Commensurability and virtual fibration for graph manifolds

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Two manifolds are ***commensurable*** if they have diffeomorphic covers. We would like invariants that distinguish manifolds up to commensurability. A collection of such ***commensurability invariants*** is ***complete*** if it always distinguishes non-commensurable manifolds.

Commensurability invariants of hyperbolic 3-manifolds are discussed in **[NRe]**. The two main ones are the invariant trace field and the invariant quaternion algebra. The latter is a complete commensurability invariant in the arithmetic case, but not in general. The set of primes at which traces fail to be integral is another commensurability invariant, and examples are given in **[NRe]** where the invariant quaternion algebras agree but this set does not. Another commensurability invariant discussed in **[NRe]** of the collection of "cusp fields" (the fields generated by cusp parameters). Craig Hodgson has pointed out that the set of $\mathrm{PSL}(2,\mathbb{Q})$-classes of a cusp parameters is a finer commensurability invariant than the cusp fields when the degree of some cusp field exceeds 3.

Here we discuss commensurability of non-hyperbolic 3-manifolds. For 3-manifolds with geometric structure the classification is known (cf. sec. 1):

Theorem A. *For each of the six "Seifert geometries" $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{S}^2\times\mathbb{E}^1$, $\mathbb{H}^2\times\mathbb{E}^1$, $\mathbb{N}il$, and $\mathbb{P}SL$ there is just one commensurability class of compact geometric 3-manifolds with the given structure (two for the last two geometries if orientation-preserving commensurability of oriented manifolds is considered). For the remaining non-hyperbolic geometry $\mathbb{S}ol$, the commensurability classes are in one-one correspondence with real quadratic number fields (such a manifold is covered by a torus bundle over a circle and the field in question is the field generated by an eigenvalue of the monodromy of this bundle).*

Non-compact finite volume non-hyperbolic geometric 3-manifolds admit geometric structures of both types $\mathbb{H}^2\times\mathbb{E}^3$ and $\mathbb{P}SL$ and form just one commensurability class, also in the oriented case.

In section 2 we define several multiplicative invariants for prime 3-manifolds. A ***multiplicative invariant*** is one that multiplies by degree for covering spaces. Our invariants are most interesting for graph manifolds. Since the ratio of two multiplicative invariants is a commensurability invariant, we get several commensurability invariants also. The following theorem, which uses two of these invariants, is a start on the commensurability classification for manifolds with non-trivial geometric decomposition.

Let M be a non-Seifert fibered oriented 3-manifold obtained by pasting two Seifert manifolds M_1 and M_2 with torus boundaries along these tori. Suppose also that neither half M_i is the total space SMb of the circle bundle over the Möbius band with orientable total space (if it is, then there is a double cover of M that either satisfies our requirements or is a torus bundle over the circle and thus covered by Theorem A). To each half M_i of M is associated a pair of numerical invariants $e_i, \chi_i \in \mathbb{Q}$ (the euler number of the fibration and orbifold euler characteristic of the base, as described in Section 1) with $\chi_i \neq 0$. Let p be the intersection number within the gluing torus of the fibers of the two pieces of M. Denote $v_i = \chi_i^2/e_i \in \mathbb{Q} \cup \{\infty\}$ and exchange M_1 and M_2 if necessary to make $|v_1| \leq |v_2|$.

Theorem B. 1. *$p^2 e_1 e_2$ and v_1/v_2 are commensurability invariants of the above M. Their product is a rational square (or indeterminate if $e_1 = e_2 = 0$).*

2. *For each $p^2 e_1 e_2 \in \mathbb{Q} - \{0\}$ and $v_1/v_2 \in [-1,1] \cap (\mathbb{Q} - \{0\})$ whose product is a rational square there is a unique commensurability class of M as above. If $e_2 = 0$ there are two commensurability classes according as e_1 is also zero or not.*

3. *The above commensurability class splits into two orientation preserving ones, determined by the sign of e_1, unless $e_1 = e_2 = 0$ or $e_1 e_2 < 0$ and $v_1/v_2 = -1$.*

We next discuss whether M is ***virtually fibered***, that is, whether some finite cover of M fibers over S^1. Thurston has conjectured that hyperbolic manifolds are always virtually fibered. Currently there is little evidence for either the truth or the falsity of this conjecture. For a geometric manifold M belonging to one of the other seven geometries the answer is easy — M is virtually fibered if it is non-compact (hyperbolic fiber) or has a geometric structure of type $\mathbb{S}^2 \times \mathbb{E}^1$ (fiber S^2), $\mathbb{E}^3$, $\mathbb{N}il$, or $\mathbb{S}ol$ (fiber T^2), or $\mathbb{H}^2 \times \mathbb{E}^1$ (hyperbolic fiber), and not virtually fibered if it is compact with $\mathbb{S}^3$, or $\mathbb{P}SL$ structure.

We shall give a complete answer for graph manifolds. In the special case of the manifold of Theorem B the answer is as follows.

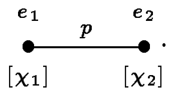
Theorem C. *The manifold M of Theorem B is virtually fibered if and only if $0 < p^2 e_1 e_2 \leq 1$ or $e_1 = e_2 = 0$.*

Before we describe the general result we describe when a graph manifold itself fibers over S^1. We also describe when it is the link of a complex surface singularity, since this turns out to be related. These results are not new (cf. e.g., **[Wi]**, **[N1]**, **[EN]**) but the present formulation in terms of a "reduced plumbing matrix" has not appeared before. We assume M is prime, since otherwise it cannot be virtually fibered.

Suppose M is a prime graph manifold whose toral decomposition (cf. **[JS]**, **[J]**) into Seifert fibered manifolds is

$$M = M_1 \cup \ldots \cup M_n .$$

Then for each i we have the invariants $e_i = e(M_i)$ and $\chi_i = \chi(M_i)$ already mentioned and for each separating torus T we have the fiber intersection number in T mentioned above, which we will denote $p(T)$. It is convenient to represent all this information by a weighted graph with a vertex for each M_i and an edge for each separating torus T. For example, the manifold M of Theorems B and C would have the graph



We call this the ***decomposition graph***. We define the ***decomposition matrix*** for M to be the symmetric $n \times n$ matrix $A(M) = (a_{ij})$ with

$$\begin{aligned} a_{ij} &= e_i + 2\sum_{iEi} \frac{1}{|p(E)|} && \text{if } i = j \\ &= \sum_{iEj} \frac{1}{|p(E)|} && \text{if } i \neq j\,, \end{aligned}$$

where iEj means E is an edge from i to j and $p(E)$ is the fiber intersection number weight for this edge.

For convenience we assume in this introduction that M is, in the terminology of section 3, a "very good graph manifold with no self-pastings." That is

- each M_i is Seifert fibered over orientable base;
- the fiber intersection numbers $p(T)$ are all positive;
- the decomposition graph $G(M)$ has no loops (edges that start and terminate at the same vertex) and no M_i is the circle bundle SMb over the Möbius band.

As we shall see, these conditions are not essential to most of our discussion, and can in any case be achieved by taking a double cover of M (except when M is a $\mathbb{S}ol$-manifold). We also assume M is not a T^2 bundle over S^1. Under these conditions we shall see that the decomposition matrix is a reduced version of the "plumbing matrix" for M of **[N1]** (see also **[vR]**, **[HNK]**). The following is a simplified version of Theorem 3.1 below.

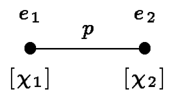
Theorem D. 1. *The above M is the link of a complex surface singularity if and only if $A(M)$ is negative definite.*

2. *The above M fibers over the circle if and only if $A(M)$ annihilates a vector with no zero component (we say $A(M)$ is "supersingular").*

3. *The above M has a "horizontal surface" (an embedded surface transverse to a Seifert fiber of one of the M_i) if and only if $A(M)$ is singular.*

The relevance for us of being the link of a complex surface singularity is that it is inherited by covers, so they too have negative definite (and hence non-singular) decomposition matrices.

For example, the graph manifold of Theorem B with decomposition graph



has decomposition matrix

$$A(M) = \begin{pmatrix} e_1 & \frac{1}{|p|} \\ \frac{1}{|p|} & e_2 \end{pmatrix}.$$

Assuming M_1 and M_2 have orientable base, it is thus a singularity link up to orientation if and only if $p^2 e_1 e_2 > 1$ and is fibered over S^1 if and only if $p^2 e_1 e_2 = 1$.

To describe the general necessary and sufficient condition for virtual fibration of a prime graph manifold M we need a definition. We shall call a block matrix

$$T = \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \cdots & \cdots & \cdots \\ T_{n1} & \cdots & T_{nn} \end{pmatrix}$$

a ***virtualizer*** if

- $T_{ij} = T_{ji}^t$ for each i, j, so T is symmetric;
- each entry of each T_{ij} is a non-negative rational number;
- T_{ii} is a nonsingular diagonal matrix whose entries sum to 1 for each i;
- the k-th row sum of T_{ij} is the k-th diagonal entry of T_{ii} for each i, j (a corresponding statement for columns follows by symmetry).

