# Regular Cocycles and Biautomatic Structures

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In **[ECHLPT]** and **[S]** it is shown that if the fundamental group of a Seifert fibred 3-manifold is not virtually nilpotent then it has an automatic structure. In the unpublished 1992 preprint **[G2]** Gersten constructs a biautomatic structure on the fundamental group of any circle bundle over a hyperbolic surface. He asks if the same can be done for the above Seifert fibered 3-manifold. We show the existence of such a biautomatic structure.

We do this in the context of a general discussion of biautomatic structures on virtually central extensions of finitely generated groups. A ***virtually central extension*** is an extension of a group $G$ by an abelian group $A$ for which the induced action of $G$ on $A$ is ***finite***, that is, given by a map $G \to \mathrm{Aut}(A)$ with finite image. The fundamental group of a Seifert fibered 3-manifold as above is a virtually central extension of a Fuchsian group $G$ by $\mathbb{Z}$. (For convenience we are using the term "Fuchsian group" for any discrete finitely generated subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ — orientable or not.)

We use a concept of "regular 2-cocycles" on a group $G$ which was suggested by Gersten's work. Here "regularity" is with respect to a (possibly asynchronously) automatic structure $L$ on $G$. If $L$ is a biautomatic structure on $G$ we show that any virtually central extension of $G$ defined by an $L$-regular cocycle also has a biautomatic structure.

As an application we show that any virtually central extension of a Fuchsian group $G$ by a finitely generated abelian group $A$ is biautomatic. In fact, if $L$ is a geodesic language on $G$, we show that all of $H^2(G;A)$ is represented by $L$-regular cocycles[1)]. In case $G$ is torsion free and $A=\mathbb{Z}$ with trivial $G$-action this is implicit in Gersten's work (loc. cit. — we give an independent treatment here that is more geometric; alternatively, it follows from his result about biautomaticity plus Theorem A below). The general case follows easily from this using Corollary 2.7 below, which says that a cohomology class for a group $G$ is regular if its restriction to some finite index subgroup of $G$ is regular.

The converse to the fact that regular cocycles lead to biautomatic structures is also true.

**Theorem A.** *Let $E$ be a virtually central extension of the group $G$ by a finitely generated abelian group $A$. Then $E$ carries a biautomatic structure if and only if $G$ has a biautomatic structure $L$ for which the cohomology class of the extension is represented by an $L$-regular cocycle.*

This strengthens the result of Lee Mosher **[M]** that biautomaticity of a central extension of $G$ implies biautomaticity of $G$. We use his work in the proof of Theorem A.

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1) March 1995: we can now prove this for any word-hyperbolic $G$, see **[NR]**.

## 1. Basic Definitions

Let $G$ be a finitely generated group and $X$ a finite set which maps to a monoid generating set of $G$. The map of $X$ to $G$ can be extended in the obvious way to give a monoid homomorphism of $X^*$ onto $G$ which will be denoted by $w \mapsto \overline{w}$. For convenience of exposition we will always assume our generating sets are ***symmetric***, that is, they satisfy $\overline{X} = \overline{X}^{-1}$. If $L \subset X^*$ then the pair consisting of $L$ and the evaluation map $L \to G$ will be called a ***language on*** $G$. Abusing terminology, we will often suppress the evaluation map and just call $L$ the language on $G$ (but therefore, we may use two letters, say $L$ and $L'$, to represent the same language $L \subset X^*$ with two different evaluation maps to two different groups). A language on $G$ is a ***normal form*** if it surjects to $G$.

A ***rational structure*** for $G$ is a normal form $L \subset X^*$ for $G$ which is a regular language (i.e., the set of accepted words for some finite state automaton).

The ***Cayley graph*** $\Gamma_X(G)$ is the directed graph with vertex set $G$ and a directed edge from $g$ to $g\overline{x}$ for each $g \in G$ and $x \in X$; we give this edge a label $x$.

Each word $w \in X^*$ defines a path $[0,\infty) \to \Gamma$ in the Cayley graph $\Gamma = \Gamma_X(G)$ as follows (we denote this path also by $w$): $w(t)$ is the value of the $t$-th initial segment of $w$ for $t = 0, \dots, \operatorname{len}(w)$, is on the edge from $w(s)$ to $w(s+1)$ for $s < t < s+1 \leq \operatorname{len}(w)$ and equals $\overline{w}$ for $t \geq \operatorname{len}(w)$. We refer to the translate by $g \in G$ of a path $w$ by $gw$.

Let $\delta \in \mathbb{N}$. Two words $v, w \in X^*$ ***synchronously*** $\delta$***-fellow-travel*** if the distance $d(w(t), v(t))$ never exceeds $\delta$. They ***asynchronously*** $\delta$***-fellow-travel*** if there exist non-decreasing proper functions $t \mapsto t', t \mapsto t'' \colon [0,\infty) \to [0,\infty)$ such that $d(v(t'), w(t'')) \leq \delta$ for all $t$.

