Irregular Links at Infinity of Complex Affine Plane Curves

Walter D. Neumann

Introduction

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial map. We describe how the link at infinity of any fiber V of f determines the regular link at infinity of f (Theorem 1.5), and hence, by **[13]**, the embedded topology of the general fiber of f. This result, which confirms a conjecture of **[11, Sect. 9]**, was announced in **[9]**.

One consequence is a simple formula for the "Milnor number at infinity" of V as a topological invariant of its link at infinity, and hence a formula (Theorem 1.8) for the total Milnor number of the critical points of f in terms of topological invariants of the irregular links at infinity of f.

We also associate to each link at infinity a topological invariant, which we call its *deficiency*, which vanishes if and only if the link is regular, and such that the sum of deficiencies over all fibers of f is a topological invariant (called the *potential*) of the regular link at infinity for f (Theorem 1.6). In particular, this formula bounds the number of fibers of f which are irregular at infinity (Corollary 1.7.1). It also tells if a given collection of fibers of f includes all fibers which are irregular at infinity. These results may help in trying to classify polynomial submersions $f\colon \mathbb{C}^2 \to \mathbb{C}$.

An amusing application is as follows (Corollary 1.7.2). It turns out that the deficiency of a link $\mathcal{L}$ equals the potential of the corresponding regular link if and only if $\mathcal{L}$ is fiberable. Thus, the link at infinity of a fiber $f^{-1}(c)$ is a fiberable link if and only if all other fibers of f are regular at infinity. The "if" was already known: in **[12]** Nemethi and Zaharia show that Milnor's map $(f-c)/||f-c||$ fibers the link at infinity of $f^{-1}(c)$ in a sufficiently large sphere in this case.

My original approach to these results, as described in **[9]**, was by Puiseux expansion of linear families of plane curve singularities. In 1992 discussions with Shulim Kaliman led to the more efficient approach using resolution given here (this approach is of course not new, having been used by Miyanishi and others since at least the 70's, see e.g., **[10]**). A version of this paper has been available as electronic preprint for some time. I originally intended to add to this paper a derivation of Suzuki's results of **[15]** and certain results of Kaliman and others by similar means, but these will wait for another paper **[11]**.

1. Terminology and Results

We refer the reader to **[13]** for proofs of the basic facts described here. An *affine plane curve* is a fiber $V = f^{-1}(c)$ of a polynomial map $f : \mathbb{C}^2 \to \mathbb{C}$. The intersection of V with any sufficiently large sphere $S^3 = S^3(R) = \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 = R\}$ is transverse, and gives a well-defined link up to isotopy, $\mathcal{L}(V, \infty) = (S^3, S^3 \cap V)$, called the *link at infinity* of V. Since $V = f^{-1}(c)$ may be a non-reduced curve, some components of $\mathcal{L}(V, \infty)$ may have multiplicities assigned. That is, $\mathcal{L}(V,\infty)$ may be a *multilink* in the terminology of [**5**] and [**13**]. For example, if $f(x,y)=x^2$, the link at infinity $\mathcal{L}(f^{-1}(c),\infty)$ consists of two unlinked unknotted components if $c \neq 1$ and is a single unknotted component of multiplicity 2 if $c=0$. A link is the special case of a multilink in which all components have multiplicity 1.

The affine plane curve $V=f^{-1}(c)$ is *regular at infinity* if nearby fibers of f "look like V near infinity." That is, for some neighborhood D of $c \in \mathbb{C}$ and some compact subset K of $\mathbb{C}^2$, the map $f|f^{-1}(D)-K$ is a locally trivial fibration. V is *regular* if it is regular at infinity and smooth. All but finitely many fibers of f are regular. The fibers which are regular at infinity all define the same link at infinity up to isotopy, which we call the *regular link at infinity* of f and denote $\mathcal{L}(f,\infty)$.

In [**13**] it was shown that the embedded smooth topology of a regular affine plane curve $V \subset \mathbb{C}^2$ is determined by the topological type of its link at infinity. It was conjectured that the same is true for any smooth fiber, but counter-examples have been found by P. Artal ([**1**], [**2**]). Nevertheless, the topology of the link at infinity of any reduced fiber $V=f^{-1}(c)$ of f determines a lot of information about V and f. For example, it determines the euler characteristic of V, corrected by the Milnor numbers of the singularities of V if V is singular. It determines whether V is regular at infinity or not. It also determines the polynomial degree of f to the extent possible, namely the set of degrees of polynomial maps $f \circ \alpha$ with $\alpha\colon \mathbb{C}^2 \to \mathbb{C}^2$ a polynomial automorphism of $\mathbb{C}^2$ (replacing f by $f\circ\alpha$ does not change the topology of f). We summarize these results in Theorem 1.2 below, in part to fix notation.

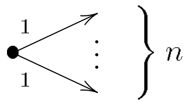
To describe these results and our new results we need to recall the language of splice diagrams. For details see [**13**].

Splice diagrams where introduced in [**5**] to classify a class of links ("graph links") which includes all *iterated torus links* — that is, links obtainable from a trivial knot by repeated cabling operations. A *splice diagram* is a finite tree, decorated as follows.

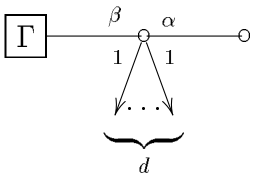
- Some valency 1 vertices are drawn as *arrowheads*; these stand for components of the link. If the diagram represents a multilink then the arrowheads should have integer weights assigned, to represent the multiplicities of the corresponding components.
- Vertices of valency 1 that are not arrowheads are called *leaves*; vertices of valency greater than 1 are *nodes*.
- At each node the ends of the edges incident at that node are given pairwise coprime integer weights.

For the special case of an iterated torus link the concept of a *rooted splice diagram* is useful, in which some non-arrowhead vertex is chosen as *root vertex*. The root vertex is considered to be a node, even if it has valency 1. As we will describe, the rooted splice diagram codes not only the link, but also the way it was constructed by iterated cabling. There may therefore be several ways of choosing a root vertex for a given splice diagram, corresponding to several different ways of constructing the given link by iterated cabling. The root vertex will be marked by a solid dot $\bullet$ in the diagram, while other non-arrowhead vertices are marked by open dots $\circ$. A rooted splice diagram for an iterated torus link can be described inductively as follows.

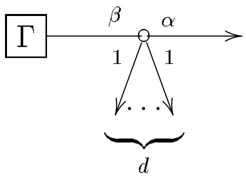
We may start with the "n-component Hopf link" $\mathcal{H}_{\backslash}=(\mathcal{S}^{\ni},\mathcal{K}_{\infty}\cup\ldots\mathcal{K}_{\backslash})$ consisting of n fibers of the Hopf fibration. Its rooted splice diagram is:



If $\boxed{\Gamma}\!\longrightarrow$ is a splice diagram for a link $\mathcal{L} = (S^3, L)$ and the indicated arrowhead corresponds to a component $K \subset L$ of this link then the diagrams



and

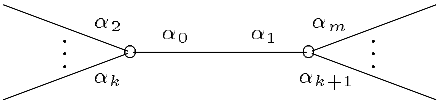


represent the result of either replacing component K by, or adding, d parallel (α, β) cables on K. (An (α, β)-cable on K is a simple curve on the boundary of a thin tubular neighborhood of K which is homologous to αK in the neighborhood and has linking number β with K.)

