Automatic structures on central extensions

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Abstract. We show that a central extension of a group H by an abelian group A has an automatic structure with A a rational subgroup if and only if H has an automatic structure for which the extension is given by a "regular" cocycle. This had been proved for biautomatic structures by Neumann and Reeves. We make a start at classifying automatic structures on such groups, but we show that, at least for automatic structures, a classification using "controlling subgroups," as done by the authors in certain other cases, is impossible.

1. Introduction

Some years ago, we embarked on a program of computing $\mathrm{S}\mathfrak{A}(G)$, the set of automatic structures on a given group G up to a natural equivalence relation (definitions are given below). We had initial successes with abelian groups, geometrically finite hyperbolic groups, and graphs of groups where the edge groups are all finite. Our method in the latter two cases is to find a family $\mathcal{C} = \{C_i\}$ of controlling subgroups of G. That is, each automatic structure on G induces an automatic structure up to equivalence on each C_i and this map from $\mathrm{S}\mathfrak{A}(G) \to \prod_i \mathrm{S}\mathfrak{A}(C_i)$ is injective. It follows then that if one knows the induced automatic structures on $\{C_i\}$, one knows the automatic structure on G. It is in this sense that $\mathcal{C}$ is controlling.

We then turned our attention towards the group $G = F \times \mathbb{Z}$ where F is a finitely generated non-abelian free group. It seems that this is a difficult problem. It quickly became apparent that the most tractable automatic structures would be those for which the central $\mathbb{Z}$ is rational. As we shall see below, even in this special case, there is no useful controlling family of subgroups. More precisely, any family of controlling subgroups contains groups commensurate to G and thus provides no reduction in the complexity of the problem.

Since we first looked into this problem, there have been two developments with direct bearing on it. One is the work of Lee Mosher on central quotients of biautomatic groups [3]. In this paper, Mosher investigates biautomatic structures

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and shows that if A is a subgroup of the center of G and L is a biautomatic structure on G then L contains a rational section for the map $G \to G/A$. Further, this section has the fellow-traveller property and thus is a quasi-isometry onto its image and provides an automatic structure on G/A.

The second development is the work of Neumann and Reeves on biautomatic structures on central extensions, in particular central extensions of hyperbolic groups [8] and [9]. Given a central extension

$$0 \to A \to G \to H \to 1,$$

they show that G is biautomatic if and only if H has a biautomatic structure L and the extension can be given by a weakly bounded L-regular cocycle (definitions are in the next section). Moreover, they show that this is always so if H is a word-hyperbolic group.

Much of this paper is concerned with generalizing these results to the case of automatic structures which are not necessarily biautomatic but have a rational central subgroup. In particular, we show that the above G has an automatic structure with A rational if and only if H has an automatic structure L and the extension is given by a right bounded and L-regular cocycle.

It is likely that G automatic implies H automatic whether or not A is a rational subgroup, but this is unknown. Of course, a counterexample would be an automatic G which is not biautomatic, which has not yet been shown to exist.

Although our results suggest that classifying the automatic structures on G may be difficult, we do have a conjectured answer in terms of regular sections when H is hyperbolic and $A = \mathbb{Z}$, at least in the biautomatic case. We provide some evidence for this answer which is of independent interest, since it describes general results about abelian subgroups of automatic groups (see Theorem 6.1 and Corollary 6.2).

2. Preliminaries

In [5] we defined what it means for two asynchronous or synchronous automatic structures on a group to be equivalent. It will be useful for our present purposes to extend this definition to arbitrary rational structures. We first recall some basic terminology.

Recall that a *finite state automaton* $\mathcal{A}$ with alphabet X is a finite directed graph on a vertex set S (called the set of *states*) with each edge labelled by an element of X and such that different edges leaving a vertex always have different labels. Moreover, a *start state* $s_0 \in S$ and a subset of *accepted states* $T \subset S$ are given. As is usual, X^* denotes the free monoid of words on the alphabet X. A word $w \in X^*$ is in the *language* L *accepted by* $\mathcal{A}$ if and only if it labels a path starting from s_0 and ending in an accept state in this graph. We may assume there is no "dead state" in S (a state not accessible from s_0 or from which no accepted state is accessible). Eliminating such states does not change the language L accepted by $\mathcal{A}$.

A language is *regular* if it is accepted by some finite state automaton.

Let G be a finitely generated group and X a finite set and $a \mapsto \bar{a}$ a map of X to a monoid generating set $\overline{X} \subset G$. The natural projection $X^* \to G$ is denoted $w \mapsto \bar{w}$. A regular language $L \subset X^*$ which surjects onto G is called a *rational structure* on G.

The *Cayley graph* $\Gamma_X(G)$ is the directed graph with vertex set G and a directed edge from g to $g\bar{a}$ for each $g \in G$ and $a \in X$; we give this edge a label a. For convenience we will take $\overline{X} = \overline{X}^{-1}$. The distance function $d(g,h)$ is defined as the length of a shortest path from g to h in $\Gamma_X(G)$.

Each word $w \in X^*$ defines a path $[0,\infty) \to \Gamma$ in the Cayley graph $\Gamma = \Gamma_X(G)$ as follows (we denote this path also by w): $w(t)$ is the value of the t-th initial segment of w for $t = 0, \dots, \operatorname{len}(w)$, is on the edge from $w(s)$ to $w(s+1)$ for $s < t < s+1 \leq \operatorname{len}(w)$ and equals $\bar{w}$ for $t \geq \operatorname{len}(w)$. We refer to the translate by $g \in G$ of a path w by gw.

Let $\delta \in \mathbb{N}$. Two words $v, w \in X^*$ *synchronously δ-fellow travel* if the distance $d(w(t), v(t))$ never exceeds δ. They *asynchronously δ-fellow travel* if there exists a non-decreasing proper function $t \mapsto t' \colon [0,\infty) \to [0,\infty)$ such that $d(v(t), w(t')) \leq \delta$ for all t.

A rational structure L for G is a *synchronous* resp. *asynchronous automatic structure* if there is a constant δ such that any two words $u, v \in L$ with $d(\bar{u}, \bar{v}) \leq 1$ synchronously resp. asynchronously fellow travel. See [5] for a discussion of the relationship of this definition with that of [1]. A synchronous automatic structure L is *synchronous-asynchronous biautomatic* if there is a constant δ such that if $v, w \in L$ satisfy $\bar{w} = \overline{av}$ with $a \in X$ then av and w asynchronously δ-fellow travel.

Given two rational structures L and L' on G defined using possibly different generating sets X and X', we may consider them both to be defined using the union $X \cup X'$ as generating set. We say L and L' are *equivalent*, written $L \sim L'$, if there exists a δ such that every L-word is asynchronously δ-fellow travelled by an L' word with the same value and vice versa. If L and L' are asynchronous automatic structures this is equivalent to requiring that $L \cup L'$ be an asynchronous automatic structure.

If L is a rational structure on G we say a subset $S \subset G$ is *L-rational* if the language

$$L_S := \{w \in L : \bar{w} \in S\}$$

is a regular language. S is *L-quasiconvex* if there exists a δ such that $w \in L$ with $\bar{w} \in S$ travels in a δ-neighbourhood of $S \subset \Gamma_X(G)$. The following is well-known in the case of automatic structures (e.g., [2], [5]).