A rational structure $L$ for $G$ is a ***synchronous*** resp. ***asynchronous automatic structure*** if there is a constant $\delta$ such that any two words $u, v \in L$ with $d(\overline{u}, \overline{v}) \leq 1$ synchronously resp. asynchronously fellow-travel. A synchronous automatic structure $L$ is ***synchronously biautomatic*** if there is a constant $\delta$ such that if $v, w \in L$ satisfy $\overline{w} = \overline{xv}$ with $x \in X$ then $\overline{x}v$ and $w$ synchronously $\delta$-fellow-travel. See **[NS1]** for a discussion of the relationship of these definitions with those of **[ECHLPT]**. In particular, as discussed there, if $L \to G$ is finite-to-one, then the definitions are equivalent; by going to a sublanguage of $L$ this can always be achieved (in fact one can always find a one-one regular sublanguage).

We define two rational structures $L$ and $L'$ on $G$ to be ***equivalent***, written $L \sim L'$, if there exists a $\delta$ such that every $L$-word is asynchronously $\delta$-fellow-travelled by an $L'$ word with the same value and vice versa. If $L$ and $L'$ are asynchronous automatic structures this is equivalent to requiring that $L \cup L'$ be an asynchronous automatic structure.

If $L$ is a rational structure on $G$ we say a subset $S \subset G$ is $L$***-rational*** if the language

$$L_S := \{w \in L : \overline{w} \in S\}$$

is a regular language. The subset $S$ is $L$***-quasiconvex*** if there exists a $\delta$ such that every $w \in L$ with $\overline{w} \in S$ travels in a $\delta$-neighborhood of $S \subset \Gamma_X(G)$. The following is well-known (e.g., **[GS]**, **[NS1]**).

**Proposition 1.1.** *If $L \sim L'$ are equivalent rational structures on $G$ then any subset $H$ of $G$ is $L$-rational if and only if it is $L'$-rational. Moreover, if $H$ is a subgroup, then it is $L$-rational if and only if it is $L$-quasiconvex.* □

We shall also need the following.

**Lemma 1.2.** *Let $L$ be an asynchronous automatic structure on $G$. If $S$ is an $L$-rational subset of $G$ then so is $Sg$ for any $g \in G$. Moreover, if $L$ is a biautomatic structure then $gS$ is also $L$-rational. In particular, if $H$ is a subgroup of finite index then its right-cosets $Hg$ are $L$-rational in the automatic case and two-sided cosets $g_1Hg_2$ are $L$-rational in the biautomatic case.*

*Proof.* Suppose $L$ is an automatic structure and $S$ is $L$-rational. It suffices to show that $S\overline{x}$ is rational for any generator $x$. We can use a standard comparator automaton (cf. **[ECHLPT]**) to see that $\{(u,v) \in L^2 : \overline{v} = \overline{ux}, \overline{u} \in S\}$ is the language of an asynchronous two-tape automaton. The projection onto the second factor is therefore a regular language, but it is just the language of words $v \in L$ that evaluate into $S\overline{x}$. Thus $S\overline{x}$ is $L$-rational. The proof that $gS$ is rational if $L$ is biautomatic is completely analogous. The final sentence of the lemma then follows since a subgroup of finite index, being quasiconvex, is rational by Proposition 1.1. □

## 2. $L$-Regular Cocycles and Biautomatic Structures.

Let $G$ be a group and $A$ be a finitely generated abelian group. Suppose

$$0 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$$

is a virtually central extension of $G$. We write $A$ additively and we denote the action of an element $g \in G$ on $A$ by $a \mapsto a^g$. Choose a section $s : G \to E$. Then a general element of $E$ has the form $s(g)\iota(a)$ with $g \in G$ and $a \in A$ and the group structure in $E$ is given by a formula

$$s(g_1)\iota(a_1)s(g_2)\iota(a_2) = s(g_1g_2)\iota(a_1^{g_2} + a_2 + \sigma(g_1, g_2)),$$

where $\sigma\colon G \times G \to A$ is a 2-cocycle on $G$ with coefficients in the $G$-module $A$. Changing the choice of section changes the cocycle $\sigma$ by a coboundary. Conversely, given a cocycle $\sigma$, the above multiplication rule defines a virtually central extension of $G$ by $A$.

**Definition.** Suppose $G$ has finite generating set $X$ and $L \subset X^*$ is an asynchronous automatic structure on $G$. We say a 2-cocycle $\sigma$ as above is ***weakly bounded*** if

1. The sets $\sigma(g, G)$ and $\sigma(G, g)$ are finite for each $g \in G$ (equivalently, $\sigma(X, G)$ and $\sigma(G, X)$ are both finite — this follows from the cocycle relation below);

and is $L$***-regular*** if in addition

2. For each $x \in X$ and $a \in A$ the subset $\{g \in G : \sigma(g, x) = a\}$ is an $L$-rational subset of $G$.

A cohomology class in $H^2(G; A)$ is $L$***-regular*** if it can be represented by an $L$-regular cocycle. The term "weakly bounded" reflects the standard terminology of ***bounded*** for a cocycle that satisfies $\sigma(G, G)$ finite.