In a rooted splice diagram we call a weight *near* or *far* according as it is on the end of its edge nearest to or furthest from the root vertex. Adjacent to any node other than the root there is thus exactly one far weight and usually several near weights. At most one of these near weights is not equal to 1.

The rooted splice diagram is an *RPI diagram* if:

- all near weights are positive and any multiplicity weights at arrowheads are positive;
- the edge determinant for any edge connecting two nodes is negative. The *edge determinant* of an edge



is $\alpha_0\alpha_1 - \alpha_2 \dots \alpha_m$.

(RPI stands for "reverse Puiseux inequalities" since the reverse condition of positive edge determinants leads to conditions equivalent to the classical Puiseux inequalities, which are necessary and sufficient for a link to be the link of a plane curve singularity. Cf **[5]**.)

An iterated cable multilink which can be represented by an RPI diagram is called an *RPI multilink*. The link at infinity of an affine plane curve is an RPI multilink (this is shown in **[13]** in the reduced case, and the non-reduced case is a consequence).

There are certain operations that can be done to simplify a splice diagram without changing the represented link.

- Any edge weighted by 1 and ending in a leaf may be deleted, with its leaf.
- Any node (other than the root node in a rooted splice diagram) of valency 2 may be deleted by replacing it and its two adjoining edges by a single edge. An RPI diagram that allows neither of the above operations is called *minimal*.

Associated to a splice diagram Γ for a multilink $\mathcal{L} = (S^3, L)$ are certain curves in $S^3 - L$, as follows. To a node v of Γ we associate the curve S_v that would be added to L by adding an additional arrow $\xrightarrow{1}$ at that node. To a leaf v of Γ we associate the curve S_v that would result by replacing this leaf by an arrowhead. These curves S_v are called *virtual components* of the multilink. For any non-arrowhead vertex of Γ, the linking number $l_v := \ell(S_v, L)$ is called the *(total) linking coefficient* at vertex v (called "multiplicity" in [**13**]). In this linking number the multiplicity of components of L are to be taken into account. An RPI diagram is *regular* if it is a diagram for a link (all arrowhead weights are 1) and no linking coefficient l_v is negative. If some RPI diagram for a link $\mathcal{L}$ is regular then every minimal RPI diagram for $\mathcal{L}$ is regular ([**13**]). We say $\mathcal{L}$ is a *regular* RPI link.

We will need the following lemma from [**5**], which enables rapid computation of the linking coefficients in terms of Γ.

Lemma 1.1. *Let $\mathcal{L} = (S^3, L)$ be a link represented by a splice diagram Γ and S_v and S_w the virtual or genuine components of $\mathcal{L}$ represented by vertices v and w of Γ. Then the linking number $\ell(S_v, S_w)$ is the product of all edge weights adjacent to, but not on, the shortest path in Γ from v to w. In particular, the linking coefficient l_v at a non-arrowhead v is the sum over all arrowheads w of the arrowhead weight times the product of edge weights adjacent to but not on the shortest path from v to w.* ∎

We recall some facts about the relationship between links at infinity and their RPI diagrams. These results were usually just proved in the reduced case (i.e., links, rather than multilinks) in the quoted references, but, in each case, the given statement either follows directly from the reduced case or has the same proof. Unless otherwise stated the results are from [**13**].

Theorem 1.2. *Suppose $f\colon \mathbb{C}^2 \to \mathbb{C}$ is a polynomial map and $V = f^{-1}(c)$.*

1. *V is regular at infinity if and only if $\mathcal{L}(V,\infty)$ is a regular RPI link* ([**6**], [**9**]). *In this case, if one takes a minimal Seifert surface of $\mathcal{L}(V,\infty)$ and extends it out to infinity by a collar in $\mathbb{C}^2$ one obtains a real surface in $\mathbb{C}^2$ which is properly isotopic to any regular fiber of f.*

2. *There is a unique minimal RPI diagram Γ representing $\mathcal{L}(V,\infty)$ with the property that the virtual component $S_\bullet$ for the root vertex represents the link at infinity of a generic line (that is, $L(V,\infty) \cup S_\bullet \subset S^3$ is the link at infinity for $V \cup \{(x,y) : ax + by = 0\}$ for generic a and b). We call this "the minimal RPI diagram at infinity for f at c". If $V = f^{-1}(c)$ is regular at infinity we call it "the minimal regular RPI diagram at infinity for f."*

3. *Every minimal RPI diagram for $\mathcal{L}(V,\infty)$ is the minimal RPI diagram at infinity for some $f \circ \alpha$ at c with α a polynomial automorphism of $\mathbb{C}^2$.* ([**14**].)

4. *If Γ is the minimal RPI diagram at infinity for f at c then the linking coefficient $l_\bullet$ of the root vertex equals the polynomial degree of f. Also the valency of the root vertex is the number of points at infinity of $V = f^{-1}(c)$ (this is $|\overline{V} \cap \mathbb{P}^1|$, where $\overline{V}$ is the closure of V in $\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{P}^1$; it depends only on f and not on c).*

5. *If V is a reduced plane curve and Γ is an RPI diagram for its link at infinity then V has euler characteristic*

$$\chi(V) = \sum_v (2 - \delta_v) l_v + \sum_p \mu_p,$$

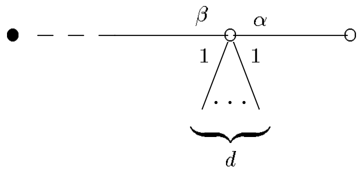
where the first sum is over all non-arrowhead vertices of Γ*, the second sum is the sum of Milnor numbers of the singularities of* V*, and* δ_v *denotes the valency of vertex* v*.* ∎

To describe our new results we need additional terminology. Let Γ be a minimal RPI diagram. In **[11, Sect. 5]** it is shown that if a node has $l_v \le 0$ then all nodes w beyond v, from the point of view of the root, have $l_w < 0$. We call any node v with $l_v = 0$ a *horizontal node*. An edge that connects a vertex v with $l_v > 0$ to an arrowhead or to a vertex w with $l_w < 0$ will be called a *horizontal edge*. The reason for this terminology will become clear later. We shall show in Sect. 3:

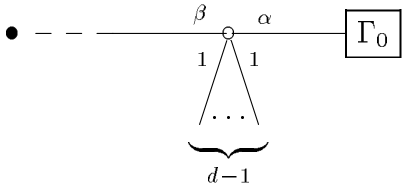
Lemma 1.3. *Given a horizontal edge of* Γ*, if one adds a vertex* u *in the middle of it, there is a unique way of adding appropriate edge weights adjacent to* u *to make* $l_u = 0$*. The result is still a (non-minimal) RPI diagram.*

Now suppose Γ is not regular. That is, Γ has a node with $l_w < 0$ or an arrowhead with weight > 1. A horizontal node or edge which has such a vertex beyond it will be called *deficient* ("beyond" will always mean from the point of view of the root). Using Lemma 1.3 we can add a horizontal node on every deficient horizontal edge of Γ to obtain an RPI diagram Γ' with no deficient edges. Γ' has at least one deficient horizontal node. We call Γ' the *working diagram*.