Proposition 2.1. *If $L \sim L'$ are equivalent rational structures on G then any subset H of G is L-rational if and only if it is L'-rational. Moreover, if H is a subgroup, then it is L-rational if and only if it is L-quasiconvex.*

Proof. The proof is the same as for automatic structures. We repeat it for completeness.

If $L \sim L'$ and $H \subset G$ is L-rational consider the set

$$\{(u,v) \in L_H \times L' : \overline{u} = \overline{v},\ u \text{ and } v \text{ asynchronously } \delta\text{-fellow travel}\},$$

where δ is the fellow-traveller constant for the equivalence $L \sim L'$. This is the language of an asynchronous two-tape automaton so the image of its projection onto the second factor is regular. This image is L'_H.

That L-rationality implies L-quasiconvexity is true for any subset $H \subset G$ with δ equal to the number of states in a finite state automaton for L_H. Indeed, any initial segment of an L_H word ends at a state that is at most this distance from an accepted state. Conversely, suppose H is an L-quasiconvex subgroup with quasiconvexity constant δ. Let B be the set of elements of H represented by words in $X \cup X^{-1}$ of length at most $2\delta+1$. Then B generates H. Indeed, if $g \in H$ is the value of $w = a_1a_2\dots a_n \in L$ then for each $t = 1, 2, \dots n$ we can find $s_t \in H$ with $d(w(t), s_t) \le \delta$ and $s_n = g$. The representation $g = (s_1)(s_1^{-1}s_2)\dots(s_{n-1}^{-1}s_n)$ represents g as a product of n elements of B. Now the set

$$\{(u,v) \in L \times B^* : \overline{u} = \overline{v},\ u \text{ and } v \text{ asynchronously } \delta \text{ fellow travel}\}$$

is the language of an asynchronous two-tape automaton and its projection onto the first factor is the language L_H, which is hence regular. □

Given a language L, a *ray* for L is an infinite word all of whose initial segments are initial segments of L-words. If L is an automatic structure on a group G we say two rays are *equivalent* if they fellow-travel with a fellow-traveller constant which may depend on the rays. We will often need the following lemma which is Lemma 6.3 of [5].

Lemma 2.2. *For any set of words of unbounded length there is a ray whose initial segments are initial segments of infinitely many words of the set.* □

We now recall the relation of central extensions with cocycles. Suppose

$$0 \to A \to G \to H \to 1$$

is a central extension. We write A additively and denote the inclusion of A in G by ι.

Choose a section $s\colon H \to G$. Then a general element of G has the form $s(h)\iota(a)$ with $h \in H$ and $a \in A$ and the group structure in G is given by a formula

$$s(h_1)\iota(a_1)s(h_2)\iota(a_2) = s(h_1h_2)\iota(a_1 + a_2 + \sigma(h_1,h_2)),$$

where $\sigma\colon H \times H \to A$ is a 2-cocycle on H with coefficients in A. Changing the choice of section changes the cocycle σ by a coboundary. Conversely, given a cocycle σ, that is, a function $H \times H \to A$ which satisfies the cocycle relation

$$\sigma(h, h_1h_2) = \sigma(h,h_1) + \sigma(hh_1,h_2) - \sigma(h_1,h_2),$$

the above multiplication rule defines a central extension of H by A.

The following terminology is slightly different from that of [8] and [9] in that the weak boundedness was there included as part of the definition of “regular.”

Definition 2.3. Suppose H has finite generating set X and $L \subset X^*$ is an asynchronous automatic structure on H. We say a 2-cocycle σ as above is *left* resp. *right bounded* if the set $\sigma(g, H)$ resp. $\sigma(H, g)$ is finite for each $g \in H$ (equivalently, $\sigma(X, H)$ resp. $\sigma(H, X)$ is finite — this equivalence follows from the cocycle relation). We shall simply say it is *weakly bounded* if it is both left and right bounded. The term “weakly bounded” reflects the standard terminology of *bounded* for a cocycle that satisfies $\sigma(H, H)$ finite.

We shall say that a cocycle σ is *L-regular* if for each $h \in H$ and $a \in A$ the subset $\{g \in H : \sigma(g, h) = a\}$ is an L-rational subset of H.

We say a cohomology class in $H^2(H; A)$ is left or right or weakly bounded or L-regular if it can be represented by a cocycle with the corresponding property.

Lemma 2.4. 1. *A right bounded cocycle σ is L-regular if and only if for any $x \in X$ and $a \in A$ the set $\{g \in H : \sigma(g, x) = a\}$ is an L-rational subset of H.*

2. *If L_1 and L_2 are equivalent asynchronous automatic structures then any L_1-regular cocycle is L_2-regular.*

Proof. This was proved in the weakly bounded case in Lemma 2.1 of [8] but the proof did not use left boundedness. □

3. Automatic structures on central extensions

Let

$$0 \to A \to G \xrightarrow{\pi} H \to 1$$

be a central extension.

Theorem 3.1. *Suppose L is an automatic structure on G such that A is L-rational. Then*

1. *L contains a rational sublanguage S which provides a quasi-isometric section to π.*

2. *The π projection of S is an automatic structure on H.*

3. *The extension is given by a right bounded regular cocycle with respect to this section.*

If H has an automatic structure and the above central extension is given by right bounded and L-regular cocycle then the argument of [8] shows that G is automatic. We thus have the following corollary, which answers a question of Neumann and Reeves [8] where the corresponding result was proved in the case of biautomatic structures.

Corollary 3.2. *A central extension G as above has an automatic structure with A rational if and only if H has an automatic structure for which the extension is given by a right bounded and regular cocycle.* □

Proof of Theorem 3.1. Results 1. and 2. were first proved in the biautomatic case by L. Mosher [3], without the assumption that A is rational. In this case Gersten and Short in [2] show that A is a subgroup of a rational central subgroup of G (see also [4] for a generalisation). In fact our results can also be generalized to the case that A has this property.

Our proof follows Mosher's for the most part, and we refer the reader to his paper for some of the details.

We shall assume that our automatic structure L is an automatic structure *with uniqueness*, that is, the evaluation $L \to G$ is bijective. By [1] this is no loss of generality, since every automatic structure contains an automatic structure with uniqueness.

The construction of the sublanguage S of L depends on the analysis of so-called central loops in a finite state automaton (FSA for short) for the language L. We review his terminology.

Definition 3.3. Let $\mathcal{A}$ be a finite state automaton. A *loop* is a closed path in $\mathcal{A}$. It is *live* if it occurs on the path of some accepted word. Since we assume our finite state automaton $\mathcal{A}$ has no dead states, all loops are live. A loop is *central* if it is labeled by a word that evaluates into A. It is *simple* if it is a simple closed path (i.e., its vertices except beginning and end are distinct). It is a *primitive central loop* if it is not a proper power of a central loop. A path in $\mathcal{A}$ is compatible with a set $\{\gamma_1, \dots, \gamma_I\}$ of loops if it intersects each of these loops. This set of loops is *live* if it is compatible with the path of some accepted word. Given a path γ in $\mathcal{A}$, $\overline{\gamma}$ denotes the element of G given by the label on γ

The following are proved by L. Mosher following ideas of [5]. The proofs only use automaticity and the fact that L is a language with uniqueness.