If the T_{ij} for $i \neq j$ are permitted to have arbitrary rational entries and last item is replaced by

- the sum of absolute values of the entries of the k-th row of T_{ij} is at most the k-th diagonal entry of T_{ii} for each i, j,

we speak of a ***sub-virtualizer***. If $A = (a_{ij})$ is an $n \times n$ symmetric matrix and T a virtualizer or sub-virtualizer as above, we shall call the matrix

$$\begin{pmatrix} a_{11}T_{11} & \cdots & a_{1n}T_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1}T_{n1} & \cdots & a_{nn}T_{nn} \end{pmatrix}$$

a ***virtualization*** or ***sub-virtualization*** of A. If the virtualizer or sub-virtualizer T has all diagonal entries equal to a fixed number $1/m$ (which, in particular, implies that the T_{ij} are all $m \times m$ matrices), we shall call it ***uniform***, and speak of a ***uniform (sub-)virtualization***.

We call a symmetric rational matrix A ***virtually singular*** if it has a virtualization which is singular. It would appear that there are variations of this concept — a considerably weaker one using sub-virtualization and a stronger one using regular virtualization — but we shall show that they are all equivalent. We call A ***supersingular*** if it annihilates a vector with no zero component and we call it ***virtually supersingular*** if it has a virtualization with this property. Again, we shall see that the "regular" and "sub-" variations of this concept are equivalent.

Theorem E. *Suppose M is a prime graph manifold with no self-pastings ("very good" is not needed) which is not a $\mathbb{S}ol$- or $\mathbb{N}il$-manifold and $A = A(M)$ is its decomposition matrix. Then M is virtually fibered if and only if its decomposition matrix A is virtually supersingular.*

A cover of M has a horizontal surface if and only if A is virtually singular.

We can give a rather simple criterion for virtual singularness of A. A is a matrix with non-negative off-diagonal entries. By reordering indices so $a_{11} = \ldots = a_{rr} = 0$, $a_{r+1,r+1}, \ldots, a_{ss} > 0$, $a_{s+1,s+1}, \ldots, a_{nn} < 0$, we may put A in the form

$$\begin{pmatrix} Q & X & Y \\ X^t & P & Z \\ Y^t & Z^t & N \end{pmatrix}$$

such that Q has zero diagonal entries, P has positive diagonal entries, N has negative diagonal entries. Let P_- be the result of multiplying the diagonal entries of P by -1.

Theorem F. *The above A fails to be virtually singular if and only if Q is void and P_- and N are both negative definite.*

For example, this says that $A = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ is virtually singular if and only if $0 \leq ac \leq b^2$. The condition for virtually supersingular is the same except that when $ac = 0$ then A is virtually supersingular if and only if $a = c = 0$. See Proposition 5.7.

The property that A have non-negative off-diagonal entries is necessary in Theorem F, but Proposition 5.2 shows that if A and A' are symmetric rational matrices that differ only in the signs of their off-diagonal entries, then one is virtually singular or supersingular if and only if the other is.

I do not as yet know as simple a condition for virtually supersingular as that of Theorem F. An additional necessary condition is given by Proposition 5.6.

Theorem 3.2 of **[LW]** gives a sufficient condition for failure of virtual fibration. In our language it is that each diagonal entry of the decomposition matrix A exceeds in absolute value the sum of absolute values of the other entries in its row. This result follows from Theorem E, since the condition is clearly inherited by virtualizations of A and is easily seen to imply that A is non-singular. The computations in the proof of the underlying Theorem 3.1 of **[LW]** can be refined to give a direct proof of our Theorem 3.1.2, not involving plumbing.

Theorem B developed out of discussions with J. Wahl in 1992 of his interesting characteristic number $P.P$ for singularity links (**[Wa]**). Discussions with Shichen Wang led me to think about virtual fibration. I am grateful to him for these discussions and for making me aware of the related papers **[LW]** and **[WW]**.

1. Geometric 3-manifolds and Theorem A

We shall only consider oriented manifolds. Let M be a Seifert fibered 3-manifold with unnormalized Seifert invariant $\{g; \beta_1/\alpha_1, \dots, \beta_r/\alpha_r\}$ (we follow the convention that $g < 0$ is non-orientable genus). Recall (e.g., **[N2]**, **[S]**) that M may be considered as a circle bundle over the orbifold with signature $\{g; \alpha_1, \dots, \alpha_r\}$. The ***euler number*** of this Seifert fibration is

$$e = -\sum_{i=1}^{r} \beta_i/\alpha_i$$

and the orbifold euler characteristic of the base is

$$\chi = e(g) - \sum_{i=1}^{r}(1 - \frac{1}{\alpha_i}),$$

where $e(g) = 2-2g$ or $2+g$ according as $g \geq 0$ or $g < 0$. M has geometric structure according to the following scheme.

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e \neq 0$	$\mathbb{S}^3$	$\mathbb{N}il$	$\mathbb{P}SL$
$e = 0$	$\mathbb{S}^2 \times \mathbb{E}^1$	$\mathbb{H}^3$	$\mathbb{H}^2 \times \mathbb{E}^1$

With a natural normalization of the metric on these geometries the volume of M is $4\pi^2\chi^2/|e|$ if $e\chi \neq 0$ and is indeterminate, depending on the geometric structure, otherwise. (cf. **[N2]**. The normalization we are using is to give $\mathbb{S}^3$ and $\mathbb{P}SL$ the natural metric on the universal cover of the unit tangent bundle of $\mathbb{S}^2$ and $\mathbb{H}^2$ respectively. For $\mathbb{S}^3$ this gives curvature $1/4$. This volume formula is just Fubini's theorem: volume $=$ (area of base) $\times$ (length of fiber) since the fiber length is $2\pi|\chi/e|$ by the construction of the geometric structure on M in **[N2]** or **[S]**.)

The fact that two Seifert manifolds belonging to the same geometry are commensurable (the commensurability can not in general be chosen to preserve the geometric structure) seems to have been first observed in the 60's by Macbeath, though not in terms of geometries. The proof is easy nowadays — the main observation is that M is covered by a circle bundle over a 2-manifold obtained by pulling back the Seifert fibration to a manifold cover of the base orbifold. For circle bundles the commensurability claim is easy.

The seventh non-hyperbolic 3-manifold geometry is $\mathbb{S}ol$. A compact orientable manifold for this geometry is either a T^2-bundle over S^1 or is double covered by one. Moreover, the monodromy $A\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ of the fibration is hyperbolic, that is it has trace $|\,\mathtt{tr}(A)| > 2$. The eigenvalues of A therefore generate a real quadratic field. It is easy to see that this field is a commensurability invariant.

One can recover M up to commensurability from this field k as follows. If $\mathtt{tr}(A) < -2$ we can take a double cover of M to replace A by A^2 which has positive trace. The two real embeddings of k give a map of k to $\mathbb{R}^2$ (alternatively, think of $\mathbb{R}^2$ as $k \otimes_{\mathbb{Q}} \mathbb{R}$) and the restriction of this map to the ring of integers $\mathcal{O}_k$ embeds $\mathcal{O}_k$ as a lattice $\Lambda \subset \mathbb{R}^2$. The group of units $\mathcal{O}_k^*$ acts by multiplication on $\mathbb{R}^2$ and Λ and hence on the torus $\mathbb{R}^2/\Lambda$. The monodromy of the fibration of M is an element of this group. Since $\mathcal{O}_k^*$ is infinite cyclic times torsion, the commensurability claim follows. This arithmetic description of these torus bundles relates to their occurrence as links of cusp singularities of Hilbert modular surfaces, see **[Hi]** for more details.

The only ones of the above geometries that admits complete finite volume non-compact geometric manifolds M are $\mathbb{H}^2 \times \mathbb{E}^1$ and $\mathbb{P}SL$. The behavior of volume in this case is as follows. Both the above geometries fiber over $\mathbb{H}^2$ and these fibrations have natural connections — the obvious horizontal connection in the product case and the lift of the riemannian connection for $\mathbb{H}^2$ in the $\mathbb{P}SL$ case. Consider a horospherical section $S^1 \times h$ at each cusp of M (h represents the Seifert fiber). The connection induces linear foliations on these tori. Thus, if we cut off the cusps along these horospherical tori we get a compact Seifert fibered 3-manifold M_0 with boundary plus a linear foliation of each boundary component. The euler number of the Seifert fibration is defined in this situation. It vanishes precisely in the $\mathbb{H}^2 \times \mathbb{E}^1$ case and otherwise the volume formula $4\pi^2|\chi^2/e|$ is still valid.