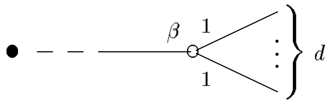
A deficient horizontal node has the form



with $\alpha > 1$, $d > 0$, $(1a_1)$

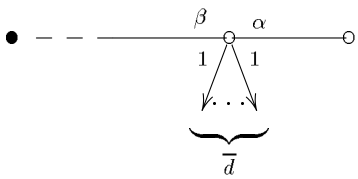


with $\alpha > 1$, $d \ge 1$, and
Γ_0 contains an arrowhead, $(1a_2)$

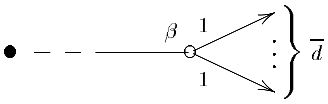
or 

with $d \ge 1$, $(1b)$

with at least one of the edges beyond the node leading to a further node or to an arrowhead with weight > 1 in each case. A horizontal node which is not deficient has no nodes beyond it, so it must have the form



with $\overline{d} \ge 1$, $(2a)$

or 

with $\overline{d} \ge 2$ $(2b)$

We shall show in Sect. 3:

Lemma 1.4. *Given a deficient horizontal node u of the working diagram Γ' as in $(1a, b)$ above, there is a unique way of replacing it and the part of Γ' beyond it by a non-deficient horizontal node as in $(2a, b)$ above in such a way that in the new diagram l_u is still 0 and l_v is unchanged for any non-arrowhead vertex of Γ' which was not beyond u. Moreover, $\overline{d} > d$ except in case $(1a_2)$, when $\overline{d} \geq d$.*

Starting with an RPI diagram Γ, we may perform the above replacement at each deficient horizontal node of the working diagram Γ'. We denote the result by Γ^{reg}. Our first main result* is:

Theorem 1.5. *If $f: \mathbb{C}^2 \to \mathbb{C}$ is a polynomial map and Γ is the minimal RPI diagram at infinity for f at c then Γ^{reg} is the minimal regular RPI diagram at infinity for f.*

If u is a deficient horizontal node of Γ' which gets replaced as above in Γ^{reg}, we define the *deficiency* at u to be the pair $\operatorname{def}(u) := (\overline{d} - d, \epsilon)$ with $d, \overline{d}$ as above and $\epsilon = 1$ if we are in case $(1a_2)$ and $\epsilon = 0$ otherwise. If u is a non-deficient horizontal node we put $\operatorname{def}(u) := (0, 0)$. The *deficiency* of Γ' is the $2n$-tuple $\operatorname{def}(\Gamma) := (\operatorname{def}(u))_{u \text{ horizontal}}$, where n is the number of horizontal nodes of Γ'. For a horizontal node u of a regular RPI diagram Γ^{reg} we define the *potential* at u to be $\operatorname{pot}(u) = (\overline{d} - 1, \epsilon)$, where ϵ is 1 or 0 according as we are in case $(2a)$ or $(2b)$. The *potential* of Γ^{reg} is the $2n$-tuple $\operatorname{pot}(\Gamma^{\mathrm{reg}}) := (\operatorname{pot}(u))_{u \text{ horizontal}}$.

Theorem 1.6. *The above invariants deficiency and potential depend (up to order) only on the links, and not on the RPI diagrams chosen to represent them. The components of the deficiency for an irregular RPI multilink are non-negative and at least one is positive. The potential of the regular link at infinity of a polynomial map is the sum of deficiencies of all its irregular links at infinity.*

Corollary 1.7. 1. *The number of fibers of the polynomial map $f: \mathbb{C}^2 \to \mathbb{C}$ which are irregular at infinity is at most equal to the total number of arrows emanating from horizontal nodes of an RPI diagram for a regular link at infinity for f, minus the number of horizontal nodes of type $(2b)$ above.*

2. *The multilink $\mathcal{L}(f^{-1}(c), \infty)$ is fiberable if and only if all fibers of f other than $f^{-1}(c)$ are regular at infinity.*

It is not unlikely that the bound of Corollary 1.7.1 is best possible, in the sense that a "generic" f with the given regular link at infinity actually realizes the bound.

Proof of 1.7 (assuming 1.6). **1.** The sum of all the components of the potential of a regular RPI link is the number stated in the Corollary, so the result follows.

2. All fibers other than $f^{-1}(c)$ are regular at infinity if and only if the deficiency of $\mathcal{L}(f^{-1}(c), \infty)$ equals the potential of a regular fiber. We claim this is so if and only if the minimal RPI diagram Γ for $\mathcal{L}(f^{-1}(c), \infty)$ has no horizontal nodes; by **[5]** this is equivalent to $\mathcal{L}(f^{-1}(c), \infty)$ being a fiberable link.

Indeed, the deficiency of a vertex of Γ' equals the potential of the corresponding vertex of Γ^{reg} if and only if it is as in $(1a_2)$ or $(1b)$ with $d = 1$. This is so if and only if the vertex has valency 2, that is, the original diagram Γ has no horizontal node there. ∎

* proofs are in section 5

Theorems 1.2.5 and 1.5 let us reformulate a formula of Suzuki [**15**]. If $V = f^{-1}(c)$ is reduced, Suzuki defines the *Milnor number at infinity* for V as the contribution to $\chi(V) - \chi(f^{-1}(c_0))$ that comes from any irregularity at infinity of V, where $f^{-1}(c_0)$ is a regular fiber. Since a singularity of V contributes its Milnor number, this Milnor number at infinity is $\chi(V) - \chi(f^{-1}(c_0)) - \sum_{p\in V}\mu_p$, where the sum is the sum of Milnor numbers of singularities of V. By Theorems 1.2.5 and 1.5 it is the following invariant of the link at infinity:

$$\mu_\infty(\mathcal{L}(V,\infty)) := \sum_{l_v<0}(2-\delta_v)l_v.$$

This Milnor number at infinity is non-negative, and vanishes if and only if $f^{-1}(c)$ is regular at infinity (this follows very easily from our above formula; it was first proved in [**15**], see also [**7**]).

Theorem 1.8 ([**15**]). *If $f\colon \mathbb{C}^2 \to \mathbb{C}$ has only isolated singularities then the euler characteristic of a regular fiber satisfies*

$$\chi(f^{-1}(c_0)) = 1 - \sum_{c\in\mathbb{C}}\mu_\infty(\mathcal{L}(f^{-1}(c),\infty)) - \sum_p \mu_p,$$

where the second sum is the sum of Milnor numbers of all singular points $p \in C^2$ of f. ∎

We can also give topological interpretation to results of Kaliman and Miyanishi-Sugie in terms of RPI diagrams. Let h be the total number of horizontal nodes and edges of a regular splice diagram for f.