**Lemma 2.1.**
1. *If $\sigma$ is an $L$-regular cocycle then for any $h \in G$ and $a \in A$ the set $\{g \in G : \sigma(g, h) = a\}$ is an $L$-rational subset of $G$*
2. *If $L_1$ and $L_2$ are equivalent asynchronous automatic structures then any $L_1$-regular cocycle is $L_2$-regular.*

*Proof.* It is enough to show that if the statement of Lemma 2.1.1 is true for $h_1$ and $h_2$ then it is true for $h = h_1 h_2$. Now the cocycle relation says

$$\sigma(g, h_1 h_2) = \sigma(g, h_1)^{h_2} + \sigma(gh_1, h_2) - \sigma(h_1, h_2).$$

Thus

$$\{g \in G : \sigma(g, h_1 h_2) = a\} = \bigcup_{a_1 + a_2 = a + \sigma(h_1, h_2)} \{g \in G : \sigma(g, h_1) = a_1^{h_2^{-1}}\} \cap \{g \in G : \sigma(gh_1, h_2) = a_2\}.$$

This is a finite union, since the sets on the right are empty for all but finitely many values of $a_1$ and $a_2$. It is a union of rational subsets since $\{g \in G : \sigma(gh_1, h_2) = a_2\} = \{g \in G : \sigma(g, h_2) = a_2\}h_1^{-1}$ is a right-translate of an $L$-rational subset and hence $L$-rational by Lemma 1.2. This proves 2.1.1.

Part 2 of the lemma follows from the fact that a subset of $G$ is $L_1$-rational if and only if it is $L_2$-rational (Proposition 1.1). Note that we also use part 1 of the lemma, since $L_1$ and $L_2$ may be languages on different generating sets. □

Suppose now that $L$ is a one-to-one biautomatic structure on $G$ (finite-to-one suffices for what follows, but one-one slightly simplifies arguments). Suppose $E$ is a virtually central extension as above given by a regular cocycle $\sigma$ determined by a section $s$. Consider the finite subset $\{s(x)\iota(-\sigma(g, x)) : g \in G, x \in X\}^{\pm 1} \subset E$ and let $Y$ be a set that bijects to this subset. If $v = x_1 x_2 \dots x_n \in X^*$ then there is a $Y$-word $v'$ whose initial segments have values $s(x_1), s(x_1 x_2), \dots, s(x_1 \dots x_n)$. Let $L'$ be the language

$$L' = \{v' : v \in L\} \subset Y^*.$$

**Proposition 2.2.** *The above language has the following properties:*
- (i) *$L'$ is regular;*
- (ii) *Evaluation maps $L'$ bijectively to the image of a section $s: G \to E$;*
- (iii) *There exists a constant $K$ such that, if $w_1, w_2 \in L'$ satisfy $\pi(\overline{w_2}) = \pi(\overline{y_1 w_1 y_2})$ with $y_1, y_2 \in Y$ then $y_1 w_1 y_2$ and $w_2$ $K$-fellow-travel in $E$.*

*Conversely, if $Y$ is a finite set which maps to a subset $\overline{Y} = \overline{Y}^{-1}$ of $E$ and $L' \subset Y^*$ is a language satisfying the above three properties then the projection $L$ of the language $L'$ to $G$ is a biautomatic structure on $G$ and the cocycle defined by the section $s$ is $L$-regular*

*Proof.* We first show the fellow-traveller property for $L'$. Let $Z$ be any $G$-invariant generating set for $A$. Denote by $d_E$ and $d_G$ the word metrics in $E$ and $G$ with respect to their generating sets $Y \cup Z$ and $X$. It is readily established that $d_E(s(g), s(g')) = d_G(g, g')$ for all $g, g' \in G$. Let $K$ be the fellow traveller constant for $L$. Given $w_1, w_2 \in L'$ with $\pi(\overline{w_2}) = \pi(\overline{y_1 w_1 y_2})$ and $y_1, y_2 \in Y$ then the fellow traveller property for $L$ tells us that

$d_G(\pi(\overline{y_1}w_1(t)), \pi(w_2(t))) \leq K$ for all $t$. Thus $d_E(s(\pi(\overline{y_1}w_1(t))), s(\pi(w_2(t)))) \leq K$. But $s(\pi(\overline{y_1}w_1(t)))$ differs from $s(\pi(\overline{y_1}))w_1(t)$ by an element of $\sigma(X,G)$ and $s(\pi(\overline{y_1}))$ differs from $\overline{y_1}$ by an element of $\sigma(G,X)$. Also $s(\pi(w_2(t))) = w_2(t)$. Thus $d_E(\overline{y_1}w_1(t), w_2(t)) \leq K + 2K'$, where $K'$ is a bound on word-length in the sets $\sigma(X,G)$ and $\sigma(G,X)$.