Lemma 3.4. 1. *A central loop is a power of a simple loop. Any other simple loop is disjoint from it.*

2. *If the set $\{\gamma_1, \dots, \gamma_I\}$ is a live set of disjoint central loops then $\{\overline{\gamma_1}, \dots, \overline{\gamma_I}\}$ is a linearly independent subset of A.* □

We shall also need the following lemma:

Lemma 3.5. *Suppose $w, w' \in L$ and $w = u\gamma^n v$ and $w' = u'xv'$ where γ is a primitive central loop. Suppose that w and w' k-fellow-travel with the x-portion of w' fellow traveling the γ^n-portion of w. Then there exists a constant K such that x has the form $x_1(\gamma')^m x_2$ with $\operatorname{len}(x_1)$ and $\operatorname{len}(x_2)$ at most K and γ and γ' evaluate to positive powers of a common central element. Moreover, K depends linearly on k.*

Proof. Assume first that w and w' end distance 1 apart.

Let C be the subgroup of A generated by $\overline{\gamma}$. Since x travels in a neighbourhood of the coset $\overline{u}C$, if it travels for a sufficiently long time, it will visit the same C-coset twice while being at the same state of $\mathcal{A}$. Let x_1 be the portion of x which first brings us to this state and coset and $x_1\gamma_0$ the portion that first returns us to this state and coset. This γ_0 is a central loop which evaluates to a power of $\overline{\gamma}$. Let γ' be the primitive central loop underlying γ_0. Let x_2 be the portion of x remaining after the last visit to this state and coset. The intervening portion of x is a central loop and must therefore be a power of γ' by Lemma 3.4.1. If x_1 does not have bounded length then the same argument shows that it includes a central loop evaluating to a power of $\overline{\gamma}$, but this is impossible by Lemma 3.4.2.

To see that K depends linearly on k we apply the result of the previous paragraph to words $w = w_0, w_1, \dots, w_k = w'$ with consecutive pairs ending at distance 1 from each other. $\square$

This lemma immediately implies

Lemma 3.6. *If $\{\gamma'_1, \dots, \gamma'_I\}$ is a live set of disjoint central loops then there is a live set $\{\gamma_1, \dots \gamma_I\}$ of disjoint central loops which is compatible with a word which evaluates into A and such that for each i the words labelling γ_i and γ'_i evaluate to powers of a common central element.* $\square$

To construct S of Theorem 3.1 we start with $L_A = \{w \in L : \overline{w} \in A\}$. By assumption, this is a regular language. By [5] there is a unique automatic structure L' on A up to equivalence equivalent to L_A. Any ray of L' fellow-travels a ray of L_A and vice versa. By [5] these rays determine points of S^{n-1}, where $n = \mathrm{rank}(A)$, and the equivalence class $[L']$ of automatic structures on A is determined by an ordered rational linear triangulation of S^{n-1} with these points as vertices. Given a simplex $\sigma = \langle \sigma_1, \dots, \sigma_I \rangle$ of this triangulation, a live set of primitive central loops $\{\gamma_1, \dots, \gamma_I\}$ is a *σ-set* if $\sigma_1, \dots, \sigma_I$ are the equivalence classes of the rays determined by $\overline{\gamma_1}, \dots, \overline{\gamma_I}$. Using slightly different terminology Mosher shows (again only using automaticity and uniqueness)

Lemma 3.7. *For each top dimensional simplex σ the number of σ-sets is finite and positive. A path of $\mathcal{A}$ is compatible with at most one σ-set.* $\square$

With these lemmas in hand, we return to the proof of Theorem 3.1. We fix a top-dimensional simplex σ and take

$$S' = \{w \in L : w \text{ is compatible with some } \sigma\text{-set, and}$$
$$w \text{ does not contain any central loop }\}.$$

We claim that

1) S' is regular.
2) $\pi(\overline{S'}) = H$.
3) There is $K > 0$ so that if $g, g' \in \overline{S'}$ with $d(\pi(g), \pi(g')) \le 1$, then $d(g, g') \le K$.

It is easy to see claim 1). The words of S' are those which are compatible with a σ-set but do not traverse any central loop. Since there are only finitely many σ-sets and only finitely many primitive central loops, it takes only finitely much data to determine if a word w is in S'. This can be checked by a finite state automaton.

To see claim 2) we take an arbitrary element $h \in H$ and check that there is a word in S' projecting to h. To do this, we consider $L_h = \{w \in L : \pi(\overline{w}) = h\}$. We pick $g \in G$ with $\pi(g) = h$. By supposition L_1 is regular, and each word of L_h ends at distance at most $\ell(g)$ from A. It follows that each word of L_h fellow travels A, and thus L_h is regular. (More generally, cosets Rg of a rational subgroup R are rational in any automatic group.) In particular, each word of L_1 is fellow traveled by a word of L_h.

Choose a σ-set $\{\gamma_1, \dots, \gamma_n\}$. By Lemma 3.6 we can assume it is chosen so that there is a word of L which evaluates into A and is compatible with this σ-set. Then L_1 contains a family of words of the form $\{w(N) = w_0\gamma_1^N w_1\gamma_2^N \dots w_{n-1}\gamma_n^N w_n : N \geq 0\}$. We now fix N sufficiently large and examine a word $w'(N) \in L_h$ which fellow travels $w(N)$. Lemma 3.5 shows that $w'(n)$ traverses central loops $\gamma_1', \dots, \gamma_n'$ so that for each i, $\overline{\gamma_i'}$ and $\overline{\gamma_i}$ are positive powers of a common central element. It follows that $\gamma_1', \dots, \gamma_n'$ is a σ-set. We now delete from $w'(n)$ each occurrence of a central loop. This produces an element of S' projecting to h.

It remains to prove claim 3). Let us first suppose that $g, g' \in \overline{S'}$ satisfy $\pi(g) = \pi(g')$. We let w and w' be their words. These are each compatible with a σ-set. Denote these σ-sets by $\{\gamma_1, \dots, \gamma_n\}$ and $\{\gamma_1', \dots, \gamma_n'\}$. Let A_σ be the subset of elements a of A that lie in the direction of σ. Since σ has maximal dimension, the g- and g'-translates of A_σ intersect. On the other hand, any element of gA_σ is within bounded distance of the value of a word which can be obtained from w by traversing the loops of its σ-set $\{\gamma_1, \dots, \gamma_n\}$. It follows that there are words w_1 and w_1' that end a bounded distance apart and are obtained from w and w' by traversing loops of their respective σ-sets. Denote this bound by c_0. Since there are only finitely many top dimensional simplices σ, we may take c_0 to be independent of the choice of g and g'.