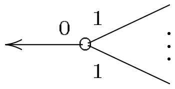
Theorem 1.9. *(*[**8**], [**10**]*.) Suppose $f\colon \mathbb{C}^2 \to \mathbb{C}$ is a primitive polynomial (i.e., not of the form $f_1(f_2(x,y))$ with f_1 of degree > 1). If r_c is the number of components of $f^{-1}(c)$ for each $c \in \mathbb{C}$ then $\sum_{c\in\mathbb{C}}(r_c - 1) \le h - 1$. Moreover, if f is a "rational polynomial" (that is, the regular fiber is a rational curve) then equality holds.*

Proof. As we discuss in more detail below, one can find compactifications X of $\mathbb{C}^2$ such that $f\colon \mathbb{C}^2 \to \mathbb{C}$ extends to a holomorphic map $\overline{f}\colon X \to \mathbb{P}^1 = \mathbb{C}\cup\{\infty\}$. (Such a compactification is not unique.) Then $D = X - \mathbb{C}^2$ is a union of rational curves (i.e., copies of $\mathbb{P}^1$) intersecting transversally. A component of D is *horizontal* if $\overline{f}$ is non-constant on it. We shall see that horizontal components correspond one-one with the horizontal nodes and edges in a regular splice diagram for f: Moreover, the degree of $\overline{f}$ on a horizontal component is the number of arrowheads at the corresponding horizontal node for a component corresponding to a horizontal node and is 1 for a component corresponding to a horizontal edge. Thus the h of the theorem is the number of horizontal components. This gives Kaliman's formulation of his results. ∎

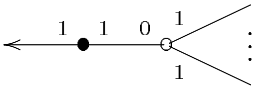
2. Examples: links at infinity which are connected sums

This section is not essential to the rest of the paper. It describes the classification of links at infinity that are non-trivial connected sums. This classification is not of great significance, but it happens to nicely illustrate our results, and also answers a question of A. Durfee. See also [4]. Another illustration is the computation in [**3**] of examples of polynomial submersions $\mathbb{C}^2 \to \mathbb{C}$ with all fibers connected.

By [**5**] a graph link which is a connected sum has minimal splice diagram of the form



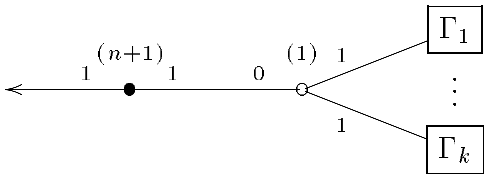
If this link has an RPI diagram, then by [**13**] some minimal RPI diagram can be obtained by adding a root vertex to the minimal splice diagram. Since near weights are positive in an RPI diagram, the only possible position for the root is:



If this is the RPI diagram for f at c then the number of points at infinity of $f^{-1}(c)$ is 2. By a linear automorphism of $\mathbb{C}^2$ we may put these points at $y=0$ and $x=0$ respectively. After multiplying by a constant if necessary, the homogeneous part of f of highest degree is then yx^n, where $n+1$ is the degree of f. Using the relationship between the RPI diagram and the Newton polygon for f (see, e.g., [**11, sect. 7**]) it is then not hard to see that f must have the form $f(x,y) = yg(x) + h(x)$ with $g(x)$ and $h(x)$ polynomials of degree n and $\leq n$ respectively. We write this as

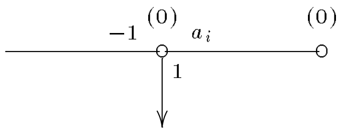
$$f(x,y) = y\prod_{i=1}^{k}(x-c_i)^{a_i} + h(x)$$

with the c_i distinct and $\sum_1^k a_i = n$. One computes that the RPI diagram for f at c is as follows. Numbers in parentheses are linking coefficients.

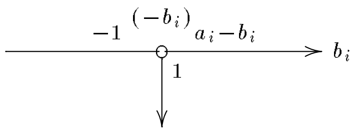


with Γ_i as follows if c_i is an b_i-fold root of $h(x)-c$:

if $a_i = 1$,



if $b_i = 0$, $a_i > 1$,



if $1 \leq b_i < a_i$,

if $b_i \geq a_i > 1$.

In the first two cases Γ_i does not contribute to irregularity while in the last two cases it does. The regular RPI diagram at infinity for f is thus the diagram with each Γ_i chosen appropriately from the first two cases. Theorem 1.2.5 shows that any regular fiber V of f has $\chi(V) = 1 - k$. Since it has k ends, it is topologically a k-punctured disk.

Let us assume $a_i > 1$ for $i \le r$ and $a_i = 1$ for $i > r$. Then we see that the potential of $\mathcal{L}(f, \infty)$ is the $2r$-tuple $\operatorname{pot}(\mathcal{L}(f,\infty)) = (0,1; \dots; 0,1)$ and the deficiency $\operatorname{def}(\mathcal{L}(f^{-1}(c), \infty))$ has $(0,1)$ in the i-th position if and only if $h(c_i) = c$. This confirms Theorem 1.6 for this case.

Every fiber of f is reduced (i.e., f has only isolated singularities) if and only if each c_i for $i \le r$ is a simple root of $h(x) - h(c_i)$, that is, it is not a branch point for $h\colon \mathbb{C} \to \mathbb{C}$. In this case we see that each deficient Γ_i contributes 1 to a Milnor number at infinity. Thus, in Theorem 1.8 the Milnor numbers at infinity contribute a total of r and the Milnor numbers of singularities of f must therefore contribute $k - r$. In particular, f has no singularities, that is, it is a polynomial submersion, if and only if no c_i is a branch point of h and $r = k$. This is easily seen by direct computation of the derivative of f, but our purpose was to illustrate the information that can easily be extracted from just the topology of the links at infinity of f.

3. Elementary lemmas

In this section we prove Lemmas 1.3 and 1.4.

Let Γ be a rooted splice diagram with positive near weights. Then Γ is an RPI diagram if and only if it has negative edge determinants. If v is a vertex of Γ, we shall denote the product of near weights adjacent to v by α_v (this is one of the near weights since at most one near weight is $\neq 1$). We denote the far weight adjacent to v by β_v.