The definition of euler number for this situation is as follows. If a section to the Seifert fibration is chosen at each boundary component of M_0 and the linear foliations parallel to these sections are used then the euler number is the euler number of the closed Seifert manifold resulting by Dehn filling the boundary components of M_0 using these sections as meridians. Suppose other linear foliations of the boundary tori T_i are given. They have slopes $r_i \in \mathbb{R}$ with respect to the chosen sections. (Slope is defined so that the sections have slope 0 and the Seifert fiber has slope ∞.) The euler number for M with respect to these foliations differs from the euler number with respect to the sections by $\sum r_i$. In the case of rational slopes this is the same as the euler number for the closed Seifert manifold obtained by Dehn filling the boundary components of M_0 using closed leaves of the boundary foliations as meridians.

M has geometric structures for any choice of linear foliations at its cusps of slope $\neq \infty$, and the validity of the volume formula follows by the same argument as in the compact case when the foliations have integral slopes and then follows by a covering argument when the slopes are rational and by continuity in general.

The covering argument uses the following lemma, which was proved in **[NRa]** in the closed case and follows similarly in general, see also **[LW]**.

Lemma 1.1. *Let $\pi\colon N \to M$ be a fiber preserving map of orientable Seifert manifolds over orientable bases which preserves linear foliations on boundary tori and let f be the degree of π restricted to a fiber and b be the degree of the induced map of base surfaces. Then $e(N) = b\,e(M)/f$.* □

2. Multiplicative invariants of geometric decomposition.

For this section M will be a closed connected orientable 3-manifold. If M is prime (i.e., not a non-trivial connected sum) the ***toral decomposition*** of Jaco-Shalen and Johannson (**[JS]**, **[J]**) cuts M open along a minimal collection of embedded incompressible tori into a collection of simple manifolds and Seifert fiberable manifolds. We make the convention that if M has a $\mathbb{S}ol$ geometric structure (i.e., it has a double cover that fibers over S^1 with hyperbolic monodromy) we do not cut it. Note that the total space SMb of the tangent circle bundle to the Möbius band admits another Seifert fibration. Namely, it fibers over the disk with two exceptional fibers of degree 2, obtained by lifting the obvious circle action on the Möbius band to SMb. When SMb occurs as a piece in the toral decomposition of a manifold, we shall always take this Seifert structure on it.

It is often more natural to cut M along tori and Klein bottles into pieces which admit finite volume geometric structures (in the case that M is non-Haken and not Seifert fibered M is not cut and its geometric structure is hyperbolic, but only conjectural). This differs from the toral decomposition above in that SMb is not allowed as a piece of the decomposition — rather than cutting along the boundary torus of such a piece we cut along its core Klein bottle, which splits it into a toral annulus. We call this the ***geometric decomposition***.

We shall use both the geometric and the toral decompositions. The geometric decomposition is the more natural decomposition in that, for example, it behaves well with respect to covering spaces. The toral decomposition is more convenient for discussing plumbing and is discussed in the next section.

M is a ***graph manifold*** if it is a connected sum of prime manifolds whose geometric decompositions consists only of pieces corresponding to the seven non-hyperbolic geometries (equivalently, their toral decompositions have only such pieces or SMb pieces).

If the geometric decomposition of a prime graph manifold M is non-trivial, that is, M is not itself geometric, then it decomposes M into pieces that belong to the $\mathbb{H}^2 \times \mathbb{E}^1$, $\mathbb{P}SL$ pair of geometries. Each piece comes with a linear foliation of its boundary tori, namely the restriction of the Seifert fibration of the adjacent piece. It thus has a well defined euler number for its fibration, and also a geometric structure of type $\mathbb{H}^2 \times \mathbb{E}^1$ or $\mathbb{P}SL$, well defined up to boundary-foliation-preserving deformation. These geometric structures lift appropriately in covers, so the sum of the volumes of the $\mathbb{P}SL$ components is a multiplicative invariant, which one might call the ***$\mathbb{P}SL$-volume*** of M. It has been studied in a different description in **[LW]**. We can split this invariant into two orientation sensitive multiplicative invariants — sum over the $\mathbb{P}SL$ components with positive euler number and sum over the $\mathbb{P}SL$ components with negative euler number. We shall call the difference of these two invariants the ***signed $\mathbb{P}SL$-volume***. If we denote by $V(M)$ and $v(M)$ respectively the sum of $|\chi^2/e|$ or χ^2/e over the $\mathbb{P}SL$ components of M, then the $\mathbb{P}SL$-volume is $4\pi^2 V(M)$ and the signed $\mathbb{P}SL$-volume is $4\pi^2 v(M)$.

We can define a further refinement of these invariants. Consider the graph with a vertex for each component of the geometric decomposition of M and an edge for each cutting torus or Klein bottle. Call this the ***geometric decomposition graph***. If it has no cycle of odd length we shall say M is a ***bipartite graph manifold***. In this case, by taking alternate components, we can partition the components of the geometric decomposition into two sets such that the pieces within each set are disjoint from each other. Then the v-invariants $v_1(M)$ and $v_2(M)$ of the two parts of this partition, ordered so $|v_1(M)| \le |v_2(M)|$, are multiplicative invariants, and similarly for the V-invariants $V_1(M)$ and $V_2(M)$. These invariants are defined for a non-bipartite graph manifold M as follows: any non-bipartite graph manifold is double covered by a bipartite one, so one takes half the invariant of the double cover. For a non-bipartite graph manifold it is easy to see that $v = v_1 = v_2$ and $V = V_1 = V_2$.

The ratio of any two multiplicative invariants is a commensurability invariant. In particular, $v_1/v_2 \in [-1,1] \cup \{\text{indeterminate}\}$ is the commensurability invariant of Theorem B.

We shall show that the other invariant $p^2 e_1 e_2$ of Theorem B is also a special case of a more general commensurability invariant.

Let M be a bipartite graph manifold (this is not essential, but leads to no restriction, as we will see, and simplifies orientation issues). Call the two classes of components of the geometric decomposition of M the "left geometric components of M" and "right geometric components of M". Each separating torus T of the geometric decomposition of M separates a left geometric component from a right one. We call them $M_1(T)$ and $M_2(T)$. Let $p(T)$ be the intersection number in T of a fiber of $M_1(T)$ and a fiber of

$M_2(T)$. Define

$$\theta(M) := \sum_{\{T: e_1(T)e_2(T)\neq 0\}} \frac{1}{p(T)} \frac{\chi(M_1(T))\chi(M_2(T))}{e(M_1(T))e(M_2(T))}.$$

We claim

Proposition 2.1. *The above invariant $\theta(M)$ is a multiplicative invariant. In particular, we can define $\theta(M)$ for an arbitrary prime graph manifold M as half of θ of the bipartite cover.*

Proof. Suppose $\pi\colon N \to M$ is a d-fold cover. Let T' be a separating torus of N and define $N_1(T')$ and $N_2(T')$ to be the geometric components that meet along T' and $p(T')$ to be the intersection number in T' of their fibers. Let $f_i(T')$ be the degree of π restricted to a fiber of $N_i(T')$ for $i = 1, 2$. Let T be the image torus in M. An easy calculation (cf. e.g., **[WW]**) shows that the intersection number of a fiber of $N_1(T')$ and a fiber of $N_2(T')$ in T' is $p(T') = p(T)f_1(T')f_2(T')/d(T')$. Lemma 1.1 implies $\chi(N_1(T'))/e(N_1(T')) = f_1(T')\chi(M_1(T))/e(M_1(T))$ and $\chi(N_2(T'))/e(N_2(T')) = f_2(T')\chi(M_2(T))/e(M_2(T))$. It follows that

$$\begin{aligned}\frac{1}{p(T')}\frac{\chi(N_1(T'))}{e(N_1(T'))}\frac{\chi(N_2(T'))}{e(N_2(T'))} &= \frac{1}{p(T')}\frac{f_1(T')\chi(M_1(T))}{e(M_1(T))}\frac{f_2(T')\chi(M_2(T))}{e(M_2(T))}\\ &= \frac{1}{p(T)}\frac{d(T')\chi(M_1(T))\chi(M_2(T))}{e(M_1(T))e(M_2(T))}.\end{aligned}$$

Summing over the T' which cover T, we see that

$$\sum_{T'\subset\pi^{-1}(T)} \frac{1}{p(T')}\frac{\chi(N_1(T'))\chi(N_2(T'))}{e(N_1(T'))e(N_2(T'))} = d\frac{1}{p(T)}\frac{\chi(M_1(T))\chi(M_2(T))}{e(M_1(T))e(M_2(T))}.$$

Summing over all T for which $e(M_1(T))e(M_2(T)) \neq 0$ now proves the proposition. □

Now for the manifold of Theorem B we have that $\theta(M) = \chi_1\chi_2/(pe_1e_2)$, $v_1 = \chi_1^2/e_1$, and $v_2 = \chi_2^2/e_2$ are all multiplicative invariants, so $p^2e_1e_2 = v_1v_2/\theta^2$ is a commensurability invariant.