Write $w_1 = v_0\gamma_1^{a_1}v_1 \dots v_{n-1}\gamma_n^{a_n}v_n$ and $w_1' = v_0'(\gamma_1')^{b_1}v_1' \dots v_{n-1}'(\gamma_n')^{b_n}v_n'$ with maximal exponents a_i and b_i. Now w_1 and w_1' fellow-travel with constant δc_0, where δ is the fellow-traveller constant for L. Thus Lemma 3.5 tells us that there is a constant c_1 so that for each i the portions $\gamma_i^{a_i}$ of w_1 and $(\gamma_i')^{b_i}$ of w_1' differ in length by at most c_1. Deleting such corresponding sections of w_1 and w_1' thus increases the fellow-traveller constant of w_1 and w_1' by at most c_1. Deleting these sections for each i thus increases the fellow traveller constant by at most nc_1. Since this gives back w and w', these words fellow travel with fellow traveller constant $nc_1 + \delta c_0$. In particular, g and g' are this close.

Now suppose that $\pi(g)$ and $\pi(g')$ are distance 1 apart. The argument of the previous paragraph goes through on observing that there are elements of the sets gA_σ and $g'A_\sigma$ which are distance 1 apart.

This completes the proofs of the three claims. We now return to the proof of Theorem 3.1.

If we take $\prec$ to be the lexicographic ordering on L, we can extract our section S by taking lex-least representatives:

$$S = \{w \in S' : \text{ if } w' \in S' \text{ and } \pi(w) = \pi(w') \text{ then } w \preceq w'\}.$$

It follows from claim 2) that S evaluates to a section. We thus have $s : H \to L$ with $\pi(\overline{s(h)}) = h$.

It follows from claims 1) and 3) that S is regular. Further, claim 3) implies that if $h, h' \in H$ with $d(h, h') \leq 1$ then $s(h)$ and $s(h')$ fellow travel. From this it follows that s is a quasi-isometry onto its image.

Finally, since S has this fellow traveler property in G, it also has this fellow traveler property under projection into H. Thus, if we evaluate into H, S is an automatic structure on H.

That part 3 of the theorem then follows was proved in [8] and [9]. This completes the proof of the theorem. □

Theorem 3.8. *Let L be an automatic structure on G for which A is rational. Let L' be the set of words in L which do not traverse a central loop. The π projection of L' consists of quasigeodesics in H.*

We first prove the following lemma.

Lemma 3.9. *Let $s = s_\sigma$ be the section constructed above. Then there is a constant c with the following property. For $h \in H$ suppose that w is a word of L with $\pi(\overline{w}) = h$ and suppose that w is not compatible with any central loop. Then $\overline{w} = \overline{s(h)}a$ where $\operatorname{len}_A(a) \leq c \operatorname{len}_H(h)$.*

Proof. We let $\{\tau\}$ be the set of top-dimensional simplices and their corresponding sections $\{s_\tau\}$. For each $h \in H$ this gives a collection of points $V_h = \{\overline{s_\tau(h)}\}$ in the fiber over h. Since each of these sections is quasi-isometric, $\operatorname{diam}(V_h)$ is linearly bounded in terms of $\operatorname{len}(h)$, say $\operatorname{diam} V_h \leq c_0 \operatorname{len}(h)$. It then follows that any element of $\pi^{-1}(h)$ is within this same distance $c_0 \operatorname{len}(h)$ of $\overline{s_\tau(h)}A_\tau$ for some τ. Suppose $w \in L$ with $\pi(\overline{w}) = h$ has $\overline{w}$ this close to $\overline{s_\tau}(h)A_\tau$. Then w fellow travels a word compatible with a τ-set with fellow-traveller constant linearly bounded in terms of $\operatorname{len}(h)$. It follows from Lemma 3.5 that w can only fellow travel $\overline{s_\tau(h)}A_\tau$ for a length of time which is linearly bounded in terms of $\operatorname{len}(h)$ before traversing a central loop.

The constant c is now found by considering the constants arising from each possible choice of τ. □

Proof of Theorem 3.8. . Suppose $w \in L'$. Then $\overline{w} = \overline{s(h)}a$ where $\operatorname{len}(a) \leq C \operatorname{len}(h)$. Then w $\delta C \operatorname{len}(h)$-fellow travels $s(h)$ where δ is the fellow-traveler constant of L. In particular $\operatorname{len}(w) \leq \operatorname{len}(s(h)) + \delta C \operatorname{len}(h)$. Thus $\operatorname{len}(w)$ is linearly bounded in terms of $\operatorname{len}(h)$. □

4. The hyperbolic case

Let

$$0 \to A \to G \xrightarrow{\pi} H \to 1$$

be a central extension with H hyperbolic.

Theorem 4.1. *Let L be an automatic structure on G. Then the following are equivalent:*

1.A is L-rational;

2. $H' < H$ is rational (with respect to the geodesic structure on H) $\iff$ $\pi^{-1}(H')$ is L-rational;

3. The projected language $\tilde{L}$ for H is an asynchronous automatic structure equivalent to the geodesic language.

Proof. This follows from Theorem 3.8 on noting that quasigeodesics in a hyperbolic group fellow-travel geodesics and that the projection of L into H asynchronously fellow travels the projection of L' into H. □

The last part of the preceding theorem can be strengthened if we make a gentleman's agreement about projection. Given a letter a evaluating into A, let us agree to treat a as the empty word on projection into H, rather than considering it as a letter that evaluates to the identity on projection into H. With this understanding, we have the following

Corollary 4.2. *Let $L \subset X^*$ be an automatic structure on G with A L-rational. Then L is equivalent to an automatic structure L' whose projection consists of $\pi(X)$ geodesics in H. In fact, we can arrange that the projection is the language of lex-least geodesics in H with respect to some ordering of X.*

Proof. We will enlarge our generating set by appending generators for A including letters a_γ evaluating to $\bar{\gamma}$ for each central loop γ. Call the resulting alphabet X'. We will include a letter for the trivial element in X' We replace each central loop of $\mathcal{A}$ with its corresponding generator. In view of Theorem 3.8 and our gentleman's agreement, our new language L_1 now projects to a language of quasi-geodesics. Let L_2 be the set of all words in X'^* which project to lex-least geodesics in H. L_2 surjects onto G and each word of L_1 is synchronously fellow travelled by a word of L_2. We take L_3 to be those words of L_2 that are fellow travelled by a word of L_1 giving the same group element. □

We have been discussing the situation when A is a rational subgroup, but it is worth giving a picture of what can happen when this fails. For this discussion we specialize to the case $A = \mathbb{Z}$.

Proposition 4.3. *Given an automatic structure L on G, if two non-commensurable $\mathbb{Z}\times\mathbb{Z}$ subgroups are rational then the central $\mathbb{Z}$ is rational and every $\mathbb{Z}\times\mathbb{Z}$ subgroup is rational.*

Proof. The $\mathbb{Z}\times\mathbb{Z}$ subgroups of G are exactly the groups of the form $\pi^{-1}(\mathbb{Z})$, with $\mathbb{Z}\le H$.