Lemma 3.1. *Let e be an edge of Γ connecting two nodes v and w, where v is the one nearer the root. Let α be the weight at the v end of e (so $\alpha = 1$ or α_v). Then the edge determinant $d(e)$ is negative if and only if $a_w l_v > \alpha l_w$.*

Proof. Let $L \subset S^3$ be the link represented by Γ and L_w the sublink represented by arrowheads beyond vertex w from the point of view of the root. A simple computation (see proof of Proposition 5.1 of **[13]**) shows

$$\alpha_w l_v - \alpha l_w = \frac{-d(e)}{\beta_w} \ell(S_w, L_w).$$

But $\frac{1}{\beta_w}\ell(S_w, L_w)$ is positive, since its computation by Lemma 1.1 only involves near weights. ∎

Proof of Lemma 1.3. Let Γ be an RPI splice diagram representing a link (S^3, L). If e is a horizontal edge, add a vertex u in the middle of this edge with adjacent near and far weights α and β. Let L_u be the sublink of L corresponding to arrowheads beyond u and let L'_u be the complementary sublink. Then, by Lemma 1.1,

$$l_u = \ell(S_u, L) = \ell(S_u, L_u) + \ell(S_u, L'_u) = \beta\gamma_u + \alpha\gamma'_u,$$

where γ_u is the sum over arrowheads beyond u of the product of weights adjacent to the simple path from u to the arrowhead, and γ'_u is the analogous sum for the remaining arrowheads. Note that the computation of γ_u only involves near weights of Γ, so $\gamma_u > 0$. The equation $l_u = 0$ thus has a unique solution in coprime $\alpha > 0$ and β, namely, $\alpha = \gamma_u/d$ and $\beta = \gamma'_u/d$, where $d = \gcd(\gamma_u, \gamma'_u)$. Moreover, once α and β have been chosen this way, the new diagram is still RPI by Lemma 3.1. ∎

Proof of Lemma 1.4. We shall just do case (1a) since (1b) is similar but simpler. Let α, β, and u be as in the lemma. If we remove vertex u then Γ falls into several components. Let L_u be the sublink represented by arrowheads in the component containing the α-weighted edge (that is, arrowheads in Γ_0), L'_u the sublink represented by arrowheads in the component of the β-weighted edge, and L''_u the sublink represented by arrowheads in the remaining components. Then

$$0 = l_u = \beta\gamma_u + \alpha\gamma'_u + \alpha\beta\gamma''_u,$$

where γ_u, γ'_u, and γ''_u are computed analogously to the previous proof. Since α and β are coprime, it follows that γ_u is divisible by α. Make the modification of Lemma 1.4 with $\overline{d} = \gamma_u/\alpha + \gamma''_u$. Then l_u is still zero. Moreover, for any vertex v that was not beyond u the contribution of $L_u \cup L''_u$ to l_v was a multiple of $\gamma_u + \alpha\gamma''_u$. It has been replaced by the same multiple of $\alpha\overline{d}$ and hence l_v is unchanged.

Finally, it is clear that $\overline{d} \geq d$. Moreover, if we are not in case (1a_2) then there must be a 1-weighted edge from the vertex u of Γ which does not end directly in an arrowhead with multiplicity 1. Then beyond that edge there must be an arrowhead with multiplicity > 1 or at least two arrowheads or a near weight > 1. That edge therefore contributes at least 2 to γ''_u, so $\overline{d} > d$. ∎

4. The resolution picture

Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be a polynomial map. Then there exist compactifications X of $\mathbb{C}^2$ such that f extends to a map $\overline{f}\colon X \to \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Such a compactification is not unique. We shall use the following way of constructing such an X. Let $V = f^{-1}(c)$ and let $\overline{V}$ be the closure of V in $\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{P}^1$. Then f extends to a map $\mathbb{P}^2 \to \mathbb{P}^1$ which is defined everywhere except at the "points at infinity of V," that is, at $\overline{V} \cap \mathbb{P}^1$. By blowing up $\mathbb{P}^2$ at these points we get a new space onto which f can be extended except possible at finitely many points. By blowing up iteratively where f is not defined, we eventually obtain the desired space X. In fact, if V_1 and V_2 are two regular fibers of f, then f fails to be defined precisely where $\overline{V}_1$ and $\overline{V}_2$ cross, and this is true also after blowing up. Thus f becomes well defined once we have blown up sufficiently to resolve the singularities of $V_1 \cup V_2$. We call the resulting map $\overline{f}\colon X \to \mathbb{P}^1$.

Let $D = X - \mathbb{C}^2$. We may modify further by additional blow-ups at points of D if desired (we will have reason to do this later). We rename to retain the notation $\overline{f}\colon X \to \mathbb{P}^1$ after such modification.

D is a union of rational curves. Denote by Δ the dual configuration graph for D. This is the weighted graph with vertices corresponding to the component curves of D, edges corresponding to intersections of two components, and numerical weights at the vertices given by the selfintersection numbers of the components. By the construction of D, this graph is a tree (in fact a simple homological calculation shows this for any curve in a compact surface whose complement is a rational homology disk).

For $t \in \mathbb{P}^1$ denote by D_t the union of those component curves of D that lie in $\overline{f}^{-1}(t)$. D_t is empty for all but a finite number of values of t. We denote by Δ_t the corresponding full subgraph of Δ. We call a component C of D *horizontal* if it lies in no D_t, or, equivalently, the map $\overline{f}|_C$ is non-constant and hence a branched covering $C \to \mathbb{P}^1$.

Lemma 4.1. *D_∞ is topologically connected, so Δ_∞ is a subtree of Δ. The "horizontal vertices" of Δ (those corresponding to horizontal components of D) are the vertices immediately adjacent to Δ_∞. Each component of the full subgraph of Δ on the non-horizontal vertices corresponds to a component of some D_t.*

Proof. First note that $\overline{f}^{-1}(\infty) \subset D$, so $D_\infty = \overline{f}^{-1}(\infty)$. If C is a horizontal component of D then it meets $\overline{f}^{-1}(\infty)$, so the corresponding vertex of Δ is adjacent to Δ_∞. On the other hand, D_t and $D_{t'}$ are disjoint for $t \neq t'$, so any vertex adjacent to Δ_∞ cannot belong to a Δ_t, so it must be horizontal.

It remains to show that D_∞ is topologically connected. This is clear by induction over our iterative construction of X, but it is in fact true even if X is not constructed as above. To see this, first note that we may assume that f is "primitive", that is, the generic fiber of f is connected. For if not, f has the form $f_1 \circ f_2$ with $f_2: \mathbb{C}^2 \to \mathbb{C}$ and $f_1: \mathbb{C} \to \mathbb{C}$ polynomial mappings such that f_2 is primitive, and we can replace f by f_2 without changing X and D_∞. Since the generic fiber of f is connected, the same holds for $\overline{f}$, and therefore every fiber of $\overline{f}$ is topologically connected. But D_∞ is the full fiber of $\bar{f}$ over ∞. ∎

We assume our D has been constructed by iteratively blowing up $\mathbb{P}^2$ as above. The original $\mathbb{P}^1$ at infinity is thus a curve of D, which we call the *root curve*. We call the corresponding vertex of Δ the *root vertex*.

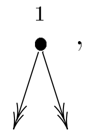
There may be points p of D where the closure $\overline{f^{-1}(t)}$ of a fiber of f does not meet D transversally. This cannot happen at a regular point of the map $\overline{f}|_C$ of a horizontal component C of D, so it can only happen at a branch point of a horizontal component or at a point of some D_t with $t \neq \infty$. There are thus just finitely many such points, and by additional blow-ups we may arrange for only transversal crossings. We shall now assume we have done this.