Remark 2.2. For a general prime 3-manifold we can still define the above invariants as well as invariants v_1^h and v_2^h analogous to v_1 and v_2 but based on hyperbolic volume of the hyperbolic components of the two parts of the bipartite decomposition. Seifert components of the geometric decomposition that are adjacent to hyperbolic components do not have linear foliations on the corresponding boundary tori, so they have indeterminate euler number and indeterminate geometry. They therefore do not contribute to the invariants $v_1, v_2, V_1, V_2, \theta$.

Note that our multiplicative invariants for graph manifolds all vanish if all components of the geometric decomposition have $e = 0$. In **[WW]** a multiplicative invariant is defined for graph manifolds which does not have this property. In fact it is insensitive to the euler numbers of the components of the geometric decomposition.

3. Toral decomposition and plumbing

If we use the toral instead of the geometric decomposition of a prime graph manifold, we have linear foliations on the boundary tori of each piece as before, induced by adjacent Seifert fibrations, so we can again define euler numbers for the pieces. Recall that we always take the Seifert fibration with two degree 2 exceptional fibers on SMb pieces. The euler number of a piece adjacent to a SMb piece differs from its euler number in the geometric decomposition, but other euler numbers are the same.

Assume now M is a graph manifold that does not belong to the $\mathbb{S}ol$ geometry. We shall say M is ***good*** if it is prime and

- every component of the toral decomposition is Seifert fibered over orientable base.

We shall say it is ***very good*** if in addition:

- the fibers of the pieces can be oriented so that when we view a separating torus from one side, a fiber in the torus of the Seifert structure on the near side has positive intersection number in the torus with a fibers from the far side.

Note that this makes sense, since if we view a torus from the opposite side we reverse its orientation and also reverse the order of the two relevant fibers, thus not changing the relevant intersection number. Finally, we say M ***has no self-pastings*** if

- there is no SMb piece in the toral decomposition (equivalently, no Klein bottles in the geometric decomposition, so toral and geometric decomposition are the same);
- no Seifert piece meets itself along a separating torus.

It is easy to see that any prime non-$\mathbb{S}ol$ graph manifold M has a very good double cover with no self-pastings. Indeed, each contravention of one of the above items leads to a generator of $H^1(M;\mathbb{Z}/2)$ and we take the cover corresponding to the sum of these generators.

Suppose M is good. Denote the pieces of the toral decomposition $M_1,\dots,M_s$, with euler numbers $e_1,\dots,e_s$. Define an $s\times s$ matrix $S=(s_{ij})$ as follows: if $i\neq j$ then s_{ij} is the sum of reciprocals of the above fiber intersection numbers over all tori that separate M_i from M_j and if $i=j$ then $s_{ii}=e_i+2\times$(sum of reciprocals of fiber intersection numbers at tori that separate M_i from itself). This matrix is called the ***reduced plumbing matrix*** for M. For a very good graph manifold with no self pastings the reduced plumbing matrix agrees with the "decomposition matrix" of the introduction.

The following is a more detailed version of Theorem D of the introduction.

Theorem 3.1. *Let M be a closed oriented graph manifold which does not fiber over the circle with torus fiber.*

1. *M is a singularity link if and only if it is very good and its reduced plumbing matrix is negative definite. (For a T^2 bundle over S^1 the condition is that the monodromy be conjugate to $\left(\begin{smallmatrix}1&0\\b&1\end{smallmatrix}\right)$ with $b>0$ or have trace >2, cf.* **[N1]***.)*

2. *M fibers over the circle if and only if it is good and its reduced plumbing matrix is supersingular.*

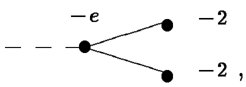
3. *If M is good then it has a horizontal surface if and only if its reduced plumbing matrix is singular.*

Proof. We may assume M is prime, since singularity links are prime by **[N1]** and oriented manifolds that fiber over the circle are prime since, except for $S^1 \times S^2$, they have universal cover $\mathbb{R}^3$.

We refer to **[N1]** for details on plumbing. The condition for M to be a singularity link is given in **[N1**, page 333**]**. It is that M can be represented by plumbing bundles over orientable surfaces according to a plumbing graph Δ with positive edge signs and negative definite intersection matrix $S(\Delta)$. Recall that a ***node*** of Δ is a vertex with non-zero genus weight or with valency ≥ 3 (valency is number of incident edges). The components of the complement of the nodes in Δ are the ***maximal chains*** of Δ.

We shall say the above negative definite plumbing graph Δ for M is ***minimal*** if it admits no "blowing down," that is there is no vertex on a chain with euler weight -1, cf. **[N1**, sect. 4**]**. This minimal plumbing graph is unique; we shall call it the ***normal form*** plumbing graph. This is not exactly what was called "normal form" in **[N1]** in that the other Seifert fibration of SMb pieces was used there. As described in **[N1]** and **[EN]**, the nodes of the normal form plumbing graph correspond to the Seifert pieces of the toral decomposition.

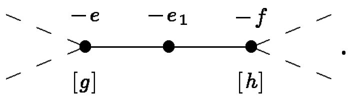
More generally, let M be any prime graph manifold that is not a torus bundle over S^1. **[N1]** gives us a unique "normal form" plumbing graph for M up to allowable changes of edge weights. As described above, we modify the normal form of **[N1]** by taking the other Seifert fibration on SMb pieces. What this means is that any valency 0 vertex with genus weight -1 should be replaced appropriately by a subgraph of the form



as described in **[N1**, Theorem 8.2**]**. The normal form plumbing diagram is then characterized by being minimal, having no valency 1 vertex with genus weight -1, and having only negative euler weights on chains. If M is good, then the plumbing diagram has no non-orientable (i.e., negative) genus weights, so the intersection matrix $S(\Delta)$ of the plumbing is defined, as in **[N1]**. Its (i,j)-entry is the sum of the edge weights ± 1 of the edges from vertex i to vertex j if $i \neq j$ and is the euler weight at vertex i plus twice the sum of edge weights of edges from i to i if $i = j$.

In **[EN**, sect. 21**]** a procedure is described to diagonalise a matrix of the form $S(\Delta)$ when Δ is a tree. We can extend this to partially diagonalise the intersection matrix of any plumbing graph as follows.

Suppose we have a chain of the plumbing graph Δ as follows (by convention, weights in brackets are genus weights, omitted genus weights are 0 and omitted edge weights are $+1$):



Then the corresponding portion of $S(\Delta)$ is

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & -e & 1 & 0 & \dots \\ \dots & 1 & -e_1 & 1 & \dots \\ \dots & 0 & 1 & -f & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

By adding suitable multiples of the middle row and column to the preceding and following row and column we get

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & -(e-\frac{1}{e_1}) & 0 & \frac{1}{e_1} & \dots \\ \dots & 0 & -e_1 & 0 & \dots \\ \dots & \frac{1}{e_1} & 0 & -(f-\frac{1}{e_1}) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

which is equivalent to

$$(-e_1) \oplus \begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & -(e-\frac{1}{e_1}) & \frac{1}{e_1} & \dots \\ \dots & \frac{1}{e_1} & -(f-\frac{1}{e_1}) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

More generally, if we start with the chain

,

then an inductive argument shows that $S(\Delta)$ is equivalent to

$$(-e_1) \oplus (-(e_2 - \frac{1}{e_1})) \oplus \dots \oplus (-[e_k, e_{k-1} \dots, e_1]) \oplus \begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & -(e-q/p) & \frac{1}{p} & \dots \\ \dots & \frac{1}{p} & -(f-q'/p) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \qquad (*)$$

where $p/q = [e_1, \dots, e_k]$ and $p/q' = [e_k, e_{k-1}, \dots, e_1]$. We are using the standard continued fraction notation

$$[e_1, e_2, \dots, e_k] = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\dots - \cfrac{1}{e_k}}}.$$

If there are c (-1)-edges on the chain (up to equivalence we can assume $c = 0$ or 1) then the analysis is the same except that the entries $1/p$ in the matrix $(*)$ are replaced by $(-1)^c/p$.

Suppose the e and f weighted vertices are nodes of the plumbing graph, so by **[N1]** they correspond to Seifert pieces for the toral decomposition of M. By Lemma 5.3 of **[N1]** the indicated chain corresponds to pasting the two Seifert pieces together along a common torus by the map with matrix

$$(-1)^c \begin{pmatrix} q & p \\ -p' & -q' \end{pmatrix}$$

with respect to (base) $\times$ (fiber) coordinates in each piece, where $p' = (qq' - 1)/p$. Hence $-q'/p$ is the appropriate contribution to euler number at the e-node, $-q/p$ is the appropriate contribution at the f-node, and $(-1)^c p$ is the intersection number of a fiber from the e-node with a fiber from the f-node. The analogous analysis holds if the f-node is absent, so the chain ended at the e_k-vertex (this case was done in **[EN]**).