We now show that the language $L' = \{w' : w \in L\}$ is regular. Denote by $W$ the finite state automaton which has accepted language $L$. Recall that $W$ may be regarded as a finite directed graph with vertex set $S$, the elements of which are referred to as states. There is a distinguished vertex, $\nu_0$, called the start state and a distinguished subset of $S$, the elements of which are known as accept states. Each edge is labelled by an element of $X$, and each vertex has exactly one outgoing edge for each element of $X$. The transition function $\tau: S \times X \to S$ is given by setting $\tau(\nu, x) = \nu'$ when there is an edge from $\nu$ to $\nu'$ labelled by $x$. A word in $X^*$ is accepted by $W$ precisely when it labels a path beginning at the start state and ending at an accept state. For $x \in X$ and $a \in \sigma(G,X)$, let $W_{x,a}$ be the finite state automaton which accepts the language $\{w \in L : \sigma(w,x) = a\}$. Denote the vertex set of $W_{x,a}$ by $S_{x,a}$ and the transition function for $W_{x,a}$ by $\tau_{x,a}$. We form a finite state automaton for $L'$ by taking as vertex set the cartesian product $S \times (\prod S_{x,a})$ together with a single extra state $\varnothing$. The edge with initial vertex $(\nu, \ldots, \nu_{x',a'}, \ldots)$ labelled by $s(x)a^{-1}$ has terminal vertex $\varnothing$, if $\nu_{x,a}$ is not an accept state of $W_{x,a}$. Otherwise, the terminal vertex is $(\tau(\nu, x), \ldots, \tau_{x',a'}(\nu_{x',a'}, x), \ldots)$. All edges with initial vertex $\varnothing$ have terminal vertex $\varnothing$. The start state is given by the vertex $(\nu, \ldots, \nu_{x',a'}, \ldots)$ which has $\nu = \nu_0$ and each $\nu_{x',a'}$ the start state of $W_{x',a'}$. The accept states are those vertices which have an accept state as the first coordinate. The finite state automaton we have defined has accepted language $L'$.

For the converse statement suppose $L'$ is a language as in the proposition. Let $L$ be the projection of this language to a language for $G$. Thus $L$ is the same formal language as $L'$ but with a different evaluation map. Then $L$ is certainly regular. The bisynchronous fellow-traveller property for $L$ is immediate from the corresponding property (iii) of $L'$. Thus $L$ is a biautomatic structure.

Thus we only need to show that the cocycle $\sigma$ for the section $s$ determined by $L'$ is regular. The facts that $\sigma(Y,G)$ and $\sigma(G,Y)$ are finite are easy consequences of the fellow-traveller property (iii) and we leave them to the reader. For the rationality statement note that the fellow-traveller property implies that the language $\{(u,v) \in L' \times L' : \overline{u}\overline{x}\iota(-a+b) = \overline{v}\}$ is the language of a (synchronous) two-tape automaton for any $x \in Y, a, b \in A$. Thus its projection onto its first factor is regular. Denote the image of $\overline{x}$ in $G$ by $\hat{x}$. Then this projection is $\{u \in L' : \exists v \in L', \overline{ux} = \overline{v}\iota(a-b)\}$, the image of which in $G$ is $\{g \in G : s(g)\overline{x} = s(g\hat{x})\iota(a-b)\}$. If we choose $b$ so $\overline{x}^{-1}s(\hat{x}) = \iota(b)$ then this is $\{g \in G : s(g)s(\hat{x}) = s(g\hat{x})\iota(a)\} = \{g \in G : \sigma(g,x) = a\}$, so this set is rational. $\square$

**Corollary 2.3.** *If, in the situation of the above Proposition, $Z$ is a finite $G$-invariant generating set for $A$ and we choose a $G$-invariant biautomatic structure $L_A \subset Z^*$ on $A$ then $M = L'L_A$ is a biautomatic structure on $E$. (Structures $L_A$ as above always exists — cf.* **[ECHLPT]** *or* **[NS2]***.)*