The intersection of two non-commensurable $\mathbb{Z}\times\mathbb{Z}$ subgroups will be a subgroup of finite index in the central $\mathbb{Z}$, so if the former are rational then so is the latter. In this case every $\mathbb{Z}\times\mathbb{Z}$ is rational by the theorem, since every $\mathbb{Z}$ subgroup of a hyperbolic group is rational. ☐

Corollary 4.4. *If L is a synchronous-asynchronous biautomatic structure on G, every rank two abelian subgroup is L-rational.*

Proof. Since $p^{-1}(1)$ is the center, it is L-rational (see [4], or [2] for the synchronous-synchronous case). ☐

Examples 4.5. Proposition 4.3 says that an asynchronous automatic structure on G with the central $\mathbb{Z}$ irrational must have zero or one rational $\mathbb{Z}\times\mathbb{Z}$ subgroup up to commensurability. Both possibilities can occur as we shall see in the case of $G=F_2\times\mathbb{Z}$, where F_2 is the free group on two generators. Let x,y be generators of F_2 and z a generator of $\mathbb{Z}$. Let $X_F=\{x^{\pm1},y^{\pm1}\}$, $X_Z=\{z^{\pm1}\}$, $X=X_F\cup X_Z$. Let $R_F\subset X_F^*$ denote the geodesic language, that is the language of reduced words.

Proposition 4.6. *The language*

$$\{(xz)^n w : n\in\mathbb{Z},\ w\in R_F\}$$

is synchronously automatic and a $\mathbb{Z}\times\mathbb{Z}$ subgroup is rational if and only if it is a subgroup of $\langle x,z\rangle$. The language

$$\{(xz)^n w : n\ge 0,\ w\in R_F\}\cup\{(yz^{-1})^n w : n\ge 0,\ w\in R_F\}$$

is synchronously automatic with no rational $\mathbb{Z}\times\mathbb{Z}$ subgroup.

Proof. Note that in both cases the language bijects to the group. It is easy to check the fellow traveller property explicitly. For example, in the case of the first language, we rewrite it as $L(x,z)L(x,y)$ where

$$L(x,z)=\{(xz)^n x^m : n,m\in\mathbb{Z}\}$$

and

$$L(x,y)=\{w\in R_F : w \text{ does not start with } x^{\pm1}\}.$$

Suppose we have $\overline{u'v'}=\overline{uva}$ with $u,u'\in L(x,z)$, $v,v'\in L(x,y)$ and a a generator. If $a=y^{\pm1}$ or v is not empty and $a=x^{\pm1}$, we have $u=u'$ and $\overline{v'}=\overline{va}$, and thus either $v'=va$ or $v=v'a^{-1}$. If $a=z^{\pm1}$, then u and u' fellow travel in $\langle x,z\rangle$ and v and v' are identical and fellow travel, one "above" the other. (We visualize $\langle z\rangle$ as the "vertical" direction.) Finally, if v is empty and $a=x^{\pm1}$, u and u' again fellow travel and v and v' are both empty.

Another way to look at this is the following. If we form the universal cover of the canonical 2-complex, X determined by the presentation

$$\langle x,y,z \mid [x,z]=[y,z]=1\rangle$$

and make each 2-cell of X isometric to the unit square, then X has a metric of nonpositive curvature. This complex can also be described as the cartesian product

$$X = \Gamma_{X_F}(F_2) \times \Gamma_{X_Z}(\mathbb{Z})$$

of the Cayley graphs of F and $\mathbb{Z}$. Each commensurability class of rank two abelian subgroups determines a totally geodesic embedded Euclidean plane. For each such plane π we have geodesic projection pr_π mapping X onto this plane, and this is a continuous map. If $u \in L(x,z)$ and $v \in L(x,y)$, then $\overline{u} = \mathrm{pr}_{\pi(\langle x,z\rangle)}(\overline{uv})$.

Note that $\langle x,z\rangle$ is a rational subgroup. It is the only rational $\mathbb{Z}\times\mathbb{Z}$ subgroup by Proposition 4.3, since $\langle z\rangle$ is not rational.

In the second example geodesic projection onto a plane is replaced by "horizontal geodesic projection" onto a pair of rational half planes, the positive (x,z) half plane and the negative (y,z) half plane. We leave the details of verifying the fellow traveller property to the reader. If some $\mathbb{Z}\times\mathbb{Z}$ subgroup were rational we could intersect it with one of these two rational half planes to see that an infinite submonoid of the center $\langle z\rangle$ is rational, which it is not. □

5. Controlling subgroups

A *controlling family of subgroups* for G is a family of subgroups $\{C_i\}$ of G with the properties that

1) for any automatic structure L on G each $\{C_i\}$ is L-rational and hence inherits an equivalence class of automatic structure $[L_i]$, and
2) the collection of such structures $\{[L_i]\}$ determines $[L]$.

We have shown in [6] and [7] that for a graph of groups with finite edge groups the conjugates of vertex groups form a controlling family of subgroups and for a geometrically finite hyperbolic group the maximal parabolic subgroups form a controlling family of subgroups.

An example will show that there is no useful family of controlling subgroups for $F_m \times \mathbb{Z}$.

Theorem 5.1. *There are automatic structures L and L' for $G = F_m \times \mathbb{Z}$ so that*

1) $H < G$ is L-rational if and only if H is L'-rational; moreover, this holds if and only if H is trivial, finite index in the center, abelian of rank two, or H is itself a nonabelian free group times $\mathbb{Z}$.

2) For each rational abelian subgroup C, L and L' induce equivalent automatic structures.

3) L and L' are not equivalent.

Notice that when H is a nonabelian free group times $\mathbb{Z}$, H is abstractly commensurate with G, though it need not lie as finite index subgroup of G.

Proof. We will take $m = 2$ for convenience. Since F_m is finite index in F_2, this will give us the general case. We take $X_F = \{x^{\pm 1}, y^{\pm 1}\}$ to be a monoid generating set bijecting to a basis for F_2. We let $\{z^{\pm 1}\}$ generate the center. For each $a \in X_F$, we take generators $a^\flat$ and $a^{\flat\flat}$ so that $\overline{a^\flat} = \overline{az^{-1}}$ and $\overline{a^{\flat\flat}} = \overline{az^{-2}}$. For $w \in X_F^*$, we let $w^\flat$ and $w^{\flat\flat}$ be the words gotten by replacing each letter a of w with the corresponding letter $a^\flat$ or $a^{\flat\flat}$. We are now prepared to describe L and L'.

Given a reduced word $w \in X_F^*$, we decompose w as $w = x^m y^n v$. Demanding that $|m|$ and $|n|$ be maximal forces this decomposition to be unique. For the purpose of this proof we shall call this decomposition of w an *xy-decomposition*. Such an xy-decomposition is necessarily reduced. In particular, if $n = 0$, v is empty. We let

$$\begin{aligned}
L_1 &= \{(x^m y^n)^\flat z^k v^{\flat\flat} : k \geq 0,\ x^m y^n v \text{ is an xy-decomposition.}\}\\
L_2 &= \{(x^m y^a)^\flat (y^b)^{\flat\flat} v^{\flat\flat} : x^m y^a y^b v \text{ is an xy-decomposition.}\}\\
L_3 &= \{(x^a)^{\flat\flat} (x^b)^\flat (y^n)^{\flat\flat} v^{\flat\flat} : x^a x^b y^n v \text{ is an xy-decomposition.}\}\\
L_4 &= \{(x^m y^n)^{\flat\flat} z^k v^{\flat\flat} : k \leq 0,\ x^m y^n v \text{ is a xy-decomposition.}\}\\
L_2' &= \{(x^a)^{\flat\flat} (x^b y^n)^\flat v^{\flat\flat} : x^a x^b y^n v \text{ is an xy-decomposition.}\}\\
L_3' &= \{(x^m)^{\flat\flat} (y^a)^\flat (y^b)^{\flat\flat} v^{\flat\flat} : x^m y^a y^b v \text{ is an xy-decomposition.}\}\\
L &= L_1 \cup L_2 \cup L_3 \cup L_4\\
L' &= L_1 \cup L_2' \cup L_3' \cup L_4
\end{aligned}$$

(see Fig. 1, where we have illustrated these languages with $\flat$ and $\flat\flat$ replaced by $\sharp$ and $\flat$ for clarity; z is vertical in the figure while the horizontal axis represents $w = x^m y^n v$). It is easy to see that each these languages is regular.

