $\langle g_1z_1, \ldots, g_4z_4 \rangle$.

Suppose we have triangulated $M$ and assigned ideal simplex shapes to the simplices as above. We also assume that we have chosen $K$ so that it admits orderings of its 3-simplices that agree on common faces, and we fix such an ordering. The arguments of this paper then go through to show

**Theorem 12.2.** There exists a flattening of $K$ as in Theorem 4.4 and the resulting element $\sum_i \epsilon_i [x_i; p_i, q_i] \in \hat{\mathcal{P}}(\mathbb{C})$ is in $\hat{\mathcal{B}}(\mathbb{C})$ and only depends on the hyperbolic manifold $M$. We denote it $\hat{\beta}(M)$. \hfill $\square$

Since $\hat{\mathcal{B}}(\mathbb{C}) \cong H_3(G; \mathbb{Z})$, this gives a “fundamental class” in $H_3(G; \mathbb{Z})$ for any oriented complete finite volume hyperbolic 3-manifold (as usual, $G = \text{PSL}(2, \mathbb{C})$ with discrete topology).

We pointed out in the Introduction that this class is easy to define algebraically if $M$ is compact, but if $M$ has cusps an algebraic definition is less clear. $M$ can be compactified to a compact manifold $\bar{M}$ with $\partial \bar{M}$ consisting of $h$ tori. The fundamental class of $M$ determines a class in $H_3(BG, \bigcup_{i=1}^h B_i; \mathbb{Z})$ where the $B_i$ are copies of $B\mathbb{Z}^2$, one for each cusp, which we may assume are disjointly embedded in $BG$. The long exact sequence for the pair gives a short exact sequence

$$0 \to H_3(BG; \mathbb{Z}) \to H_3(BG, \bigcup_{i=1}^h B_i; \mathbb{Z}) \to \mathbb{Z}^h \to 0.$$  

If this sequence had a natural splitting, we could use this splitting to get a class in $H_3(G; \mathbb{Z})$, but we see no obvious one.

**Example 12.3.** Fig. 8 gives a well-known ideal triangulation for the complete hyperbolic structure on the figure eight knot complement (see, e.g., [22, 15]). This manifold is denoted m004 in the Callahan-Hildebrand-Weeks cusped census (see [2, 21, 9]). Both simplices are regular ideal simplices but the vertex-order of the second
simplex is opposite to its orientation. The cross-ratio parameters are therefore \( \omega := e^{\pi i/3} \) and \( \omega^{-1} \) respectively, and \( \hat{\beta}(m004) = [\omega; p, q] - [\omega^{-1}; r, s] \) for suitable \( p, q, r, s \). In the next section we will see that we get a flattening of the complex \( K \) if and only if \( q = -1 - 2p \) and \( (r, s) = -(p, q) \). Choosing, e.g., \( p = 0 \), we see

\[
\hat{\beta}(m004) = [\omega; 0, -1] - [\omega^{-1}; 0, 1].
\]

If

\[
V_0 = 1.014941606409653625021202554\ldots
\]

denotes the volume of the regular ideal tetrahedron, then

\[
R(\omega; p, q) = \frac{(2p - 2q - 1)\pi^2}{12} + V_0 i,
\]

\[
R(\omega^{-1}; -p, -q) = \frac{(2p - 2q - 1)\pi^2}{12} - V_0 i.
\]

Thus

\[
(vol + i\text{cs})(m004) = 2V_0,
\]

recovering the known volume and Chern-Simons invariant of the figure eight knot complement.

We now describe what happens when we deform our hyperbolic structure by hyperbolic Dehn surgery to perform a Dehn filling on \( M \) (see, e.g., \[22, 15\]). Topologically, the Dehn filled manifold \( M' \) differs from \( M \) in that some of the cusps have been filled by solid tori, which adds new closed geodesics at these cusps (the cores of the solid tori). Let \( \lambda_j \) be the complex length of the geodesic (length plus \( i \) times torsion) added at the \( j \)-th cusp. If no geodesic has been added at the \( j \)-th cusp we put \( \lambda_j = 0 \).