We thus see that if we do the above procedure to every maximal chain of the plumbing graph then the resulting matrix $(*)$ is precisely what we called the "reduced plumbing matrix" for M. Thus the plumbing matrix $S(\Delta)$ is equivalent to the direct sum of a negative definite diagonal matrix and the reduced plumbing matrix. Thus the latter is negative definite if and only if the former is, completing the proof of the first part of the theorem.

Part 2 of the theorem was proved in reference **[14]** of **[N1]**, which was never published, but most of the ingredients have appeared elsewhere. We describe that proof here. As mentioned in the introduction, another proof can essentially be extracted from the proof of Theorem 3.1 of **[LW]**.

By **[EN**, Theorem 4.2**]** a map of a graph manifold to S^1 is homotopic to a fibration if and only if its restriction to each Seifert component of the toral decomposition is homotopic to a fibration. It is well known that a map of a Seifert fibered 3-manifold is homotopic to a fibration if and only if it is transverse to some Seifert fibration of the manifold, that is, it has non-zero degree on the fibers (cf. e.g., **[NRa]** — this goes back to the Conner Raymond theory of injective circle actions in the '60's). Since we are not considering torus bundles over S^1, the Seifert manifolds we need to consider all have unique Seifert fibered structures except in the case of SMb, in which case we have specifically chosen the Seifert fibered structure that fibrations to the circle are transverse to. Thus, for our graph manifold M, a map to the circle is homotopic to a fibration if and only if it has non-zero degree on each fiber of a Seifert component of the toral decomposition. In particular, this forces these Seifert components to have orientable base, so M is good. The theorem will thus be proven once we show the following lemma.

Lemma 3.2. *If M is a good graph manifold and the Seifert pieces of its toral decomposition are $M_1, \dots, M_s$, then the following are equivalent for an integer tuple $(l_1, \dots, l_s)$:*

i) $(l_1, \dots, l_s)$ *occurs as the tuple of degrees on the typical fibers of the M_i of some map $M \to S^1$;*

ii) $(l_1, \dots, l_s)$ *occurs as the tuple of intersection numbers of some oriented embedded surface $S \subset M$ with the typical fibers of the M_i;*

iii) $(l_1, \dots, l_s)$ *is annihilated by the reduced plumbing matrix for M.*

Proof. The equivalence of i) and ii) results by associating to a map $f : M \to S^1$ a smooth fiber of a smooth map homotopic to f and conversely, associating to an embedded oriented surface $S \subset M$ its dual cohomology class and then using the identification $H^1(M; \mathbb{Z}) = [M, S^1]$ to associate a map to S^1.

To see the equivalence with statement iii), consider the normal form plumbing for M and let X be the corresponding 4-manifold obtained by plumbing disk bundles, so $M = \partial X$. The long exact cohomology sequence for the pair (X, M) gives an exact sequence

$$H^1(M) \to H^2(X, M) \to H^2(X).$$

By Poincaré duality, $H^2(X, M) = H_2(X)$ which is the free abelian group on the fundamental classes of the base surfaces of the bundles being plumbed. Identifying $H^1(M)$ with $[M, S^1]$, the map $H^1(M) \to H^2(X, M)$ thus associates to a homotopy class $[f] \in [M, S^1]$ a tuple $(m_1, \dots, m_k)$ of numbers, one to each bundle being plumbed. We claim m_i is the degree of f on the fiber of the i-th bundle. Indeed, this follows from the commutative diagram

$$\begin{tikzcd}
H^1(M) \arrow[r] \arrow[d] & H^2(X, M) \arrow[d] \\
H^1(S) \arrow[r] & H^2(D, S)
\end{tikzcd}$$

where D is a fiber of the corresponding disk bundle piece of X and $S = \partial D$. Thus, if we number vertices of the plumbing diagram Δ so that vertices $1, \dots, s$ are the nodes, then $m_1, \dots, m_s$ become the numbers $l_1, \dots, l_s$ of the lemma.

On the other hand, rewriting $H^2(X, M) \to H^2(M)$ as $H_2(X) \to \mathrm{Hom}(H_2(X, \mathbb{Z})$ via Poincaré duality, it becomes the intersection form, so $(m_1, \dots, m_k)$ occurs as the image of some $[f] \in [M, S^1]$ if and only if it is in the kernel of the plumbing intersection matrix $S(\Delta)$. Since, with the given ordering of the vertices of Δ, our reduction of $S(\Delta)$ to the reduced plumbing matrix plus a diagonal matrix only added multiples of rows and columns beyond the first s of them to other rows and columns, the first s components of any vector annihilated by $S(\Delta)$ are not changed during the reduction. This proves the lemma. $\square$

4. Proof of Theorem B

Let M be an oriented non-Seifert-fibered manifold obtained by pasting two Seifert manifolds M_1 and M_2 with boundaries T^2 along their torus boundaries. If M_1 and M_2 both equal SMb then M is double covered by a torus bundle over S^1, while if just one of them, say M_1, is SMb, then M is double covered by a manifold obtained by pasting two copies of M_2. Thus the assumption of Theorem B that neither part is SMb is no real restriction. We now make that assumption. Then the decomposition $M = M_1 \cup M_2$ is its geometric decomposition.

We have already seen in section 2 that v_1/v_2 is a commensurability invariant and that $p^2 e_1 e_2$ also is if $e_1 e_2 \neq 0$. Note that the properties that one or both of e_1 and e_2 are zero are also commensurability invariant since they are equivalent for a cover to the properties that some or all the components of the geometric decomposition have $e = 0$.

If $e_1 \neq 0$ then the sign of e_1 is an orientation preserving commensurability invariant except perhaps when $|v_1/v_2| = 1$, since in this case we can exchange the indices 1 and

2 which will change the sign of e_1 if e_1 and e_2 have opposite signs. Thus part 3 of the theorem follows from part 2.

To see that the invariants determine M up to commensurability the following lemma will be useful.

Lemma 4.1. *Let $X \to F$ be a Seifert fibration with oriented total space, with ∂F consisting of r copies of S^1, and with a section to the fibration given on ∂F, so $e(X)$ and $\chi(X)$ are defined (recall $\chi(X)$ means the orbifold euler characteristic of F). Assume also that $\chi(X) < 0$. Then there exist positive integers d_0, n_0 such that $d_0 n_0 e$ is integral and the following is true. For any positive integers d, n, m with d_0 dividing d, n_0 dividing n, and m dividing dne, there exists a circle bundle $X' \to F'$ with connected orientable base F' and a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\ F' \arrow[r] & F \end{tikzcd}$$

satisfying:

i. $X' \to X$ *is a covering of degree* dnm;

ii. $\partial F'$ *consists of* rd *circles;*

iii. *restricted to each boundary component* $T^2 = Base \times Fiber$ *of* X' *the map* π *is* $p_n \times p_m$ *where* p_n *is the connected* n*-fold cover of the circle;*

iv. $\chi(X') = dn\chi(X)$.

v. $e(X') = dne(X)/m$.

Proof. Let $F_0 \to F$ be a finite normal covering of the orbifold F by a smooth oriented 2-manifold. We may choose it to have the same degree n_0 on each boundary component (e.g., by taking a characteristic cover). Then each boundary component of F is covered by d_0 boundary components of F_0, where $d_0 n_0$ is the degree of $F_0 \to F$. We may assume the cover is chosen so rd_0 is even. We may also assume n_0 is sufficiently large that $n_0\chi(X) < -r$. Since the pullback of X to F_0 is a circle bundle of euler number $d_0 n_0 e(X)$, this number is integral.

Let d and n be multiples of d_0 and n_0. We shall show first that a covering as in the Lemma exists for $m = 1$. The euler characteristic of F_0 is $d_0 n_0 \chi(X)$ which is less that rd_0, so if we fill in the boundary of F_0 by disks we get a surface $\bar{F}_0$ of negative euler characteristic. Thus $\bar{F}_0$ admits a connected (d/d_0)-fold cover, and we let F_1 be the inverse image of F_0 in this cover. Then F_1 has rd boundary components. Choose a map of $H^1(F_1)$ to $\mathbb{Z}/(n/n_0)$ which maps each boundary component to a generator (we made this easy by arranging that dr is even — map half the boundary components to $1 \in \mathbb{Z}/(n/n_0)$ and half to $-1 \in \mathbb{Z}/(n/n_0)$). Let $F' \to F_1$ be the induced cyclic cover. Let $X' \to F'$ be the pull-back of $X \to F$ via the composite map $F' \to F$. It is easy to see that this satisfies the Lemma for $m = 1$.