*Proof.* $M$ is certainly a regular language. Suppose $w_1, w_2 \in L'$ and $v_1, v_2 \in L_A$ satisfy $\overline{xw_1v_1y} = \overline{w_2v_2}$ with $x, y \in (Y \cup Z)$. Then $\overline{x}w_1$ $K$-fellow-travels $w_2$. Hence there exists $a \in A$ with $d_E(1,a) \leq K+1$ with $a\overline{v_1^y} = \overline{v_2}$. Then $d_A(1,a)$ is bounded by some constant $c$ say. It follows that $v_1^y$ and $v_2$ are $cK_A$-fellow-travellers, where $K_A$ is the fellow-traveller constant for $L_A$. Now, since $L'$ is injective it has a "departure function" (cf. **[ECHLPT]**), so there exists a constant $\delta$ so that any subword $u$ of length at least $\delta$ of an $L'$-word has $d(1,\overline{u}) > 2K$. Since $\overline{x}w_1$ $K$-fellow-travels $w_2$, the lengths of $w_1$ and $w_2$ can differ by at most $\delta$. It follows easily that $\overline{x}w_1v_1$ fellow-travels $w_2v_2$ with constant $cK_A + c + 1 + K + \delta$. □

**Corollary 2.4.** *Let $A \to A'$ be an equivariant map of finitely generated abelian groups with finite $G$-actions. Suppose this map has finite kernel and cokernel. Let $L$ be a biautomatic structure on $G$. Then a class in $H^2(G;A)$ is $L$-regular if and only if its image in $H^2(G;A')$ is $L$-regular.*

*Proof.* The "only if" holds even if $A \to A'$ does not have finite kernel and cokernel and is easy, so we shall just prove the "if". A homomorphism with finite kernel and cokernel is a composition of a surjection with finite kernel and an injection with finite cokernel, so it suffices to prove these two special cases.

Let $E$ and $E'$ be the virtually central extensions determined by the cohomology classes in $H^2(G;A)$ and $H^2(G;A')$ in question. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \longrightarrow & A' & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1 \end{array}.$$

Let $\sigma'$ be an $L$-regular cocycle representing the class in $H^2(G;A')$. Recall that $\sigma'$ is determined by some section $s'\colon G \to E'$ and we have a regular language $L' \subset Y^*$ as in Proposition 2.2 bijecting onto $s'(G)$, where $Y$ is some finite set with an evaluation map to a symmetric subset of $E'$.

We first consider the case that $A \to A'$ is surjective with finite kernel. Then the same holds for $E \to E'$. Pick any lift of $Y \to E'$ to a map $Y \to E$ and interpret $L'$ as a language on $E$. Then $L'$ clearly satisfies the condition of Proposition 2.2, proving the corollary in this case (in Proposition 2.2 we asked that the image $\overline{Y} \subset E$ satisfy $\overline{Y} = \overline{Y}^{-1}$, but we can enlarge $Y$ as necessary to achieve this).

Next suppose $A \to A'$ is injective with finite cokernel. Choose coset representatives $a_1, \dots, a_k \in A'$ for $A$ in $A'$. Then $\iota(a_1), \dots, \iota(a_k)$ are coset representatives for $E$ in $E'$. Let $c\colon E' \to \{a_1, \dots, a_k\}$ be the map which picks the coset representative. Then the section $s\colon G \to E$ given by $s(g) = s'(g)\iota(-c(s'(g))$ has cocycle $\sigma(g,h) = \sigma'(g,h) + c(s'(gh)) - c(s'(g))^{s'(h)} - c(s'(h))$. This is clearly weakly bounded and is easily seen to be regular. □

Applying this corollary to the map $A \to A$ given by multiplying by a non-zero integer shows:

**Corollary 2.5.** *A cohomology class in $H^2(G;A)$ is "virtually $L$-regular" (that is, some non-zero multiple can be represented by an $L$-regular cocycle) if and only if it is regular.* □

Now suppose $G$ is biautomatic with biautomatic structure $L$ and $H < G$ is a subgroup of finite index. Then there is an induced biautomatic structure $L_H$ on $H$ which is unique up to equivalence. Let $S$ be a set of right coset representatives for $H$ in $G$ and let $r\colon G \to S$ be the map that takes an element to its coset representative. The ***transfer map*** $H^2(H;A) \to H^2(G;A)$ is defined on the level of cocycles by the formula

$$T\sigma(g_1, g_2) = \sum_{y\in S} \sigma(yg_1(r(yg_1))^{-1}, r(yg_1)g_2(r(yg_1g_2))^{-1})^y.$$

**Proposition 2.6.** *Suppose $H < G$ is a subgroup of finite index and $\sigma$ is an $L_H$-regular cocycle on $H$ with coefficients in $A$. Then $T(\sigma)$ is an $L$-regular cocycle on $G$.*