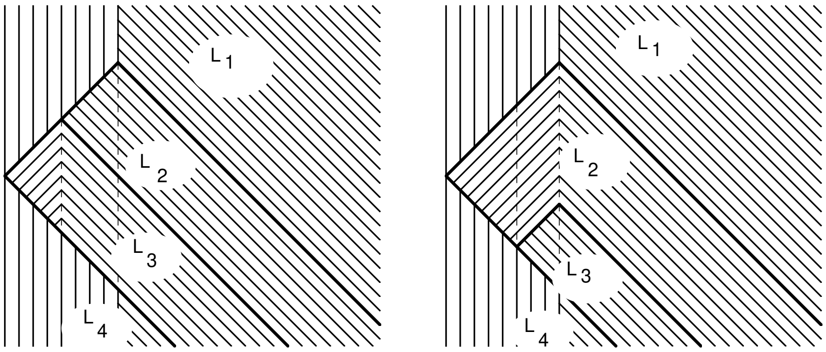

Figure 1: The languages L and L'

In the following we suppose $x^m y^n v$ is always an xy-decomposition. Then L_1, L_2, L_3, L_4, L_2' and L_3' biject to the following subsets S_1, S_2, S_3, S_4, S_2' and S_3' of

G.

$$
\begin{aligned}
S_1 &= \{\overline{z^r x^m y^n v} : -|m| - |n| - 2\operatorname{len}(v) \le r\} \\
S_2 &= \{\overline{z^r x^m y^n v} : -|m| - 2|n| - 2\operatorname{len}(v) \le r \le -|m| - |n| - 2\operatorname{len}(v)\} \\
S_3 &= \{\overline{z^r x^m y^n v} : -2|m| - 2|n| - 2\operatorname{len}(v) \le r \le -|m| - 2|n| - 2\operatorname{len}(v)\} \\
S_4 &= \{\overline{z^r x^m y^n v} : r \le -2|m| - 2|n| - 2\operatorname{len}(v)\} \\
S_2' &= \{\overline{z^r x^m y^n v} : -2|m| - |n| - 2\operatorname{len}(v) \le r \le -|m| - |n| - 2\operatorname{len}(v)\}. \\
S_3' &= \{\overline{z^r x^m y^n v} : -2|m| - 2|n| - 2\operatorname{len}(v) \le r \le -2|m| - |n| - 2\operatorname{len}(v)\}
\end{aligned}
$$

Notice that for $i = 1, 2, 3$ the languages L_i and L_{i+1} agree on the overlap of S_i and S_{i+1}, so the language L is a one-one rational structure for G. The same holds for L'. Moreover, each language L_i and L_i' has one parameter's worth of freedom for each choice of $w = x^m y^n v$. In L_1 and L_4, this parameter is k, while for the remaining languages it is the decomposition of m or n into $a + b$. To check that L and L' are automatic structures, one must check what happens after right-multiplying g by a generator. In each case, this can change the number of letters $\flat$-ed or $\flat\flat$-ed by at most one and can only change the final letter of w. It is easy to see that this implies that L and L' each have the fellow traveller property.

To see that L and L' are inequivalent, consider the words for $\overline{x^m y^m z^{-3m}}$ in each of these languages. In L this element is represented by $(x^m)^\flat (y^m)^{\flat\flat}$, while in L' is represented by $(x^m)^{\flat\flat}(y^m)^\flat$. Letting m grow without bound shows that these languages are inequivalent.

We now examine the L- and L'-rational subgroups of G. We claim that for both L and L' a non-trivial finitely generated subgroup of $H \subset G$ is rational if and only if $\pi|_H$ is not a monomorphism.

Indeed, suppose $\pi|_H$ is not monomorphic. Then H is commensurate with $\pi^{-1}(\pi(H))$. The center of G is clearly rational, so by Theorem 3.1, H is a rational subgroup of G if $\pi(H)$ is a rational subgroup of F_2. But any finitely generated subgroup of a free group is rational.

Conversely, suppose $\pi|_H$ is a monomorphism. We must show that if H is rational then H is trivial. Suppose not. Suppose $g \in F_2$ is in the image of $\pi|_H$. Let C_g be the cyclic subgroup generated by g. Then $H \cap \pi^{-1}(C_g)$ is an intersection of rational subgroups and hence rational. Also $H \cap \pi^{-1}(C_g)$ is cyclic. But a rational cyclic subgroup of $\pi^{-1}(C_g)$ must follow a pair of rays of $\{w \in L : \pi(\bar{w}) \in C_g\}$ (respectively, $\{w \in L' : \pi(\bar{w}) \in C_g\}$). But if r is such a ray which does not lie in the center, then the projection of r onto the center is strictly monotone down. Since one of the two rays for $H \cap \pi^{-1}(C_g)$ must be monotone up, this is a contradiction.

Now it is not hard to see that $\pi|_H$ is not monomorphic if and only if H is finite index in $\pi^{-1}(\pi(H))$, and this holds if and only if H is finite index in the center, free abelian group of rank two, or a nonabelian free group times $\mathbb{Z}$. We have thus proved the first part of the theorem.

It remains to see that L and L' induce equivalent structures on the rank two abelian subgroups. Any such subgroup is commensurate with one of the form $C_g \times \mathbb{Z}$ with $C_g \subset F_2$ maximal cyclic, so we need only consider such subgroups. Now g is represented by a reduced word $w = uw_0u^{-1}$ with u maximal in this decomposition. The words $uw_0^n u^{-1}$ are then also reduced words which represent the powers of g. Since C_g is a maximal cyclic, w_0 is not a proper power of a smaller word.

Suppose first that $w = x^{\pm 1}$, so u is empty and $w_0 = w$. Then both L and L' restrict on $C_g \times \mathbb{Z}$ to the language $\{(x^n)^\flat z^r : n, r \in \mathbb{Z}, r \geq 0\} \cup \{(x^a)^{\flat\flat}(x^b)^\flat : a, b \in \mathbb{Z}, ab \geq 0\} \cup \{(x^n)^{\flat\flat} z^r : n, r \in \mathbb{Z}, r \leq 0\}$.

Next suppose $w_0 = y^{\pm 1}$ and $u = x^\alpha$ is a power of x. Then the restrictions of L and L' to $C_g \times \mathbb{Z}$ are both equivalent to the language $\{x^\alpha (y^n)^\flat x^{-\alpha} z^r : n, r \in \mathbb{Z}, r \geq 0\} \cup \{x^\alpha (y^a)^\flat (y^b)^{\flat\flat} x^{-\alpha} z^r : a, b \in \mathbb{Z}, ab \geq 0\} \cup \{x^\alpha (y^n)^{\flat\flat} x^{-\alpha} z^r : n, r \in \mathbb{Z}, r \leq 0\}$.