Now if $m \neq 1$ is a divisor of $e(X') = dne(X)$ then we replace the above X' by a fiberwise m-fold cyclic cover to get the desired X'. □

The above proof shows:

Scholium to 4.1. *If, in the above lemma, F is a smooth oriented surface with r boundary components (so $X \to F$ is a smooth fibration) and r is even and $\chi(X) < -r$, then we may choose $d_0 = n_0 = 1$.* □

We call the cover given by Lemma 4.1 a (d,n,m)-cover. Note that the choice of section on ∂X affects the behavior of this cover on ∂X as well as the value of $e(X)$. In applications of the lemma to geometric components of graph manifolds this $e(X)$ is usually not the euler number with respect to the boundary foliation that we are interested in.

Returning to our manifold $M = M_1 \cup M_2$, we choose sections to the Seifert fibrations on $\partial M_1 = \partial M_2$. Let n_0 and d_0 satisfy Lemma 4.1 for both M_1 and M_2, with d_0 even and $n_0\chi_i < -1$ for $i = 1,2$. If n is a multiple of n_0 and d a multiple of d_0 then M_1 and M_2 have $(d,n,1)$ covers. They therefore have (dn,n,n) covers, which we will denote $\widehat{M}_1$ and $\widehat{M}_2$. Since an (n,n)-cover of the torus is compatible with any gluing map, we can paste $\widehat{M}_1$ to $\widehat{M}_2$ to get a manifold $\widehat{M} = \widehat{M}_1 \cup \widehat{M}_2$ that covers M. The fiber intersection number in each separating torus is still p. That is, in each such torus a fiber of $\widehat{M}_1$ intersects a fiber of $\widehat{M}_2$ in an orbit of the $\mathbb{Z}/p$-action on fibers. If we factor by this action on each side we get $\widetilde{M} = \widehat{M}/(\mathbb{Z}/p) = \widehat{M}_1/(\mathbb{Z}/p) \cup \widehat{M}_2/(\mathbb{Z}/p)$ in which the fiber intersection number in each torus is 1. Lemma 1.1 tells us how the euler numbers relevant to our geometric decompositions behave (these are not the euler numbers with respect to the sections chosen above). We summarize the result by representing $\widetilde{M}$ by the plumbing diagram

$$\underset{[\tilde\chi_1]}{\overset{\tilde e_1}{\bullet}} \overset{dn}{=\!=\!=\!=\!=} \underset{[\tilde\chi_2]}{\overset{\tilde e_2}{\bullet}},$$

with $\tilde e_1 = dnpe_1$, $\tilde e_2 = dnpe_2$, $\tilde\chi_1 = dn^2\chi_1$, $\tilde\chi_2 = dn^2\chi_2$, and where the double line with label dn is shorthand for dn connecting edges.

We can now use the fibers of each half of $\widetilde{M}$ as sections in the boundary components of the other half. We apply the above Scholium to Lemma 4.1 to the two halves of $\widetilde{M}$, taking a (d',n_1,n_2)-cover of the left half and a (d',n_2,n_1) cover of the right half. Then the results still match up with fiber intersection numbers 1 in the boundary tori to give a cover $\overline{M}$ of $\widetilde{M}$. $\overline{M}$ is represented by a plumbing diagram

$$\underset{[\bar\chi_1]}{\overset{\bar e_1}{\bullet}} \overset{\delta n}{=\!=\!=\!=\!=} \underset{[\bar\chi_2]}{\overset{\bar e_2}{\bullet}},$$

with $\delta = dd'$, $\bar e_1 = \delta npn_1e_1/n_2$, $\bar e_2 = \delta npn_2e_2/n_1$, $\bar\chi_1 = \delta n^2 n_1\chi_1$, $\bar\chi_2 = \delta n^2 n_2\chi_2$. The only restriction on δ, n, n_1, and n_2 by our construction was that n be a multiple of n_0 and δ a multiple of d_0 for some fixed n_0 and d_0.

Now suppose $M' = M_1' \cup M_2'$ is another manifold as in Theorem B with the same invariants $p^2e_1e_2$ and v_1/v_2 as M. We construct $\overline{M}'$ commensurable with M' as above, but using numbers δ', n', n_1', and n_2', and we need to show that we can choose these numbers so $\overline{M}$ and $\overline{M}'$ are isomorphic.

Suppose first that $p^2e_1e_2 \neq 0$. Let p', e_1', e_2', χ_1', χ_2' be the invariants of M', so ${p'}^2e_1'e_2' = p^2e_1e_2$ and $({\chi_1'}^2/e_1')/({\chi_2'}^2/e_2') = ({\chi_1}^2/e_1)/({\chi_2}^2/e_2)$. We first choose n and δ as above that work for both M and M' and fix $n' = n$, $\delta' = \delta$. We choose n_1,

n_1', n_2, n_2' to satisfy $n_1/n_1' = \chi_1'/\chi_1$ and $n_2/n_2' = \chi_2'/\chi_2$. We claim that $\overline{M}$ and $\overline{M}'$ then agree up to orientation.

By reversing the orientation of M' if necessary we may assume e_1 and e_1' have the same sign. Then e_2 and e_2' do also. The equations $\bar\chi_1 = \bar\chi_1'$ and $\bar\chi_2 = \bar\chi_2'$ are immediate from our choices. To show $\bar e_1 = \bar e_1'$ we consider the product of $(\bar e_1/\bar e_1')^2$, $({p'}^2 e_1' e_2'/p^2 e_1 e_2)$, and $(v_1/v_2)/(v_1'/v_2')$. It suffices to show this product equals 1, since the second two multiplicands are 1 by assumption. Applying the definitions of all the ingredients in terms of the e_i, n_i, etc. and simplifying leads to $(n_1^2 {n_2'}^2 \chi_1^2 {\chi_2'}^2)/(n_2^2 {n_1'}^2 \chi_2^2 {\chi_1'}^2)$, which is 1 by the choice of the n_i and n_i'. Similarly $\bar e_2 = \bar e_2'$.

The above argument also works if $e_1 = e_2 = 0$. If $e_2 = 0$ and $e_1 \neq 0$ we again arrange that e_1 and e_1' have the same sign. Then choose a suitable $n = n'$ and then choose $\delta/\delta' = (\chi_2' e_1' \chi_1)/(\chi_2 e_1 \chi_1')$, $n_2/n_2' = (e_1\chi_1')/(e_1'\chi_1)$, and $n_1/n_1' = ({\chi_1'}^2 \chi_2 e_1)/(\chi_1^2 \chi_2' e_1')$. It is easily verified that this does what is required.

To complete the proof of Theorem B we must show that all possible values of $p^2 e_1 e_2$ and v_1/v_2 whose product is a square can be realized. The cases when $e_1 e_2 = 0$ are trivial, so assume $e_1 e_2 \neq 0$ and let the above square be r^2/s^2. Once one has chosen e_1, e_2, and p realizing $p^2 e_1 e_2$, to realize v_1/v_2 one must choose χ_1 and χ_2 so that $\chi_1/\chi_2 = |r/(spe_2)|$. But, it is easy to see that for given e, the set of χ for which a Seifert manifold with invariants e and χ exists includes almost all negative integers. So appropriate χ_1 and χ_2 can be found as integers. $\square$

5. Virtual Fibration of Graph Manifolds.

In this section we will prove Theorem E and deduce Theorem C from it. We first prove some basic results about virtualization of matrices. Proposition 5.2 was promised in the Introduction.

Lemma 5.1. **1.** *A virtualization of a virtualization of A is a virtualization of A.*

2. *If A is singular or supersingular then so is any virtualization of A.*

3. *If $\overline{A}$ and $\overline{A}'$ are virtualizations of A then there exists a matrix $\widehat{A}$ which is a virtualization of both $\overline{A}$ and $\overline{A}'$.*

Proof. Part 1 is easy and left to the reader. For part 2 note that if A annihilates $(v_1, \dots, v_n)^t$ and $T = (T_{ij})$ is a virtualizer, then the T-virtualization of A annihilates $(v_1, \dots, v_1, \ \dots \dots \ , v_n, \dots, v_n)^t$ with n_i repeats of v_i, where $n_i \times n_i$ is the size of T_{ii}. Finally, for part 3 observe that if $T = (T_{ij})$ and $T' = (T_{ij}')$ are virtualizers then $T'' = (T_{ij} \otimes T_{ij}')$ is a virtualizer which is a virtualization of both T and T'. $\square$

Proposition 5.2. *If A is a symmetric rational matrix then the following are equivalent:*
i). *A has a supersingular regular virtualization;*
ii). *A has a supersingular virtualization;*
iii). *A has a supersingular sub-virtualization.*
The corresponding statements with "supersingular" replaced by "singular" are also mutually equivalent.

Proof. We shall prove the supersingular case. The singular case is the same proof.

Clearly i) ⇒ ii) ⇒ iii). The fact that ii) ⇒ i) follows easily using parts 1 and 2 of Lemma 5.1. We leave this to the reader. More interesting is the implication iii) ⇒ ii). We shall give a purely algebraic proof of this implication and then explain the topology underlying it. First some preparatory comments.