The deformation deforms the representation \( \Gamma = \pi_1(M) \to G \). If we keep the points \( \tilde{p}_i \) at fixed points of the images of the cusp subgroups of \( \Gamma \) then our triangulation deforms to what is called a “degree one ideal triangulation” in \[14\] (it is not a genuine triangulation because the topology of \( K \) has not changed, so its topology is not the topology of \( M' \)). By deforming the flattened ideal simplex parameters continuously we obtain flattened ideal simplex parameters \( [x'_i; p'_i, q'_i] \) after deformation.

**Theorem 12.4.** With the above notation, the element in \( \hat{\mathcal{B}}(\mathbb{C}) \) represented by the Dehn filled manifold \( M' \) is

\[
\hat{\beta}(M') = -\sum_{j=1}^{h} \chi(e^{\lambda_j}) + \sum_i \epsilon_i [x'_i; p'_i, q'_i]
\]

with \( \chi \) as in Theorem 5.6. Moreover,

\[
(vol + i\text{cs})(M') = -\frac{\pi}{2} \sum_{j=1}^{h} \lambda_j - i \sum_i \epsilon_i R(x'_i; p'_i, q'_i)
\]

The second formula of this theorem was proved up to a constant (depending on \( K \) but independent of the Dehn filling) in \[12\]. The constant was conjectured to be a multiple of \( i\pi^2/6 \). Since \[12\] used unordered simplices, the constant in that version is indeed a multiple of \( i\pi^2/6 \). Versions prior to 1.10.2 of the program Snap \[9, 4\] use this formula but ignores the parity condition, giving answers accurate to \( i\pi^2/12 \). Snappea \[21\] uses a version, also from \[12\], that is accurate to an unknown
constant. Both programs then bootstrap this to an accurate computation of the Chern-Simons invariant for any manifold that can be related by a sequence of Dehn drillings and fillings to one with known Chern-Simons invariant. The value they print is \( \text{cs}(M)/2\pi^2 \), hence well defined modulo \( 1/2 \).

Snap now uses the formulae of this paper and can compute Chern-Simons for any manifold. For compact manifolds accessible by the bootstrapping method, Snap also computes the eta-invariant using a related formula of [11]; from this the Riemannian Chern-Simons (defined modulo \( 2\pi^2 \), hence modulo 1 in their normalization) is then computed.

**Proof of Theorem 12.4.** For simplicity of exposition we assume \( M \) has just one cusp represented by the ideal vertex \( p \) of \( K \).

The proof depends on the fact that the deformed flattening parameters \( [x_i; p_i', q_i'] \) still satisfy our flattening conditions around edges, but there is a natural flattening condition at the filled cusps that is not satisfied, which leads to the “correction term” \( \sum_{j=1}^h \chi(e^{\lambda_j}) \) in the theorem.

The link \( L_p \) of vertex \( p \) of \( K \) is a torus. Let \( C \) be a simplicial curve in \( L_p \) that is a meridian of the solid torus that the cusp has been replaced by. We can obtain a genuine triangulation of \( M' \) by the blow-up procedure of Section 9, blowing up using the curve \( C \). As in that section, we want to extend the flattening after blowing up so that matching pairs of added simplices have the same flattening and therefore cancel in the computation of \( \hat{\beta}(M') \). To be able to extend the flattening of \( K \) after blowing up, we need that the log-parameter along a normal curve parallel to \( C \) is zero. As it stands, it is not zero, since it represents the logarithm of the derivative of the holonomy of the meridian curve. It was zero before deformation, but after deformation the meridian curve represents a full rotation about the core curve of the solid torus added by the Dehn surgery, so the logarithm of the derivative of its holonomy is \( 2\pi i \) (see, e.g., [22, 15] for more detail). We can use the procedure of Section 9 to correct this: we must modify flattenings on the simplices traversed by a normal curve representing a longitude of the added solid torus as indicated in Fig. 7 (the longitude runs horizontally across the center of the figure). The effect on each modified simplex is to change its contribution to \( \sum_i \epsilon_i [x_i; p_i, q_i] \) by something of the form \( [x; p, q \pm 1] - [x; p, q] \) or the equivalent after a permutation of vertices.

By the calculations of Proposition 5.5 this is \( \pm \chi(z) \), where \( z \) is the cross-ratio parameter corresponding to the edge of the 3-simplex that the curve passes (so \( z \) is one of \( x, 1/(1 - x), 1/x, \) etc.). As we sum these contributions over the affected simplices with appropriate signs, they sum to \( -\chi(L) \), where \( L \) is product of the corresponding cross-ratio parameters or their inverses for the edges passed by the normal curve. This \( L \) is the derivative of holonomy of the longitude and \( \log(L) \) is the complex length \( \lambda \) of the core curve of the solid torus added by the Dehn filling (see, e.g., [22, 15]). This proves the first equation of the theorem.