For $c \in \mathbb{C}$ we define an extended dual graph $\Delta(c)$ by adding an arrow to Δ for each point where $\overline{f^{-1}(c)}$ meets D. If $\overline{f^{-1}(c)}$ is non-reduced, we give these arrows appropriate multiplicity weights. The result is then a plumbing graph for the link at infinity $\mathcal{L}(f^{-1}(c), \infty)$ in the sense of **[5]** (except for orientation — see proof of Proposition 5.1). In the next section we recall from **[5]** how to construct the splice diagram from this plumbing graph. We close this section by computing an example.

Example: $f(x, y) = xy^2 + y$. If we use homogeneous coordinates $[x, y, z]$ in $\mathbb{P}^2$ then the above f extends to the rational map $\overline{f}: \mathbb{P}^2 \to \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ given by

$$\overline{f}([x, y, z]) = \frac{xy^2 + yz^2}{z^3},$$

which is undefined only at the points $[0,1,0]$ and $[1,0,0]$ of the root curve $E_0 = \mathbb{P}^2 - \mathbb{C}^2$. We can represent this situation by the following dual graph, where we are using double-headed arrows to indicate these two points of indeterminacy.



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To make $\overline{f}$ well defined we must blow up these two points of indeterminacy. We start with the point $[1,0,0]$.

Near $[1,0,0]$ we may work in the affine chart $U = \{x = 1\}$ and use y and z as coordinates there. Then $\overline{f}$ is given by

$$\overline{f}(y,z) = \frac{y^2 + yz^2}{z^3}.$$

When we blow up we replace U by two charts $\tilde{U}_1$ with coordinates (u_1, v_1) and $\tilde{U}_2$ with coordinates (u_2, v_2). The map $\sigma : \tilde{U}_1 \cup \tilde{U}_2 \to U$ is given by $\sigma(u_1, v_1) = (u_1 v_1, v_1)$, $\sigma(u_2, v_2) = (u_2, u_2 v_2)$. The exceptional divisor $E_1 = \sigma^{-1}(0,0)$ is given by $v_1 = 0$ in the first chart and $u_2 = 0$ in the second. The proper transform of the root curve E_0 (which we still denote E_0) does not meet $\tilde{U}_1$ and is given in the second chart by $v_2 = 0$.

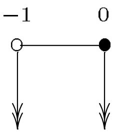
As is customary, we denote the map $\overline{f} \circ \sigma$ still by $\overline{f}$. In these two charts it is given by

$$\overline{f}(u_1, v_1) = \frac{u_1^2 + u_1 v_1}{v_1}$$

$$\overline{f}(u_2, v_2) = \frac{1 + u_2 v_2^2}{u_2 v_2^3}$$

respectively. Thus we see that $\overline{f}$ takes the value ∞ on E_1 except at the point $(u_1, v_1) = (0,0)$ of the first chart, where it is still undefined.

Recalling that the exceptional divisor of a blow-up has selfintersection number -1, while the selfintersection number of a curve is reduced by 1 each time a point is blown up on it, we see that the new situation has the following dual graph:



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Since the point of indeterminacy is in the (u_1, v_1)-chart, we now work in this chart and rename its coordinates (y, z) in order to free up the notations (u_1, v_1) and (u_2, v_2) for another blow-up. We are thus looking at a function given locally as

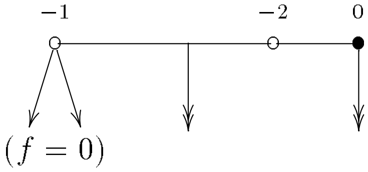
$$\overline{f}(y,z) = \frac{y^2 + yz}{z}$$

and blow up at the point $(y,z) = (0,0)$ just as before. Our new exceptional divisor is $E_2 = \{v_1 = 0\} = \{u_2 = 0\}$ and the proper transform of E_1 only meets the second chart and is there given by $v_2 = 0$.

In these coordinates the new $\overline{f}$ is given by

$$\overline{f}(u_1, v_1) = u_1^2 v_1 + u_1 v_1$$
$$\overline{f}(u_2, v_2) = \frac{u_2 + u_2 v_2}{v_2}.$$

We thus see that $\overline{f}$ takes the value 0 on E_2 except at the point $(u_2, v_2) = (0, 0)$ of the second chart, where it is still undefined. This is the point $E_1 \cap E_2$. Moreover, the fiber $\bar{f}^{-1}(0)$ consists of $E_2 \cup \{u_1 = 0\} \cup \{u_1 = -1\}$. The dual graph is now



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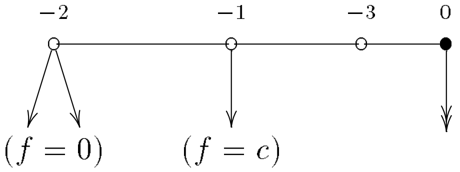
A blow-up at the point $E_1 \cap E_2$ gives us, after renaming coordinates as before, two new coordinate systems (u_1, v_1) and (u_2, v_2), a new exceptional divisor $E_3 = \{v_1 = 0\} = \{u_2 = 0\}$ and $\overline{f}$ given as

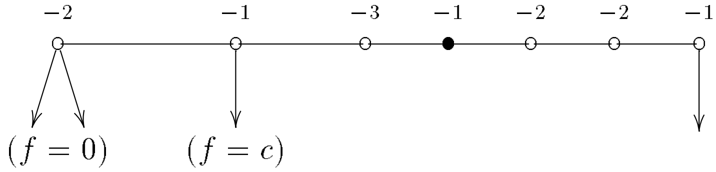
$$\overline{f}(u_1, v_1) = u_1 + u_1 v_1$$
$$\overline{f}(u_2, v_2) = \frac{1 + u_2 v_2}{v_2}.$$

Thus $\overline{f}$ is now well-defined, and E_3 is a horizontal curve that is mapped isomorphically by $\overline{f}$ to $\mathbb{P}^1$. The dual graph is now



$(c \neq 0)$.

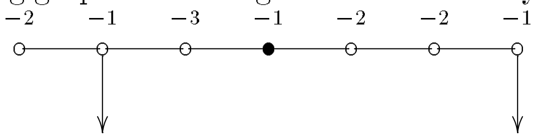
A similar computation shows that to resolve the indeterminacy of $\overline{f}$ at $[0, 1, 0]$ we must blow up this point, giving an exceptional divisor E_4 on which $\overline{f}$ takes the value ∞ except for one point of indeterminacy, then blow up this point of E_4 to get E_5 on which $\overline{f}$ again takes the value ∞ except for one point of indeterminacy, and then finally blow up this point of E_5 to get a horizontal curve E_6 on which $\overline{f}$ is an isomorphism to $\mathbb{P}^1$. This gives the following final diagram:



$(c \neq 0)$.