*Proof.* Since $T\sigma(g,x) = \sum_{y\in S} \sigma(yg(r(yg))^{-1}, r(yg)x(r(ygx))^{-1})^y$, the set $\{g \in G : T\sigma(g,x) = a\}$ is the union over all sums of the form $\sum_{y\in S} a_y = a$ of the sets $\bigcap_{y\in S}\{g \in G : \sigma(yg(r(yg))^{-1}, r(yg)x(r(ygx))^{-1}) = a_y^{y^{-1}}\}$. This is a finite union of finite intersections, so it suffices to show that the sets involved in the intersections are rational. Now $\{g \in G : \sigma(yg(r(yg))^{-1}, r(yg)x(r(ygx))^{-1}) = a_y^{y^{-1}}\} = \bigcup_{b\in S}(\{g \in G : \sigma(ygb^{-1}, bx(r(bx))^{-1} = a_y^{y^{-1}}\} \cap \{g \in G : r(yg) = b\})$. The set $\{g \in G : \sigma(ygb^{-1}, bx(r(bx))^{-1} = a_y^{y^{-1}}\}$ is a two-sided translate of the rational set $\{g \in G : \sigma(g, bx(r(bx))^{-1} = a_y^{y^{-1}}\}$ and is hence rational, while $\{g \in G : r(yg) = b\}$ is a translate of a subgroup of finite index and is hence rational. □

**Corollary 2.7.** *If $H < G$ is of finite index then the restriction of a cohomology class $x \in H^2(G;A)$ to $H^2(H;A)$ is $L_H$-regular if and only if $x$ is $L$-regular.*

*Proof.* The "if" is easy so we prove the "only if." Thus, assume the restriction of $x$ is regular. Since the composition of restriction and transfer $H^2(G;A) \to H^2(H;A) \to H^2(G;A)$ is multiplication by the index $[G : H]$ (see e.g., **[E**, Theorem 7**]**), it follows that the element $[G : H]x \in H^2(G;A)$ is regular, so $x$ is virtually regular. Thus the result follows from Corollary 2.5. □

*Proof of Theorem A.* Corollary 2.3 is one direction of Theorem A in the introduction. To prove the other direction we appeal to the work of Lee Mosher **[M]**. He proves that if a central extension $E$ of a group $G$ has a biautomatic structure then so does $G$. His main argument is the construction of a language $L'$ satisfying the conditions of Proposition 2.2 above, in the case of a central extension

$$0 \to \mathbb{Z} \to E \to G \to 1.$$

In particular, in this situation Proposition 2.2 then says the cohomology class for the extension is regular.

We first consider the case of a central extension

$$0 \to A \to E \to G \to 1,$$

such that $E$ has a biautomatic structure. Let $x \in H^2(G;A)$ be its cohomology class. Write $A$ as a direct sum of a finite group $F$ and copies of $\mathbb{Z}$ as follows: $A = F \oplus \coprod_{i=1}^n \mathbb{Z}$. Then $H^2(G;A) = H^2(G;F) \oplus \coprod_{i=1}^n H^2(G;\mathbb{Z})$. For each $j = 1, \dots, n$ we can form

$K_j = E/(F \oplus \coprod_{i \neq j} \mathbb{Z})$ and we have the induced extension

$$0 \to \mathbb{Z} \to K_j \to G \to 1. \qquad (*)$$

Lee Mosher's results say firstly that $K_j$ is biautomatic (since $E$ is a central extension of $K_j$) and therefore secondly, via the above remarks, that the cohomology class of $(*)$ is regular. That is, the image of $x$ in the $j$-th summand $H^2(G;\mathbb{Z})$ of $H^2(G;A)$ is regular for each $j = 1,\dots,n$. By Corollary 2.4 the same is true for the image of $x$ in $H^2(G;F)$. It follows that $x$ is regular.

Now if the extension is only a virtually central extension we take $H$ to be the kernel of the action of $G$ on $A$ and consider the restriction of our extension: $0 \to A \to E_0 \to H \to 1$. This is a central extension, so we can apply the case just proven to it and then apply Corollary 2.7 to complete the proof of Theorem A. □

**Remark 2.8.** If one replaces "biautomatic" by "automatic" or "asynchronously automatic" in the above discussion, then it is appropriate to replace the concept of "regular" cocycle by a concept "right regular" obtained by dropping the condition that $\sigma(X,G)$ be finite. The analogs of the results 2.1–2.5 then go through, though we do not know if the analog of Theorem A holds.

## 3. Biautomatic structures for virtually central extensions of Fuchsian groups

**Theorem 3.1.** *Let $G$ be a finitely generated Fuchsian group. Then any virtually central extension of $G$ by a finitely generated abelian group has a biautomatic structure.*

*Proof.* We shall use the geodesic language $L$ with respect to any finite generating set as a biautomatic structure on $G$. Let $A$ be any finitely generated abelian group with finite $G$-action. It suffices to show that every class in $H^2(G;A)$ is $L$-regular. By Corollary 2.7 we may replace $G$ by a subgroup of finite index as desired. Thus there is no loss of generality in assuming $G$ is torsion free and acts trivially on $A$, so we will do so. As in the previous section, we can then split $A$ as the sum of copies of $\mathbb{Z}$ and a finite group $F$. Any class in $H^2(G;F)$ is regular by Corollary 2.4, so it suffices to prove that any class in $H^2(G;\mathbb{Z})$ is regular.