If neither of the above cases pertains, then it is not hard to see that the xy-decomposition of uw_0^2 as $x^m y^n v$ has non-trivial v. The xy-decomposition of w^a for $a \geq 2$ is therefore $x^m y^n (v w_0^{a-2} u^{-1})$. Similarly, the xy-decomposition of uw_0^{-2} is $x^{m'} y^{n'} v'$ with v' non-trivial and the xy-decomposition of w^a for $a \leq -2$ is then $x^{m'} y^{n'} (v' w_0^{2-a} u^{-1})$. It then follows easily that the restrictions of L and L' to $C_g \times \mathbb{Z}$ are both equivalent to the language $\{z^r (w^s)^{\flat\flat} : r, s \in \mathbb{Z}\}$. □

6. Structure results

Theorem 6.1. *Let L be a finite-to-one asynchronous automatic structure on a group G and A an abelian subgroup of G with centralizer $Z(A)$. Suppose $w_1, \ldots, w_n$ are mutually inequivalent L-rays such that it happens infinitely often that $w_1, \ldots, w_n$ all pass through elements of $Z(A)$ which are equal modulo A. Then n is bounded by the number of states of any machine for L.*

Proof. By induction on $t = 1, 2, \ldots$ we can choose increasing functions $T_i(t)$ for $i = 1, \ldots, n$ such that for each t the n elements $w_i(T_i(t))$ are in $Z(A)$ and are equal modulo A. Let $\mathbb{A}$ be a finite state automaton for L and for each i and t let $s_i(t)$ be the state of $\mathbb{A}$ reached by w_i at time $T_i(t)$. There exists an n-tuple $(s_1, \ldots, s_n)$ which occurs infinitely often as $(s_1(t), \ldots, s_n(t))$. Let $t_0, t_1, \ldots$ be the sequence of values of t for which this happens. Suppose n exceeds the number of states of $\mathbb{A}$. Then $s_j = s_k$ for some $j \neq k$. Let $d = d(w_j(T_j(t_0)), w_k(T_k(t_0)))$ and let δ be the fellow-traveller constant for L. Since the rays w_j and w_k are inequivalent, we can find a t_p such that neither of the words $w_j\langle T_j(t_p)\rangle$ and $w_k\langle T_k(t_p)\rangle$ asynchronously $(\delta + d)$-fellow-travels an initial segment of the other. Write $w_i\langle T_i(t_p)\rangle = w_i\langle T_i(t_0)\rangle u_i$ for $i = j, k$. Then the words $w_j\langle T_j(t_0)\rangle u_j u_k$ and $w_j\langle T_j(t_0)\rangle u_k u_j$ are both in L, they have the same value, but they fail to δ-fellow-travel. This contradiction proves the theorem. □

Corollary 6.2. 1. *If L is a finite-to-one asynchronous automatic structure on a group G and $w_1, \ldots, w_n$ are inequivalent rays which visit an ϵ-neighbourhood of a finitely generated abelian subgroup A of G infinitely often then n is bounded by a number that only depends on L and ϵ.*

2. *Suppose $0 \to A \to G \to H \to 1$ is a central extension and L is a finite-to-one asynchronous automatic structure on G. Then there exists a bound $K = K(\epsilon)$ such that if $w_1, \ldots, w_n$ are inequivalent rays whose projections to H asynchronously ϵ-fellow-travel each other then $n \leq K$.*

Proof. 1. Choose a set of elements Y and a word u in these elements such that the initial segments of u run through all the elements in a ϵ-neighbourhood of $1 \in G$ and $\bar{u} = 1$. Let L' be the language obtained from L by replacing any word $a_1 a_2 \ldots a_m$ by the word $a_1 u a_2 u \ldots a_m u$. Then L' is an asynchronous automatic structure equivalent to L and the L'-rays obtained from $w_1, \ldots, w_n$ visit A infinitely often. We can thus apply the theorem to L'.

2. We again replace L by L' constructed as above and the theorem applies. □

We now restrict to the case of a central extension

$$0 \to \mathbb{Z} \to G \to H \to 1$$

with H hyperbolic. We assume L is an automatic structure on G with $\mathbb{Z}$ L-rational. By Corollary 4.2 we may assume that L projects to a language of lex least geodesics on H. An L-ray r' which does not fellow travel the center projects to an H ray r. We will say that r' lies over r. Corollary 6.2 implies that there are a bounded number of inequivalent L-rays in G lying over rays equivalent to r and the bound does not depend on the ray in question. The following lemma shows that up to fellow-travelling we need only consider those that actually lie over r.

Lemma 6.3. *Suppose r' is an L-ray lying over the geodesic ray r in H. There exists a fellow-traveller bound such that if r_1 is a geodesic ray in H which is equivalent to r then there is an L-ray r'_1 that lies over r_1 and fellow-travels r'.*

Proof. There exists a bound ϵ such that equivalent geodesic H-rays fellow travel with bound ϵ. Thus, for each initial segment of r we can find an element of G lying over r_1 and within distance ϵ of r'. Take the L-words for these elements of H. Applying Lemma 2.2 to the set of L-words we have just constructed gives the desired ray r'_1. □

Now the preimage of a ray in H is a half plane in G. We will see that the automatic structure in each of these half planes "looks like" part of a structure for $\mathbb{Z}^2$. The following will help us see this.

Lemma 6.4. *There is a N with the following property. Suppose that $g \in G$ lies over a ray r in H, and there are rays r^+ and r^- lying over r with r^+ above g and r^- below g. Then there is a ray r' over r which passes within N of g. In fact, r' passes through $g' = gz^m$ with $|m| \leq N$.*

Proof. Denote the generator of $\mathbb{Z}$ by z. Recall that each element of G is within S of some ray where S is the number of states of a FSA for L. For each $t > \ell(\pi(g)) + S$, we consider the element $r(t)$ and suppose that r^- and r^+ pass through $r^-(t)$ and $r^-(t)z^{M(t)}$ respectively. For each value of k between 0 and $M(t)$ consider the L-word for $r^-(t)z^k$. This word fellow-travels an initial segment of r and therefore passes within distance δ of some element of the form gz^p where $p = p(t,k)$. The fellow traveller property ensures that each time we change the value of k by 1, p changes by at most a bounded amount. Since gz^p is below g for $k = 0$ and above g for $k = M(t)$, we have some value $k(t)$ such that the word r_t for $r^-(t)z^{k(t)}$ passes a bounded distance from g. As in the previous proof, we now apply Lemma 2.2 to find a ray r', each of whose initial segments is an initial segment of some r_t. This r' almost does what is required; since it may lie over a ray that fellow travels r rather than r itself, we must apply Lemma 6.3 to make it lie over r. ☐

We next observe:

Proposition 6.5. *If r is a geodesic ray in H then there exists a ray r' lying over it. In fact, we may choose r' to fellow travel the section constructed in Theorem 3.1.*

Proof. For each t let $g(t)$ be the point of the section lying over $r(t)$. Since the section is rational any initial segment of an L-word for $g(t)$ ends a bounded distance from the section. Thus, if we apply Lemma 2.2 to the set of L-words for these elements $g(t)$, we get a ray that fellow-travels the section. Now this ray lies over a ray of H that fellow-travels r, since the initial segments of this H-ray are initial segments of geodesics that end on r. We can now apply Lemma 6.3 to get a ray that actually lies over r. ☐

Now let r be a ray of H and let $r_1, \ldots, r_m$ be representatives for the equivalence classes of rays that lie over r, ordered from lowest to highest. Let r_0 be a ray which fellow-travels $z^{-\infty}$ and r_{m+1} be a ray which fellow-travels z^{∞}. Let $\pi^{-1}(r)$ be the half-plane over r in the Cayley graph. We are interested in rays which fellow-travel this half-plane. The following proposition says that the behaviour is as in abelian groups, cf. [5].

