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On normal embedding of complex algebraic surfaces

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Dedicated to our friends Maria (Cidinha) Ruas and Terry Gaffney in connection to their 60-th birthdays

Abstract

We construct examples of complex algebraic surfaces not admitting normal embeddings (in the sense of semialgebraic or subanalytic sets) with image a complex algebraic surface.

1. *Introduction*

Given a closed and connected subanalytic subset $X \subset \mathbb{R}^m$ the *inner metric* $d_X(x_1, x_2)$ on X is defined as the infimum of the lengths of rectifiable paths on X connecting x_1 to x_2. This metric defines the same topology on X as the Euclidean metric on $\mathbb{R}^m$ restricted to X (also called *"outer metric"*). This follows from the famous Lojasiewicz inequality and the subanalytic approximation of the inner metric [**6**]. But the inner metric is not necessarily bi-Lipschitz equivalent to the Euclidean metric on X. To see this it is enough to consider a simple real cusp $x^2 = y^3$. A subanalytic set is called *normally embedded* if these two metrics (inner and Euclidean) are bi-Lipschitz equivalent.

Theorem 1.1 [**4**]. *Let $X \subset \mathbb{R}^m$ be a connected and globally subanalytic set. Then there exist a normally embedded globally subanalytic set $\tilde{X} \subset \mathbb{R}^q$ and a global subanalytic homeomorphism $p\colon \tilde{X} \to X$ bi-Lipschitz with respect to the inner metric. The pair $(\tilde{X}, p)$ is called a normal embedding of X.*

The original version of this theorem (see [**4**]) was formulated in a semialgebraic language, but it easy to see that this result remains true for a global subanalytic structure or, moreover, for any o-minimal structure. The proof remains the same as in [**4**]

Complex algebraic sets and real algebraic sets are globally subanalytic sets. By the above theorem these sets admit globally subanalytic normal embeddings. Tadeusz Mostowski asked if there exists a complex algebraic normal embedding when X is a complex algebraic set, i.e., a normal embedding for which the image set $\tilde{X} \subset \mathbb{C}^n$ is a complex algebraic set. In this note we give a negative answer for the question of Mostowski. Namely, we prove that a Brieskorn surface $x^b + y^b + z^a = 0$ does not admit a complex algebraic normal embedding if $b > a$ and a is not a divisor of b. For the proof of this theorem we use the ideas of the remarkable paper of A. Bernig and A. Lytchak [**3**] on metric tangent cones and the paper of the authors on the (b, b, a) Brieskorn surfaces [**2**]. We also briefly describe other examples based on taut singularities.

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2. *Proof*

Recall that a subanalytic set $X \subset \mathbb{R}^n$ is called *metrically conical* at a point x_0 if there exists a Euclidean ball $B \subset \mathbb{R}^n$ centered at x_0 such that $X \cap B$ is bi-Lipschitz homeomorphic, with respect to the inner metric, to the straight cone over its link at x_0. When such a bi-Lipschitz homeomorphism is subanalytic we say that X is *subanalytically metrically conical* at x_0.

EXAMPLE 1. The Brieskorn surfaces in $\mathbb{C}^3$

$$\{(x, y, z) \mid x^b + y^b + z^a = 0\}$$

$(b > a)$ are subanalytically metrically conical at $0 \in \mathbb{C}^3$ (see [**2**]).

We say that a complex algebraic set admits a *complex algebraic normal embedding* if the image of a subanalytic normal embedding of this set can be chosen complex algebraic.

EXAMPLE 2. Any complex algebraic curve admits a complex algebraic normal embedding. This follows from the fact that the germ of an irreducible complex algebraic curve is bi-Lipschitz homeomorphic with respect to the inner metric to the germ of $\mathbb{C}$ at the origin (e.g., [**8**], [**5**]).

THEOREM 2.1. *If $1 < a < b$ and a is not a divisor of b then no neighborhood of 0 in the Brieskorn surface in $\mathbb{C}^3$*

$$\{(x, y, z) \in \mathbb{C}^3 \mid x^b + y^b + z^a = 0\}$$

admits a complex algebraic normal embedding.

We will need the following result on tangent cones.

THEOREM 2.2. *If (X_1, x_1) and (X_2, x_2) are germs of subanalytic sets which are subanalytically bi-Lipschitz homeomorphic with respect to the induced Euclidean metric, then their tangent cones $T_{x_1}X_1$ and $T_{x_2}X_2$ are subanalytically bi-Lipschitz homeomorphic.*

This result is a weaker version of the results of Bernig-Lytchak([**3**], Remark 2.2 and Theorem 1.2). We present here an independent proof.

Proof of Theorem 2.2. Let us denote

$$S_xX = \{v \in T_xX \mid |v| = 1\}.$$

Since T_xX is a cone over S_xX, in order to prove that $T_{x_1}X_1$ and $T_{x_2}X_2$ are subanalytically bi-Lipschitz homeomorphic, it is enough to prove that $S_{x_1}X_1$ and $S_{x_2}X_2$ are subanalytically bi-Lipschitz homeomorphic.