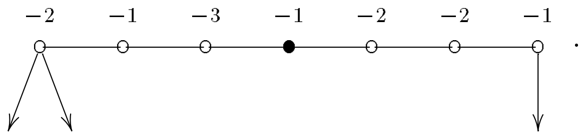
In other words, a plumbing graph for the regular link at infinity is

$\Delta(c) =$

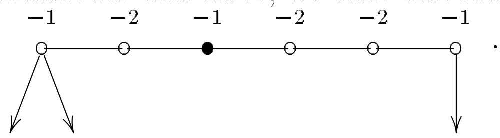


$(c \neq 0)$. (4.1)

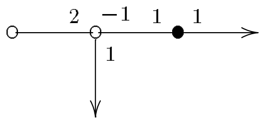
The fiber $f^{-1}(0)$, which is irregular at infinity, has link at infinity determined by

$\Delta(0) =$ 

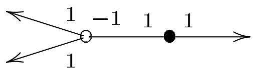
Since the third blow-up was redundant for this fiber, we take instead

$\Delta(0) =$  (4.2)

For comparison we give the RPI splice diagrams for the regular and irregular links at infinity. In the next section we describe the relation between the plumbing graph and the splice diagram. The regular link at infinity has RPI diagram

 (4.3)

and the irregular link at infinity has RPI diagram

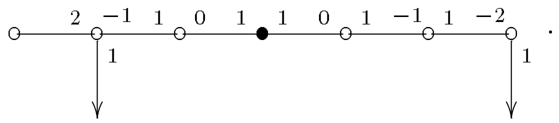
 (4.4)

5. From resolution to RPI diagram

We recall the construction of a splice diagram from our plumbing graph, as described in [**5**]. Thus let $\Delta(c)$ be a plumbing graph, constructed as in the previous section, and Δ the same graph with arrows deleted, so Δ just represents the curve configuration D.

The *intersection matrix* $A(\Delta(c)) = A(\Delta)$ is the square matrix of intersection numbers of the curves of the curve configuration D. Thus the i-th diagonal entry is the selfintersection weight at the i-th vertex and for $i \neq j$ the (i,j)-entry is 1 or 0 according as the i-th and j-th vertices are connected by an edge or not.

Our initial version of the RPI splice diagram will be the same underlying graph as $\Delta(c)$, but with the vertex weights replaced by edge weights that are computed as follows. Given an edge e of $\Delta(c)$, its removal splits $\Delta(c)$ into two graphs Δ_1 and Δ_2, one of which, say Δ_2, contains the root vertex. The near and far weights on e will then be the determinants $\det(-A(\Delta_1))$ and $\det(-A(\Delta_2))$ respectively. Of course, if Δ_1 consists of a single arrowhead then the far weight is 1, while if it consists of a single non-arrowhead vertex then the far weight is irrelevant and may be omitted. We will denote the resulting splice diagram by $\Gamma_0(c)$. For example, from the graph (4.1) above we get

$\Gamma_0(c) =$ 

$\Gamma_0(c)$ is generally a non-minimal splice diagram (in particular it usually has many vertices of valency 2). We shall denote by $\Gamma(c)$ the result of reducing $\Gamma_0(c)$ to get a minimal rooted splice diagram. Thus, in the above example we get the RPI splice diagram (4.3) claimed in the previous section.

Proposition 5.1. *$\Gamma(c)$ is the minimal RPI diagram at infinity for f at c in the sense of Theorem* 1.1.2.

Proof. Note first that the effect on $A(\Delta)$ of a blow-up is to replace $A(\Delta)$ by $A(\Delta) \oplus (-1)$ and then perform some row and column operations. It therefore follows by induction that $\det(-A(\Delta)) = -1$.

According to **[5, sect. 20]** the splice diagram $\Gamma(c)$ constructed above is a splice diagram for the link represented by the given plumbing graph if $\det(-A(\Delta)) > 0$ and for the mirror image link if $\det(-A(\Delta)) < 0$. Thus $\Gamma(c)$ represents the mirror image of the link represented by the plumbing diagram. This is $\mathcal{L}(f^{-1}(c), \infty)$, since the plumbing diagram represents the link in S^3 with S^3 oriented as the boundary of a neighbourhood of the curve configuration while $\mathcal{L}(f^{-1}(c), \infty)$ has S^3 oriented as the boundary of a large disk in $\mathbb{C}^2$.

We next verify that $\Gamma(c)$ is an RPI diagram. It is an easy calculation to see that reducing a non-minimal RPI diagram to its minimal representative preserves the RPI property. It thus suffices to show that $\Gamma_0(c)$ is an RPI diagram.

If we remove the root vertex from Δ then Δ falls into a collection of graphs, each of which represent the result of iterated blowing up starting from nothing. Each of these graphs thus has negative definite intersection matrix, so any subgraph Δ_1 of Δ that does not include the root vertex has negative definite intersection matrix. Each near weight of $\Gamma_0(c)$ is $\det(-A(\Delta_1))$ for some such Δ_1 and is therefore positive. On the other hand, a simple induction over the blowing up process shows that each edge determinant of $\Gamma_0(c)$ is -1. Thus $\Gamma_0(c)$ is an RPI diagram.

We must finally verify that $\Gamma(c)$ is the RPI diagram that corresponds to f in the sense of Theorem 1.2.2. That theorem says it is the correct one if the virtual component of the root vertex is the link at infinity of a generic line. But a generic line meets the root curve E_0 transversally in a single point, so it is represented by an arrow at the root vertex, so its link at infinity is indeed the virtual component of the root. ∎

We shall need the following observation about the relationship between the non-minimal diagram $\Gamma_0(c)$ and the minimal diagram $\Gamma(c)$. Recall that, by Lemma 4.1, the removal of the horizontal vertices makes Δ fall into pieces which are the components of the subgraphs Δ_t. Let Δ'_t be such a component with $t \neq \infty, c$.

Lemma 5.2. *Δ'_t reduces to a single vertex of valency 1 in $\Gamma(c)$.*

Proof. Δ'_t corresponds to a connected subgraph of $\Gamma(c)$ with no arrowheads which is connected to the rest of Γ_c by a single edge. In a minimal RPI diagram such a subgraph must consist of a single vertex. ∎

We next interpret the linking coefficients l_v for $\Gamma(c)$ in terms of the plumbing diagram. Again, it is convenient to work also with the unreduced diagram $\Gamma_0(c)$.

Any non-horizontal curve E of D occurs in $\overline{f}^{-1}(t)$ for some $t \in \mathbb{P}^1$. Define m_E to be the multiplicity of E as a component of $\overline{f}^{-1}(t)$.

Proposition 5.3. *Given a vertex v of $\Gamma(c)$ or $\Gamma_0(c)$, let E be the corresponding component* *curve of D. Then the linking coefficient l_v is as follows*

$$\begin{aligned} l_v &= m_E && \text{if } \overline{f}(E) = \infty \\ &= -m_E && \text{if } \overline{f}(E) = c \\ &= 0 && \text{if } \overline{f}(E) = t \neq \infty, c \\ &= 0 && \text{if } E \text{ is horizontal.} \end{aligned}$$

Note that the third case means v is as in Lemma 5.2.