Proposition 6.6. *Let $0 \le i \le m$ and let P be the portion of the half-plane $\pi^{-1}(r)$ that lies on or between the rays r_i and r_{i+1}. Then there exists a constant K such that one of the following situations holds:*

- *if r' is a ray that fellow-travels P and does not lie in a K-neighbourhood of r_i then $r' \sim r_{i+1}$;*
- *if r' is a ray that fellow-travels P and does not lie in a K-neighbourhood of r_{i+1} then $r' \sim r_i$.*

Proof. The ray r_1 or r_m may be taken to be the ray of Proposition 6.5 (depending on whether we have taken a positive or negative central loop). We consider the case $i = m$. Note that every initial segment of r_m is an initial segment of a word compatible with the central loop. If we meet the central loop along r_m then we can first travel around the central loop before we travel along the rest of r_m so the region above r_m is reached by travelling vertically and then outwards. Otherwise we can reach the central loop from any point along r_m so we reach the part of the plane above r_m by travelling along r_m and then travelling vertically. The proposition thus follows in both cases.

The case $i = 0$ is completely analogous to the above, so fix i with $1 \le i \le m-1$. For each t let $g^+(t)$ be the lowest point of the fiber over $r(t)$ that lies on a ray r^+ that fellow travels r_{i+1} and $g^-(t)$ the highest point that lies on a ray r^- that fellow travels $r_i(t)$. If $g^+(t)$ is below $g^-(t)$ then the two rays r^+ and r^- must eventually cross, so by the fellow traveller property, $g^+(t)$ can be at most a bounded distance below $g^-(t)$. On the other hand Lemma 6.4 implies that $g^+(t)$ can be at most a bounded distance above $g^-(t)$, since any point between them is a bounded distance from some ray over r. We now consider a ray r' which has the property that each of its initial segments is an initial segment of the L-word for $g^+(t)$ for some t. This exists again by Lemma 2.2. This ray r' must either fellow-travel r_i or r_{i+1}.

We claim that r' passes within a bounded distance of the points $g^+(t)$ for each t. To see this, we consider rays s_t and $s_{t'}$ which fellow travel r_{i+1} and pass through $g^+(t)$ and $g^+(t')$ respectively. We assume $t < t'$. Now $s_{t'}(t')$ lies below or at least as low as $s_t(t')$ since $s_{t'}(t') = g^+(t')$. On the other hand $s_{t'}(t)$ lies at least as high as $s_t(t)$. Consequently s_t and $s_{t'}$ cross within a bounded distance of each other for some time between t and t'. It follows that they can not have been far apart at time t. Since the initial segments of r' fellow-travel increasingly long initial segments of s_t for increasing t, the claim follows.

Since the ray r' fellow travels either r_i or r_{i+1} and is always within a bounded distance of the highest points on rays fellow traveling r_i and the lowest points on rays fellow traveling r_{i+1}, the proposition follows. ☐

7. Conjectures

We assume as above that G is given by a central extension

$$0 \to \mathbb{Z} \to G \to H \to 1$$

with H hyperbolic and not virtually cyclic. Fix a generating set $X = X^{-1}$ for G that includes the generators $z^{\pm 1}$ of $\mathbb{Z}$.

We consider languages S that project (in the sense of Corollary 4.2) to the geodesic biautomatic structure on H. As discussed in section 3, this language evaluates to a quasi-isometric section if and only if it determines a right bounded regular cocycle. We shall call two such languages *equivalent* if they fellow-travel. If S is equivalent in this sense to its translates we call it *invariant.* It is proved in [8] that S is invariant if and only if the cocyle is also left bounded.

If S_1 and S_2 are two such languages, we say $S_1 \le S_2$ if the second section lies everywhere above or on the first with respect to the natural ordering of each $\mathbb{Z}$ coset. We now show how a sequence $S_1 \le S_2 \le \cdots \le S_m$ of such languages induces a biautomatic structure on G. We denote by S_0 and S_{m+1} the languages $\{z^j : j \le 0\}$ and $\{z^j : j \ge 0\}$ respectively. We let $\epsilon = 0$ or 1 and consider the language

$$M = N \cap \bigcup_{n \equiv \epsilon \pmod 2} S_n S_{n\pm 1},$$

where N is the language of words that project to geodesics in H. It is not hard to see that this is a biautomatic structure on G whose equivalence class depends only on the equivalence classes of the S_i and on ϵ.

We conjecture

Conjecture 7.1. *Up to equivalence all biautomatic structures on G arise in the above manner. Moreover, if m is chosen as small as possible then the S_i and ϵ are determined by the biautomatic structure on G up to equivalence.*

Our example that shows that there is no useful family of controlling subgroups for $F \times \mathbb{Z}$ was automatic but not biautomatic.

Conjecture 7.2. *If G is a central extension of a word hyperbolic group then maximal abelian subgroups form a controlling family of subgroups for biautomatic structures.*

We close with a cautionary example. Let $G = F \times \mathbb{Z}$, where F is the free group on generators x and y. We take as generating set $\{x, X, y, Y, z, Z, x^\sharp, X^\sharp, x^\flat, X^\flat\}$, where z is a generator of the center and capital letters evaluate to the inverses of small letters, and $x^\sharp, X^\sharp, x^\flat, X^\flat$ evaluate to $xz, x^{-1}z, xz^{-1}, x^{-1}z^{-1}$. For any word w in x, X, y, Y we denote by $w^\sharp$ the word obtained by replacing each occurrence of x or X by $x^\sharp$ resp. $X^\sharp$; similarly for $w^\flat$. Let $L \subset \{x, X, y, Y\}^*$ be the geodesic language (language of reduced words) for F and denote by $L^\sharp = \{w^\sharp : w \in L\}$ and $L^\flat = \{w^\flat : w \in L\}$. We put

$$S_0 = \{Z\}^*, \quad S_1 = S_2 = L^\flat, \quad S_3 = L, \quad S_4 = S_5 = L^\sharp, \quad S_6 = \{z\}^*,$$

and put $\epsilon = 1$. The above construction gives a biautomatic structure. A ray r of F has three inequivalent rays above it if and only if it does not end in y^∞ or Y^∞, in which case it only has one ray above it up to equivalence. We thus have a dense set of points at infinity of F corresponding to rays with one ray above them up to equivalence and a dense set of points at infinity corresponding to rays with three.

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