Suppose B is a sub-virtualization of A which is supersingular. Then B is obtained from a virtualization B_0 of A by multiplying each off-diagonal entry by a suitable rational number between -1 and 1. We temporarily call this operation on B_0 "reduction." It suffices to show that B_0 has a virtualization which is supersingular, since a virtualization of a virtualization of A is a virtualization of A. For easier notation we replace B_0 by A and thus assume A has a reduction A_0 which is supersingular.

Suppose A is an $n \times n$ matrix with entries a_{ij}. Write the entries of A_0 as $(1-2r_{ij})a_{ij}$ with $0 \le r_{ij} \le 1$. In particular, $r_{ij} = 0$ if $i = j$. For each i, j let

$$T_{ij} = \frac{1}{2}\begin{pmatrix} 1 - r_{ij} & r_{ij} \\ r_{ij} & 1 - r_{ij} \end{pmatrix}.$$

Then $T = (T_{ij})_{1 \le i,j \le n}$ is a virtualizer. Since A_0 is supersingular it annihilates some $(u_1, \dots, u_n)^t$ with $u_i \ne 0$ for each i. The T-virtualization

$$\bar{A} = \begin{pmatrix} a_{11}T_{11} & \cdots & a_{1n}T_{1n} \\ \vdots & \dots & \vdots \\ a_{n1}T_{n1} & \cdots & a_{nn}T_{nn} \end{pmatrix}$$

of A annihilates $(u_1, -u_1, \dots, u_n, -u_n)^t$, so A is virtually supersingular. □

The topology underlying this proof is as follows. By multiplying by a suitable positive integer we may assume A is an integral matrix and each $r_{ij}a_{ij}$ is integral. We can realize A as the reduced plumbing matrix of some graph manifold, which we again denote M, with Seifert components $M_1, \dots, M_n$ and, for each $i \ne j$, with precisely $|a_{ij}|$ tori joining M_i to M_j with fiber intersection number $\operatorname{sign}(a_{ij})$ in each of these tori. If, for each $i \ne j$, we cut along $r_{ij}|a_{ij}|$ of these tori and repaste by the map $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ to change the sign of the fiber intersection number, then we obtain a graph manifold M_0 with reduced plumbing matrix A_0. We claim that M and M_0 have a common double cover $\overline{M}$. Indeed, consider the decomposition graph Γ for M. That is, Γ has a vertex for each M_i and an edge for each separating torus. If an edge corresponds to a torus where we have cut and re-pasted as above, call it a (-1)-edge. Map $H_1(\Gamma)$ to $\mathbb{Z}/2$ by taking any cycle in Γ to the number modulo 2 of (-1)-edges on it. This map induces a 2-fold cover $\overline{\Gamma}$ of Γ which induces the desired covers $\overline{M}$ and $\overline{M}_0$ of M and M_0. These covers are diffeomorphic as follows: choose some vertex of $\overline{\Gamma}$ and for each Seifert component of $\overline{M}$ use either the identity map or a map that reverses both base and fiber orientations, according as the corresponding vertex in $\overline{\Gamma}$ is separated from the chosen one by an even or odd number of (-1)-edges. The reduced plumbing matrix of this $\overline{M}$ is in fact a multiple of the above virtualization $\bar{A}$ of A. Topologically, the reason $\bar{A}$ is supersingular is because M_0 fibers by Theorem 3.1.2, so its cover $\overline{M}$ does.

Proof of Theorem E. For simplicity we will just discuss the case of virtual fibration and supersingularness. The arguments apply without change to prove the analogous statements for existence of a horizontal surface and virtual singularness.

We shall first consider the case that M is a very good graph manifold with no self-pastings, so its reduced plumbing matrix A is the same as its decomposition matrix. The following is the basic ingredient for our discussion.

Lemma 5.3. 1. *If B is any virtualization of A then there exists a cover $N \to M$ such that the reduced plumbing matrix of N is a positive multiple of B.*

2. *If $N \to M$ is a covering then there exists a covering $\overline{N} \to N$ such that the reduced plumbing matrix of $\overline{N}$ is a positive multiple of some uniform virtualization of A.*

Proof. Since M is very good the geometric and toral decompositions of M agree. Let M_i, $i = 1, \dots, n$, be the Seifert components of this decomposition and denote $e_i = e(M_i)$, $\chi_i = \chi(M_i)$, for each i. We first prove statement 1.

Choose n_0 and d_0 to satisfy Lemma 4.1 for each M_i and so that, in addition, d_0 is even and $n_0\chi_i < -1$ for each i. As in the proof of Theorem B, we can choose (d_0n_0, n_0, n_0)-covers $\widehat{M_i}$ of each M_i and paste these together to get a manifold $\widehat{M}$ that covers M with degree $d_0n_0^3$. Note that $e(\widehat{M_i}) = d_0n_0e_i$ and each separating torus for M has been replaced in $\widehat{M}$ by d_0n_0 separating tori, all with the same fiber intersection number as the original torus. Thus the reduced plumbing matrix $\widehat{A}$ for $\widehat{M}$ is simply $\widehat{A} = d_0n_0A$. Note that each $\widehat{M_i}$ is a Seifert manifold to which the Scholium to Lemma 4.1 applies, so they have $(d,1,1)$-covers for arbitrary d.

Let

$$T = \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \cdots & \cdots & \cdots \\ T_{n1} & \cdots & T_{nn} \end{pmatrix}$$

be a virtualizer and multiply it by a suitable positive integer to get an integral matrix

$$D = \begin{pmatrix} D_{11} & \cdots & D_{1n} \\ \cdots & \cdots & \cdots \\ D_{n1} & \cdots & D_{nn} \end{pmatrix}.$$

Denote the diagonal entries of the matrix D_{ii} by $d_{ii11}, \dots, d_{iin_in_i}$. For any i, j, the sum of the entries $d_{ijk1}, \dots, d_{ijkn_j}$ in the k-th row of D_{ij} is d_{iikk}. For $k = 1, \dots, n_i$ let N_{ik} be a $(d_{iikk}, 1, 1)$-cover of $\widehat{M_i}$. Then we can glue all the N_{ik} together as follows: for each torus separating $\widehat{M_i}$ from $\widehat{M_j}$, glue d_{ijkm} of the d_{iikk} tori covering it in N_{ik} to corresponding tori in N_{jm}. The resulting N is a cover of $\widehat{M}$ and hence of M and has reduced plumbing matrix

$$d_0n_0 \begin{pmatrix} a_{11}D_{11} & \cdots & a_{1n}D_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1}D_{n1} & \cdots & a_{nn}D_{nn} \end{pmatrix}.$$

Thus part 1 of the Lemma is proved.

To prove statement 2 we need the following:

Lemma 5.4. *If $X \to F$ is as in Lemma 4.1 and Y is any cover of X then there exists d_1 and n_1 such that for any multiples d and n of d_1 and n_1 a cover $\overline{Y}$ of Y exists which is a (dn,n,n)-cover of X.*

Proof. Suppose the degree of $Y \to X$ restricted to a Seifert fiber is m. Let $Z \to X$ be the cover corresponding to the largest characteristic subgroup of $\pi_1(X)$ contained in the image of $\pi_1(Y) \to \pi_1(X)$. Let r be the degree of this cover $Z \to X$. It still has degree m on Seifert fibers. The induced cover of base orbifolds has the same degree p say over every boundary component of F. By Lemma 4.1 we may find a (d_0n_0pm, n_0m, n_0p)-cover $\widehat{Y}$ of Z for some (d_0, n_0). This $\widehat{Y}$ is then a (d_1n_1, n_1, n_1)-cover of X with $d_1 = rd_0$ and $n_1 = n_0pm$. If $d = d_1d_2$ and $n = n_1n_2$ we now take a further (d_2n_2, n_2, n_2)-cover to obtain $\overline{Y}$. □

Returning to the proof of Lemma 5.3.2, suppose $N \to M$ is the given cover. Choose d and n such that for each $i = 1, \dots, n$ and each Seifert component N_{ij} of N that covers the Seifert component M_i of M, there is (dn,n,n)-cover $\overline{N}_{ij}$ of M_i covering N_{ij} as in Lemma 5.4. Let d_{ij} be the degree of $N_{ij} \to M_i$. Then the disjoint union $d_{ij}\overline{N}_{ij}$ of d_{ij} copies of N_{ij} is a dn^3-fold cover of N_{ij}. The $d_{ij}\overline{N}_{ij}$ can be pasted together to give a dn^3-fold cover $\overline{N}$ of N. The reduced plumbing matrix of $\overline{N}$ is then dn times a uniform virtualization of A. □

Theorem E now follows easily for M very good without self-pastings. Suppose some cover N of M fibers over S^1. Then by Lemma 5.3.2 we may lift to a cover $\overline{N}$ of N whose reduced plumbing matrix is a uniform virtualization of A and apply Theorem 3.1.2, showing that A has a supersingular uniform virtualization. Conversely, if A has a supersingular virtualization, then by Lemma 5.3.1 plus Theorem 3.1.2, M has a cover which fibers over S^1.