If $\mathbb{H}^2/G$ is non-compact then $G$ is free, so $H^2(G;\mathbb{Z}) = 0$. Thus assume that $\mathbb{H}^2/G$ is compact. Then Gersten in **[G2]** in effect constructed a regular cocycle $\sigma$ representing the generator of $H^2(G;\mathbb{Z})$ (we give a different construction below). Thus every element of $H^2(G;\mathbb{Z})$ is regular. □

Gersten has informed us that his construction of the biautomatic structure in the torsion free case will remain unpublished. We therefore give a treatment of his result here for completeness.

Our construction is rather different from his and yields a regular cocycle for a multiple of the generator of $H^2(G;\mathbb{Z})$ when $G$ is a closed surface group of genus $g > 1$, rather than for the generator.

Fix a presentation

$$G = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle,$$

and let $P$ be the hyperbolic $4g$-gon with angles $\pi/2g$ and sides labelled by the $a_i$ and $b_i$ in such a way that the a word corresponding to a circuit of $P$ is the relator for the above presentation. Identifying corresponding sides of $P$ gives a hyperbolic structure on the closed surface of genus $g$, and there is a tessellation of $\mathbb{H}^2$ by copies of $P$ given by the universal cover of the surface. The 1-skeleton $\Gamma$ of this tessellation is the Cayley graph of $H$ with respect to the generating set $X = \{a_1, b_1, \ldots, a_g, b_g\}^{\pm 1}$. Suppose $w = x_1 \ldots x_n \in X^*$ is a word. We consider a point moving along the path of this word. The tangent vector at the point is well defined except at vertices of the path. As we pass from the $x_i$ edge to the $x_{i+1}$ edge of the path the tangent vector swings through an angle of $\theta_i$ with $-\pi < \theta_i \leq \pi$ (here $\theta_i = \pi$ only occurs if $w$ is non-reduced, namely $x_{i+1} = x_i^{-1}$). Define an integer $n(w)$ by

$$n(w) = \sum_{i=1}^{n-1} \frac{2g}{\pi} \theta_i .$$

Notice that $n(w^{-1}) = -n(w)$ if (and only if) $w$ is a reduced word. If $w$ is a closed path, and we set $\theta_n$ equal to the angle that the tangent vector swings through from the $x_n$ edge to the $x_1$-edge, then it is a standard result of hyperbolic geometry that

$$\sum_{i=1}^{n} \theta_i = A(w) + 2\pi\tau(w),$$

where $A(w)$ is the "signed area" enclosed by $w$ (with area that is multiply enclosed taken with appropriate multiplicity) and $\tau(w) \in \mathbb{Z}$ is the "turning number" of $w$, that is, the total rotation number of the tangent vector as it moves along the path (we measure this either by parallel translating all the tangent vectors back to some fixed base point in $\mathbb{H}^2$ or by following the motion of a point at infinity determined by the moving tangent vector). Thus

$$\begin{aligned} n(w) &= 2gA(w)/\pi + 4g\tau(w) - \frac{2g}{\pi}\theta_n \\ &= 8g(g-1)N(w) + 4g\tau(w) - \frac{2g}{\pi}\theta_n , \end{aligned} \tag{$**$}$$

where $N(w)$ is the signed number of copies of $P$ enclosed by $w$.

Let

$$E_k = \langle A_1, B_1, \ldots, A_g, B_g, Z \mid Z \text{ central}, \prod_{i=1}^{g} [a_i, b_i] = Z^k \rangle,$$

where $k = 8g(g-1)$. The central extension

$$0 \to \mathbb{Z} \xrightarrow{\iota} E_k \xrightarrow{\pi} G \to 1$$

where $\pi(A_i) = a_i$, $\pi(B_i) = b_i$ and $\pi(Z) = 1$, represents $k$ times a generator of $H^2(G;\mathbb{Z})$. We shall construct a section for which the corresponding cocycle is regular.

Let $L \subset X^*$ be a language which bijects to $G$ and comprises only geodesic words. For $w = x_1 \dots x_n \in L$ denote by $W = X_1 \dots X_n$ the word obtained by replacing each $a_i^{\pm}$ by $A_i^{\pm}$ and each $b_i^{\pm}$ by $B_i^{\pm}$. Define a section $s\colon G \to E_k$ by $s(\overline{w}) = \overline{W} Z^{-n(w)}$.