For the second equation we use the fact that \( R(\hat{\beta}(M')) = i(\text{vol} + i\text{cs})(M') \). We know this if \( M' \) is compact and we will deduce it below if \( M' \) is non-compact, so we assume it for now. \( R \circ \chi: \mathbb{C}^* \to \mathbb{C}/\pi^2\mathbb{Z} \) is the map \( z \mapsto \frac{\pi}{2\lambda} \log z \), so \( R\chi(e^{\lambda}) = \frac{\pi}{2\lambda} \lambda \).

Applying \( \frac{1}{i} R \) to the first equation of the theorem thus gives the second.

Meyerhoff’s extension of Chern-Simons invariant to cusped hyperbolic manifolds in [10] is given by defining \( (\text{vol} + i\text{cs})(M) \) to be the limit of \( (\text{vol} + i\text{cs})(M') + \)
\[ \sum_{j=1}^{h} \lambda_j \] as \( M' \) approaches \( M \) in hyperbolic Dehn surgery space. The case that \( M' \) is compact in the above Theorem thus implies:

**Corollary 12.5.** For any oriented complete finite volume hyperbolic 3-manifold \( M \)

\[ i(\text{vol} + i\text{cs})(M) = R(\hat{\beta}(M)) \in \mathbb{C}/\pi^2\mathbb{Z}, \]

where \( \text{cs}(M) \) is Meyerhoff’s extension of the Chern-Simons invariant if \( M \) is non-compact.

The “correction term” \( -\sum_{j=1}^{h} \chi(e^{\lambda_j}) \) in Theorem 12.4 arises because the sum of flattening parameters corresponding to a meridian of the a solid torus added at a cusp by Dehn filling is \( 2\pi i \) rather than zero. The proof of the theorem shows that flattening parameters can be chosen so that this sum is zero, and then the correction term is not there. This gives an analog of Theorem 4.4 for these triangulations:

**Theorem 12.6.** Consider the degree one triangulation of \( M' \) of Theorem 12.4. Then there exists flattenings \([x'_i; p''_i, q''_i]\) of the ideal simplices which satisfy the conditions

- parity along normal paths is zero;
- log-parameter about each edge is zero;
- log-parameter along any normal path in the neighborhood of a 0-simplex that represents an unfilled cusp is zero;
- log-parameter along a normal path in the neighborhood of a 0-simplex that represents a filled cusp is zero if the path is null-homotopic in the added solid torus.

For any such choice of flattenings we have

\[ \hat{\beta}(M') = \sum_i [x'_i; p''_i, q''_i] \in \hat{\mathcal{B}}(\mathbb{C}) \]

so

\[ i(\text{vol} + i\text{cs})(M') = \sum_i R(x'_i; p''_i, q''_i). \]

13. Example and discussion

We return to the figure 8 knot complement \( m004 \) of Example 12.3 to illustrate the above formulae. Denote the cross-ratio parameters of the two simplices in Fig. 8 by \( z \) and \( y \), and choose flattenings \([z; p, q]\) and \([y, r, s]\). The consistency condition about each edge can be read off from the figure. The condition for the edge labeled with a single arrow is:

\[ 2 \log z + \log z' - \log y' - 2 \log y'' = 2\pi i \]

and the condition for the other edge turns out to be equivalent to this. The log-parameters for the flattenings are

\[ (\log z + p\pi i, \log z' + q\pi i, \log z'' + (-1 - p - q)\pi i), \]
\[ (\log y + r\pi i, \log y' + s\pi i, \log y'' + (1 - r - s)\pi i). \]

Thus the flattening condition about the edge is

\[ 2p + q - s - 2(1 - r - s) = -2. \]
Figure 9. Fundamental domain of the cusp torus, triangulated as the link of the 0-simplex of the ideal triangulation of $m004$.