Proof. It suffices to give the proof for $\Gamma_0(c)$ since the linking coefficient at a vertex of $\Gamma(c)$ is equal to the linking coefficient at the corresponding vertex of $\Gamma_0(c)$.

Let N be a neighbourhood of D in X that is plumbed together from disk-bundle neighbourhoods of the curves of D and let B_E be a disk fiber of the disk-bundle neighbourhood of E. Let $\partial_E = \partial B_E$ and let A_E be a 2-chain in the 4-ball $D^4 = X - \operatorname{int}(N)$ with boundary $-\partial_E$ (here $-\partial_E$ means ∂_E with reversed orientation). Then $A_E \cup B_E$ is a 2-cycle in X. Since any two fibers of $\overline{f}$, considered as divisors on X, are homologous to each other, we have equality of intersection numbers

$$(A_E \cup B_E) \cdot (\overline{f}^{-1}(\infty)) = (A_E \cup B_E) \cdot (\overline{f}^{-1}(c)). \tag{5.1}$$

Suppose now first that $E \subset \overline{f}^{-1}(\infty)$. Then the left side of (5.1) is $B_E \cdot \overline{f}^{-1}(\infty) = l_E$. The right side is $A_E \cdot (f^{-1}(c) \cap D^4) = \ell(\partial A_E, \partial(f^{-1}(c) \cap D^4)) = \ell(-\partial_E, L)$, where L is our link at infinity. On the other hand, $-\partial_E$ is the virtual component S_v for the vertex v of $\Gamma_0(c)$ corresponding to E (cf. **[5, sect. 23]**; the orientation reversal here is the same one as in the proof of Proposition 5.1). Thus $l_E = \ell(S_v, L) = l_v$.

Suppose next that $E \subset \overline{f}^{-1}(t)$ with $t \neq \infty$. Then the left side of (5.1) vanishes. By the computation of previous paragraph, the right side is l_v if $t \neq c$ and $l_v + l_E$ if $t = c$. Thus $l_v = -l_E$ or 0 according as $t = c$ or not.

Finally, if E is a horizontal curve then the left side of (5.1) vanishes and the right side is l_v, so $l_v = 0$. ∎

Proof of Theorem 1.5. It will be convenient to prove a modified version of Theorem 1.5.

If Γ is an RPI splice diagram, we will denote by Γ'' the result of using Lemma 1.3 to add a horizontal vertex on *every* horizontal edge (rather than just the deficient horizontal edges), and call it the *extended RPI diagram*. We can apply the process of Lemma 1.4 to Γ'' rather than Γ and formulate Theorem 1.5 for extended minimal RPI diagrams rather than minimal RPI diagrams. Lemma 1.4 remains valid if we include all horizontal vertices, rather than just deficient horizontal vertices, in the replacement process — non-deficient horizontal vertices are simply unaffected by the process. It follows that Theorem 1.5 is equivalent to the analogous theorem for extended RPI diagrams.

If instead of reducing $\Gamma_0(c)$ all the way to $\Gamma(c)$, we retain all horizontal vertices, we obtain the extended RPI diagram $\Gamma''(c)$. Let $\Gamma''(c)_{\geq 0}$ be the full subdiagram of $\Gamma''(c)$ on the vertices with $l_v \geq 0$ along with the weights l_v at its vertices. $\Gamma''(c)_{\geq 0}$ depends only on the plumbing graph Δ and the multiplicities m_E, so it is independent of the value of c. The uniqueness statement of Lemma 1.4 can be formulated that $\Gamma''(c)^{\text{reg}}$ only depends on $\Gamma''(c)_{\geq 0}$. Thus $\Gamma''(c)^{\text{reg}}$ is independent of c. Since $\Gamma''(c)^{\text{reg}} = \Gamma''(c)$ if $\Gamma''(c)$ is regular, Theorem 1.5 (for extended diagrams) follows. ∎

Proof of Theorem 1.6. In **[13]** it is described how to obtain all minimal RPI diagrams for a given RPI link from a certain minimal (unrooted) splice diagram by adding certain subgraphs, all of whose vertices have positive linking coefficients. It follows that the invariants "deficiency" and "potential", which depend only on vertices with $l_v \leq 0$, do not depend on the particular minimal RPI diagram used.

Let Γ'' be the minimal extended regular RPI diagram for a polynomial $f: \mathbb{C}^2 \to C$. Given a horizontal node u of Γ'', let E be the corresponding horizontal curve of D. Consider the branched covering map $\overline{f}: E \to \mathbb{P}^1$. For $t \in \mathbb{P}^1$ denote by d_t the number of inverse images of t in E. Let $\overline{d}$ be the generic value of d_t, that is, the degree of the branched covering map. Then the standard formula for euler characteristic of a branched cover gives

$$2\overline{d} - 2 = \sum_{t \in \mathbb{P}^1} (\overline{d} - d_t).$$

Now the inverse image of ∞ in E consists of the unique point where E meets D_∞, so $d_\infty = 1$. By separating $t = \infty$ from the above sum we get

$$\overline{d} - 1 = \sum_{t \in \mathbb{C}} (\overline{d} - d_t). \tag{5.2}$$

If $t \in \mathbb{C}$ is a branch point, that is $d_t < \overline{d}$, then u is a deficient node in the splice diagram $\Gamma''(t)$, so it will be as in one of the figures $(1a, b)$ preceding Lemma 1.4. At each point of E that maps to t there must be a component of $\overline{f}^{-1}(t)$ meeting E, so the number d of fig. $(1a, b)$ is just d_t. Moreover, if u is as in case $(2a)$ in the regular diagram Γ'' then there is a $\tau \in \mathbb{C}$ for which a component of D_τ meets E and leads to the outgoing edge with near weight α at v. For this value $t = \tau$, and this value alone, the vertex u of $\Gamma''(t)$ is in case $(1a_2)$. Thus we see that the deficiency $(\overline{d} - d, \epsilon)$ of u is non-vanishing precisely if $t \in \mathbb{C}$ is either a branch point or $t = \tau$ (or both). Moreover, we see that the sum over all $t \in \mathbb{C}$ of the deficiencies at u is the potential at u as follows: for the first component this is just equation (5.2), while for the second component ϵ it is the uniqueness of the value $t = \tau$ that leads to $\epsilon = 1$. ∎

The only things remaining to be checked are the statements about horizontal curves in the proof of Theorem 1.9. Consider the extended regular RPI splice diagram Γ'' in which all horizontal edges have been replaced by horizontal nodes. As observed above, the horizontal nodes of Γ'' correspond one-one with horizontal curves of D, and the degree of $\overline{f}$ on a horizontal curve E is the number of arrows at the corresponding vertex of Γ''. This is what was claimed in the proof of Theorem 1.9.

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Department of Mathematics
The University of Melbourne
Parkville, Vic 3052
Australia
Email: neumann@maths.mu.oz.au