**Proposition 3.2.** *With the above definitions, the cocycle $\sigma$ defined by $\iota(\sigma(g_1, g_2)) = s(g_1)s(g_2)s(g_1g_2)^{-1}$ is a bounded regular cocycle.*

*Proof.* Note that the number $N(w)$ in $(**)$ can also be described as follows. Since $\overline{w} = 1$, we can write $w$ in the free group on $X$ as

$$w = \prod_{j=1}^{r} u_j r^{n_j} u_j^{-1},$$

where $r = \prod_{i=1}^{g}[a_i, b_i]$. Then $N(w) = \sum_{j=1}^{r} n_j$. Now if $w$ is a word with $\overline{w} = 1$ and $W$ is the corresponding word in the $A_i$ and $B_i$ then equation $(**)$ implies that

$$\overline{W} Z^{-n(w)} = Z^{-4g\tau(w) + \frac{2g}{\pi}\theta_n}.$$

We first show that the cocycle $\sigma$ is a bounded cocycle. Let $g_1, g_2, g_3 \in G$ with $g_1g_2 = g_3$ and let $w_1, w_2, w_3 \in L$ be the words representing them. Then $s(g_i) = \overline{W_i} Z^{-n(w_i)}$, so

$$\begin{aligned}\iota(\sigma(g_1, g_2)) &= s(g_1)s(g_2)s(g_3)^{-1}\\ &= \overline{W_1 W_2 W_3^{-1}} Z^{-n(w_1)-n(w_2)+n(w_3)}.\end{aligned}$$

Denote $w = w_1 w_2 w_3^{-1}$. It is not hard to see that the path determined by $w$ has $|\tau(w)| \le 2$. Denote by $\phi_1$ the angle between the tangent vectors to $w$ at the last edge of $w_1$ and the first edge of $w_2$. Similarly, $\phi_2$ denotes the angle from $w_2$ to $w_3$, and $\phi_3$ the angle from $w_3$ to $w_1$. We choose these with $-\pi < \phi_i \le \pi$. Since $w_3$ is reduced, we have $n(w_3^{-1}) = -n(w_3)$, so $n(w_1) + n(w_2) - n(w_3)$ differs from $n(w)$ just by $\frac{2g}{\pi}(\phi_1 + \phi_2)$. Thus

$$\iota(\sigma(g_1, g_2)) = Z^{-4g\tau(w) + \frac{2g}{\pi}(\phi_1+\phi_2+\phi_3)},$$

and it follows that $\sigma$ is a bounded cocycle.

To prove that the cocycle is regular we consider the above formula in case $g_2 = \overline{x}$, where $x$ is a generator. The language $\{(w_1, w_3) \in L \times L : \overline{w_3} = \overline{w_1 x}\}$ is regular. Suppose that $x_1 \dots x_m$ and $y_1 \dots y_n$ is a pair of words in this language. The values of $\phi_1$ and $\phi_2$ are determined by $x_m$ and $y_n$ respectively. Similarly, the value of $\phi_3$ is given by $x_1$ and $y_1$. It is not hard to see that the turning number is also determined by the same data in this case (namely $\tau(w) = -1$ if all three of the $\phi_i$ are negative, and otherwise $\tau(w) = 0$ if $\phi_3$ is negative or both $\phi_1$ and $\phi_2$ are negative, and $\tau(w) = 1$ in all other cases). It is clear that these data can be checked by finite state automata, and therefore one can construct a finite state automaton which will accept the language $\{w \in L : \sigma(\overline{w}, \overline{x}) = a\}$. □

## 4. Questions

S. Gersten, in **[G1]**, shows that if a central extension $E$ of a bicombable group $G$ by a finitely generated abelian group $A$ is given by a bounded cocycle then $E$ is bicombable. His argument is the same as the argument of our section 2 — the only difference being that regularity of languages is not important. It follows that his result is valid even if the cocycle is only weakly bounded. A natural question therefore is whether a weakly bounded cohomology class on a finitely generated group is always bounded.

This question is also relevant to quasi-isometry. Gersten shows that if the cocycle is bounded then $G \times A$ is quasi-isometric to $E$, but it is again not hard to see that weakly bounded suffices. In fact in this case only the condition that $\sigma(G, X)$ is bounded is needed — the map $g \times a \mapsto s(g)\iota(a)$ then gives a quasi-isometry. Moreover, if a quasi-isometry $G \times A$ to $E$ exists such that the composition $G \times \{1\} \to E \to G$ is a quasi-isometry then the central extension is determined by a cocyle with $\sigma(G, X)$ bounded. But we know no example of a cohomology class for a group which is represented by such a cocycle and is not bounded.

Thurston has claimed (unpublished) that central extensions of word-hyperbolic groups by finitely generated abelian groups are automatic. Are they in fact biautomatic? In fact, might every 2-dimensional cohomology class on a word-hyperbolic group be representable by a bounded regular 2-cocycle?

*Added March 1995*: We now have a proof of this; see **[NR]**. The proof also shows for any finitely generated group $G$ that a cocycle with either $\sigma(X, G)$ or $\sigma(G, X)$ bounded is cohomologous to one with both of these sets bounded.

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