The log holonomy for a meridian and longitude can be read off from the fundamental domain of the cusp torus illustrated in Fig. 9 (taken from [15] but modified since orientation conventions differ; in particular, $w, w', w''$ there are our $(y'')^{-1}, (y')^{-1}, y^{-1}$ because of different vertex-ordering for the second simplex). They are:

$$u := \log z'' + \log y'' \quad v := 2 \log z - 2 \log z''.$$  

Their vanishing thus gives flattening conditions

$$(1 - r - s) + (-1 - p - q) = 0$$

$$2p - 2(-1 - p - q) = 0$$

which simplify to

$$-p - q - r - s = 0$$

$$2 + 4p + 2q = 0.$$  

Solving equations (19), (21), (22) gives

$$q = -1 - 2p, \quad (r, s) = -(p, q).$$  

Simultaneously solving the consistency condition (18) and the cusp conditions $u = v = 0$ (see (20)) gives $z = \omega, y = \omega^{-1}$. Thus, as promised earlier, we see that $\hat{\beta}(m005) = [\omega; p, q] - [\omega^{-1}; -p, -q]$ with $q = -1 - 2p$. Choosing $p = 0$, we get:

$$\hat{\beta}(m005) = [\omega; 0, -1] - [\omega^{-1}; 0, 1].$$

Now suppose we deform our parameters to perform hyperbolic $(\alpha, \beta)$ Dehn filling on this manifold. Let $z, y$ now denote the deformed values of the simplex cross-ratios (they simultaneously satisfy the consistency condition (18) and the Dehn filling condition $\alpha u + \beta v = 2\pi i$). Choose $\gamma, \delta \in \mathbb{Z}$ so

$$\begin{vmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{vmatrix} = 1.$$
As described in [15], the complex length of the added geodesic is \( \lambda := -(\gamma u + \delta v) \), so Theorem 12.4 gives the formulae

\[
\hat{\beta}(m004(\alpha, \beta)) = -\chi(e^{-\gamma u - \delta v}) + [z; 0, -1] - [y; 0, 1]
\]

\[
(\text{vol } + \text{cs})(m004(\alpha, \beta)) = \frac{\pi}{2} (\gamma u + \delta v) - i(R(z; 0, -1) - R(y; 0, 1)),
\]

with \( u, v \) given by (20).

On the other hand, if we use Theorem 12.6, the Dehn filling condition \( \alpha u + \beta v = 2\pi i \) gives the flattening condition

\[
\alpha(-p - q - r - s) + \beta(2 + 4p + 2q) = -2
\]

while the parity condition says \( p + q + r + s \) must be even. A simultaneous solution of these and equation (19) is:

\[
q = \gamma - 1 - 2p, \quad r = -2\delta - p, \quad s = -\gamma + 4\delta + 1 + 2p.
\]

Choosing, e.g., \( p = 0 \), we get

\[
\hat{\beta}(m004(\alpha, \beta)) = [z; 0, \gamma - 1] - [y; -2\delta, -\gamma + 4\delta + 1],
\]

\[
(\text{vol } + \text{cs})(m004(\alpha, \beta)) = -i(R(z; 0, \gamma - 1) - R(y; -2\delta, -\gamma + 4\delta + 1)).
\]

It is not hard to verify directly the equivalence of (24) and (25).

13.1. Other fields. Although we have worked over \( \mathbb{C} \) in this paper, most of what we have done works for a subfield \( K \subset \mathbb{C} \) if we replace \( G = \text{PSL}(2, \mathbb{C}) = \text{PGL}(2, \mathbb{C}) \) by \( G = \text{PGL}(2, K) \). In Theorem 5.6 we must replace the group \( \mu \) of roots of unity by the group \( \mu_K \) of roots of unity in \( K \). The homomorphism \( \lambda: H_3(\text{PGL}(2, K)) \to \widehat{B}(K) \) is still defined. For a hyperbolic 3-manifold \( \mathbb{H}^3/\Gamma \) with \( \Gamma \subset \text{PGL}(2, K) \) the element \( \hat{\beta}(M) \) naturally lies in \( \widehat{B}(K) \), but we have numerical evidence that it lies in \( \widehat{B}(k) \), where \( k \) is the invariant trace field of \( M \) (\( k \) is always contained in \( K \) and is generally smaller). The arguments of [14] show that \( \hat{\beta}(M) \in \widehat{B}(k) \) if \( M \) is non-compact, while some power of 2 times \( \hat{\beta}(M) \) lies in \( \widehat{B}(k) \) in general.

References


Department of Mathematics, Barnard College, Columbia University
E-mail address: neumann@math.columbia.edu