It remains to show that the general case of Theorem E follows from the very good case. Suppose M is a prime graph manifold with no self-pastings and let A be its decomposition matrix. One can take a double cover of M that makes all Seifert components have orientable base and then a further double cover, if necessary, to make fiber intersection numbers in tori positive. The resulting reduced plumbing matrix is a virtualization of A, so it is virtually singular or supersingular if and only if A was (this uses Lemma 5.1). Thus Theorem E holds using A. □

We now discuss simpler criteria for a matrix A to fail to be virtually singular. Our aim is Theorem F.

Proposition 5.5. *Let A be a negative definite rational matrix with non-negative off-diagonal entries. Then any sub-virtualization of A is still negative definite and is therefore non-singular.*

Proof. We first prove the proposition for a virtualization. We can find a very good graph manifold M with reduced plumbing matrix A. Then M is a singularity link by Theorem 3.1.1. By Lemma 5.3 any virtualization of A is a positive multiple of the reduced plumbing matrix of a cover of M. Since a cover of M is still a singularity link, the virtualization is negative definite by Theorem 3.1.1 again.

Now let A_0 be a reduction of A in the sense of the proof of Proposition 5.2 and and let $\bar{A}$ be the virtualization of A constructed in that proof. Let $X = (X_{ij})_{1\le i,j\le n}$ with $X_{ii} = \left(\begin{smallmatrix}1&1\\1&-1\end{smallmatrix}\right)$ and $X_{ij} = \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$ for $i \ne j$. Then

$$X^t\bar{A}X = \begin{pmatrix} a_{11} & 0 & a_{12} & 0 & \cdots & a_{1n} & 0\\ 0 & (1-2r_{11})a_{11} & 0 & (1-2r_{12})a_{12} & \cdots & 0 & (1-2r_{1n})a_{1n}\\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ a_{n1} & 0 & a_{n2} & 0 & \cdots & a_{nn} & 0\\ 0 & (1-2r_{n1})a_{n1} & 0 & (1-2r_{n2})a_{n2} & \cdots & 0 & (1-2r_{nn})a_{nn} \end{pmatrix}.$$

By what we have just shown, this is a negative definite matrix. After rearranging rows and columns it is $A \oplus A_0$, so A_0 is negative definite. □

Proof of Theorem F. We first show the "if." Thus, suppose A can be written

$$A = \begin{pmatrix} P & Z\\ Z^t & N \end{pmatrix}$$

as in Theorem E with N and P_- negative definite. Using Proposition 5.5 we see that any virtualization $\bar{A}$ of A still has the same property. Thus we may rename $\bar{A}$ as A and it suffices to show A is non-singular.

If it is singular then

$$\begin{pmatrix} P & Z\\ Z^t & N \end{pmatrix}\begin{pmatrix} u\\ v \end{pmatrix} = \begin{pmatrix} 0\\ 0 \end{pmatrix}$$

has a non-zero solution. We would then have $Pu + Zv = 0$ and $Z^tu + Nv = 0$ which imply $u^tPu = -u^tZv = -v^tZ^tu = v^tNv$. Since u^tPu is non-negative and v^tNv is non-positive, a solution of this must have $u^tPu = v^tNv = 0$ and hence $u = v = 0$.

For the converse we show first that if A has a singular principal minor then A is virtually singular. Indeed, suppose that after reindexing,

$$A = \begin{pmatrix} A_0 & X\\ X^t & A_1 \end{pmatrix}$$

with A_0 singular. Then

$$A = \begin{pmatrix} A_0 & 0\\ 0 & A_1 \end{pmatrix}$$

is a singular sub-virtualization of A. Thus if the Q of Theorem F is non-void, we may use the minor $A_0 = (a_{11})$ to see that A is virtually singular. Thus assume Q is void. To complete the proof we must show A is virtually singular if one of P_- and N is not definite. If we replace A by $-A$ and then reverse the signs of the off-diagonal entries (which is a sub-virtualization), we reverse the roles of P_- and N. Thus with no loss of generality it is N which is indefinite. We shall show some sub-virtualization of N has a singular principal minor.

By replacing N by an appropriate minor of N we may assume

$$N = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & M & \\ a_{1k} & & \end{pmatrix}$$

with M negative definite but N not negative definite. Then $(-1)^{k-1}\det(M) > 0$ and $(-1)^k \det(N) \geq 0$. Let

$$N_t = \begin{pmatrix} a_{11} & ta_{12} & \cdots & a_{1k} \\ ta_{12} & & & \\ \vdots & & M & \\ ta_{1k} & & & \end{pmatrix} \quad \text{for } 0 \leq t \leq 1.$$

Then $\det(N_t) = a_{11}\det(M) + t^2 q$ for some rational q, so any solution t of $\det(N_t) = 0$ has t^2 rational. There is such a solution with $0 < t \leq 1$ since $(-1)^k \det(N_1) \geq 0$ and $(-1)^k \det(N_0) < 0$. With this t the matrix

$$\begin{pmatrix} (1-t^2)a_{11} & 0 & 0 & \cdots & 0 \\ 0 & t^2 a_{11} & t^2 a_{12} & \cdots & t^2 a_{1k} \\ 0 & t^2 a_{21} & & & \\ 0 & \vdots & & M & \\ 0 & t^2 a_{k1} & & & \end{pmatrix}$$

is a singular sub-virtualization of N. □

The first part of the above proof also show the following necessary condition for virtual supersingularness. A symmetric matrix N is ***negative*** if $v^t N v \leq 0$ for all vectors v. We need the analog of Proposition 5.5 for negativeness, but this follows by a simple continuity argument.

Proposition 5.6. *If A can be written in the form*

$$A = \begin{pmatrix} P & Z \\ Z^t & N \end{pmatrix}$$

with P_- and N both negative then A is virtually supersingular if and only if P_- and N are each virtually supersingular. If, for example, $N = N_1 \oplus N_2$ with N_1 and N_2 both non-trivial and negative and one of them definite but the other not, then N is virtually singular but not virtually supersingular and so the same holds for the above A. □

Proof of Theorem C. Theorem C follows from Theorem E and the following

Proposition 5.7. *The rational matrix $A = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ is virtually singular if and only if $0 \leq ac \leq b^2$. It is virtually supersingular if and only if it is virtually singular and either neither or both of a and c is zero.*

Proof. If $ac > b^2$ then A is definite. By multiplying by -1 if necessary we can make A negative definite and then, if $b < 0$, we multiply the first row and column by

-1 so $b > 0$. Then Theorem F applies to show A is virtually singular if and only if $0 \le ac \le b^2$.

Proposition 5.6 shows that A is not virtually supersingular if one but not both of a and c is zero.

To complete the proof we must show that $0 < ac \le b^2$ or $a = c = 0$ implies A is virtually supersingular. If ac happens to be a rational square, say $ac = d^2$, then this is easy, since $\left(\begin{smallmatrix} a & d \\ d & c \end{smallmatrix}\right)$ is a sub-virtualization which is supersingular. We may thus assume ac is not a rational square, in particular $0 < ac < b^2$. By multiplying A by $1/b$ we may assume $b = 1$. Then A is the reduced plumbing matrix of a manifold M as in Theorem B. Since virtual supersingularity of A is equivalent to virtual fibration of M, which is a commensurability property, we may replace A by the reduced plumbing matrix of a commensurable manifold. Thus, without loss of generality we may assume

$$A = \begin{pmatrix} e & 1 \\ 1 & 1 \end{pmatrix}.$$

Write $e = p/q$ and $r = q + 1$. Define

$$\begin{aligned} \alpha &= \frac{q^3}{q^3+1} \\ \beta &= \frac{pr^2}{pr^2+1} \\ \gamma &= \frac{pqr + pq^3r^2}{(q^3+1)(pr^2+1)}. \end{aligned}$$

All we really need about r is that $r > q$ and $q^2 - pr > 0$. The latter is because $q^2 - pr \ge q^2 - (q-1)(q+1) \ge 1$. One then verifies easily that $\alpha - \gamma$, $\beta - \gamma$, and $1 + \gamma - \alpha - \beta$ are all positive. Thus

$$\begin{pmatrix} \alpha e & 0 & \gamma & \alpha - \gamma \\ 0 & (1-\alpha)e & \beta - \gamma & 1 + \gamma - \alpha - \beta \\ \gamma & \beta - \gamma & \beta & 0 \\ \alpha - \gamma & 1 + \gamma - \alpha - \beta & 0 & 1 - \beta \end{pmatrix}$$

is a virtualization of A. It is supersingular, since it annihilates $(r, -q^3r, -q, pqr^2)^t$, as direct computation shows. □

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