By Corollary 0.2 in [**9**], there exists a subanalytic bi-Lipschitz homeomorphism with respect to the induced Euclidean metric:

$$h\colon (X_1, x_1) \to (X_2, x_2)\,,$$

such that $|h(x) - x_2| = |x - x_1|$ for all x. Let us define

$$dh\colon S_{x_1}X_1 \to S_{x_2}X_2$$

as follows: given $v \in S_{x_1}X_1$, let $\gamma\colon [0,\epsilon) \to X_1$ be a subanalytic arc such that

$$|\gamma(t) - x_1| = t \ \forall\ t \in [0,\epsilon) \quad \text{and} \quad \lim_{t\to 0^+} \frac{\gamma(t) - x_1}{t} = v\,;$$

we define

$$dh(v) = \lim_{t\to 0^+} \frac{h\circ\gamma(t) - x_2}{t}.$$

Clearly, dh is a subanalytic map. Define $d(h^{-1})\colon S_{x_2}X_2 \to S_{x_1}X_1$ the same way. Let $k > 0$ be a Lipschitz constant of h. Let us prove that k is a Lipschitz constant of dh. In fact, given $v_1, v_2 \in S_{x_1}X_1$, let $\gamma_1, \gamma_2\colon [0,\epsilon) \to X_1$ be subanalytic arcs such that

$$|\gamma_i(t) - x_1| = t \ \forall\ t \in [0,\epsilon) \quad \text{and} \quad \lim_{t\to 0^+} \frac{\gamma_i(t) - x_1}{t} = v_i \text{ for } i = 1,2.$$

Then

$$\begin{aligned}
|dh(v_1) - dh(v_2)| &= \Big|\lim_{t\to 0^+} \frac{h\circ\gamma_1(t) - x_2}{t} - \lim_{t\to 0^+} \frac{h\circ\gamma_1(t) - x_2}{t}\Big| \\
&= \lim_{t\to 0^+} \frac{1}{t}|h\circ\gamma_1(t) - h\circ\gamma_2(t)| \\
&\le k \lim_{t\to 0^+} \frac{1}{t}|\gamma_1(t) - \gamma_2(t)| \\
&= k|v_1 - v_2|.
\end{aligned}$$

Since $d(h^{-1})$ is Lipschitz by the same argument and dh and $d(h^{-1})$ are mutual inverses, we have proved the theorem. $\square$

Corollary 2.3. *Let $X \subset \mathbb{R}^n$ be a normally embedded subanalytic set. If X is subanalytically metrically conical at a point $x \in X$, then the germ (X,x) is subanalytically bi-Lipschitz homeomorphic to the germ $(T_xX, 0)$.*

Proof. The tangent cone of the straight cone at the vertex is the cone itself. So the corollary is a direct application of Theorem 2.2. $\square$

Proof of Theorem 2.1. Let $X \subset \mathbb{C}^3$ be the complex algebraic surface defined by

$$X = \{(x,y,z) \mid x^b + y^b + z^a = 0\}.$$

We are going to prove that the germ $(X,0)$ does not have a normal embedding in $\mathbb{C}^N$ which is a complex algebraic surface. In fact, if $(\tilde{X},0) \subset (\mathbb{C}^N,0)$ is a complex algebraic normal embedding of $(X,0)$ and $p\colon(\tilde{X},0) \to (X,0)$ is a subanalytic bi-Lipschitz homeomorphism, since $(X,0)$ is subanalytically metrically conical [**2**], then $(\tilde{X},0)$ is subanalytically metrically conical and by Corollary 2.3 $(\tilde{X},0)$ is subanalytically bi-Lipschitz homeomorphic to $(T_0\tilde{X},0)$. Now, the tangent cone $T_0\tilde{X}$ is a complex algebraic cone, thus its link is an S^1-bundle. On the other hand, the link of X at 0 is a Seifert fibered manifold with b singular fibers of degree $\frac{a}{\gcd(a,b)}$. This is a contradiction because the Seifert fibration of a Seifert fibered manifold (other than a lens space) is unique up to diffeomorphism. $\square$

The following result relates the metric tangent cone of X at x and the usual tangent cone of the normally embedded sets. See [**3**] for a definition of a metric tangent cone.

THEOREM 2.4 [**3**], Section 5. *Let $X \subset \mathbb{R}^m$ be a closed and connected subanalytic set and $x \in X$. If $(\tilde{X}, p)$ is a normal embedding of X, then $T_{p^{-1}(x)}\tilde{X}$ is bi-Lipschitz homeomorphic to the metric tangent cone $\mathcal{T}_x X$.*

REMARK 1. We showed that the metric tangent cones of the above Brieskorn surface singularities are not homeomorphic to any complex cone.

2.1. *Other examples*

We sketch how taut surface singularities give other examples of complex surface germs without any complex analytic normal embeddings.

Both the inner metric and the outer (euclidean) metric on a complex analytic germ (V, p) are determined up to bi-Lipschitz equivalence by the complex analytic structure (independent of a complex embedding). This is because $(f_1, \dots, f_N): (V, p) \hookrightarrow (\mathbb{C}^N, 0)$ is a complex analytic embedding if and only if the f_i generate the maximal ideal of $\mathcal{O}_{(V,p)}$, and adding to the set of generators gives an embedding which induces the same metrics up to bi-Lipschitz equivalence.

A *taut* complex surface germ is an algebraically normal germ (V, p) (to avoid confusion we say "algebraically normal" for the algebro-geometric concept of normality) whose complex analytic structure is determined up to isomorphism by its topology. So if its inner and outer metrics are not bi-Lipschitz equivalent then it has no complex analytic normal embedding with algebraically normal image. Taut complex surface singularities were classified by Laufer [**7**] and include, for example, the simple singularities. A simple singularity (V, p) of type B_n, D_n, or E_n has non-reduced tangent cone, from which follows easily that it has non-equivalent inner and outer metrics. Thus (V, p) admits no complex algebraic normal embedding as an algebraically normal germ.

If we drop the requirement that the image be algebraicly normal, (V, p) still has no complex analytic normal embedding. Indeed, suppose we have a subanalytic embedding $(V, p) \to (Y, 0) \subset (\mathbb{C}^n, 0)$ whose image Y is complex analytic but not necessarily algebraically normal (see also [**1**]). By tautness, the normalization of Y is isomorphic to V, which has non-reduced tangent cone. So Y also has non-reduced tangent cone, so it is not normally embedded.

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