1 Motivation and Background

My research is in symplectic and contact geometry with a focus on moduli problems in contact topology. I have been working to provide foundations and computations of contact homology invariants. These Floer theoretic invariants are closely related to symplectic homology and have applications to singularity theory, dynamics, string theory, mirror symmetry, knot theory, and symplectic embedding problems. My research combines techniques from differential topology and geometry, partial differential equations, complex geometry, and algebraic topology and geometry.

Symplectic and contact structures first arose in the study of classical mechanical systems such as those describing planetary motion and wave propagation; see [Ar78, Ne16]. Solutions of these systems are flow lines of either the Hamiltonian vector field on a symplectic manifold or the Reeb vector field on a contact manifold, see Figure 1.

Understanding the evolution and distinguishing transformations of these systems necessitated the development of global invariants of symplectic and contact manifolds which encode structural aspects of the Reeb and/or Hamiltonian flows. These invariants are built out of moduli spaces of pseudoholomorphic curves [Gr85], which are solutions to a nonlinear Cauchy-Riemann equation. They originated from Floer’s breakthrough [Fl88] to combine classical variational methods, Gromov’s theory of pseudoholomorphic curves, and Witten’s interpretation of Morse theory [Wi82].

Formally, a contact structure $\xi$ is a maximally non-integrable hyperplane distribution. This is in contrast to an integrable hyperplane distribution, whose hyperplanes are given by the tangent spaces of a submanifold; see Figure 2. Any 1-form $\lambda$ whose kernel defines a contact structure is called a contact form. The Reeb vector field $R_\lambda$ depends on the choice of contact form $\lambda$ and is defined by $\lambda(R_\lambda) = 1, \ d\lambda(R_\lambda, \cdot) = 0$. A closed Reeb orbit of period $T > 0$ is defined up to parametrization by:

$$\gamma : \mathbb{R}/T\mathbb{Z} \to M, \ \dot{\gamma}(t) = R_\lambda(\gamma(t)).$$

A closed Reeb orbit is said to be simple, or equivalently embedded, whenever (1) is injective. The linearized Reeb flow for time $T$ defines a symplectic linear map of $(\xi, d\lambda)$ and if this map does not have 1 as an eigenvalue then $\gamma$ is said to be nondegenerate. The contact form $\lambda$ is called nondegenerate if all Reeb orbits are nondegenerate; generic contact forms have this property.

Constructions of moduli based invariants looked promising with the advent of a comprehensive symplectic field theory, announced in 2000 by Eliashberg, Givental, and Hofer [EGH00]. Symplectic field theory generalizes Floer and Gromov-Witten theories by studying moduli spaces of pseudoholomorphic curves from punctured Riemann surfaces to certain noncompact symplectic manifolds with asymptotics on periodic orbits of the Hamiltonian or Reeb vector field. There is a Fredholm theory describing these moduli spaces as the zero set of an infinite dimensional bundle. To obtain contact and symplectic invariants one must first regularize these moduli spaces so that the (compactified) moduli spaces can be given the structure of a smooth manifold or orbifold, possibly with boundary and corners.

When multiply covered curves are present, as in symplectic field theory, there is typically a failure of transversality of the zero set which describes the moduli space these curves. The most general approach to regularize these moduli spaces is to abstractly perturb the standard nonlinear Cauchy-Riemann equation. However, the use of abstract perturbations is a lengthy technical endeavor which is often not suitable for applications and in some cases not well understood by the community.
My research, in part joint with Hutchings, makes use of more direct methods to extend the transversality theory for the standard pseudoholomorphic curve equation and to isolate precise phenomena which can be accounted for with established geometric and analytic methods. In particular, we have provided foundations for a subset of symplectic field theory known as (cylindrical) contact homology, whose chain complex is generated by certain closed Reeb orbits and whose differential counts pseudoholomorphic cylinders interpolating between these Reeb orbits, by extending the methods of my thesis [Ne13, Ne15]. Our work [HN16, HN2, HN3, HN4] gives a rigorous definition of cylindrical contact homology for dynamically convex contact forms in three dimensions (cf. Definition 4.1), invariance in the dynamically convex case, a definition of local contact homology in any dimension, and a substitute for cylindrical contact homology in higher dimensions in the absence of contractible Reeb orbits. These foundational results were not previously available in the literature, cf. [Bo02, Bo09, BO09, BCE07, Us99, MLY04].

Section 2 gives an overview of my foundational results with Hutchings. A discussion of the computational methods I developed in [Ne] is given in Section 3. Applications of these computational methods to singularity theory are given in Section 3.1 and to dynamics in Section 3.2. A brief discussion of research carried out by my undergraduate mentees [AHNS17, CN18] can be found in Section 3.3. Finally, technical aspects of my foundational work in defining contact homology and related invariants can be found in Section 4.

2 Overview of foundational results

Cylindrical contact homology is in principle an invariant of contact manifolds \((Y, \xi)\) that admit a non-degenerate contact form \(\lambda\) without Reeb orbits of certain gradings. The cylindrical contact homology of \((Y, \xi)\) is defined by choosing a nondegenerate contact form \(\lambda\), taking the homology of a chain complex over \(\mathbb{Q}\) which is generated by “good” Reeb orbits and denoted by \(\mathbb{C}^\infty_{EGH}(Y, \lambda, J; \mathbb{Q})\), and whose differential \(\partial_{EGH}\) counts \(J\)-holomorphic cylinders in \(\mathbb{R} \times Y\) for a suitable almost complex structure \(J\). The grading on the complex is given by the Conley-Zehnder index, a winding number associated to the path of symplectic matrices obtained from linearizing the flow along \(\gamma\), restricted to \(\xi\). The Conley-Zehnder index is denoted by \(\text{CZ}_\tau(\gamma)\) and typically depends on a choice of trivialization \(\tau\). Unfortunately, in many cases there is no way to choose \(J\) so as to obtain the transversality for holomorphic cylinders needed to define \(\partial_{EGH}\), to show that \((\partial_{EGH})^2 = 0\), and to prove that the homology is invariant of the choice of \(J\) and \(\lambda\).

In [HN16] we extended the methods from my thesis [Ne13, Ne15] and showed that the cylindrical contact homology differential \(\partial_{EGH}\) can be defined by directly counting pseudoholomorphic cylinders in dimension 3 for the class of dynamically convex contact forms, cf. Definition 4.1 without any abstract perturbation of the Cauchy-Riemann equation. In conjunction with the axiomatic Morse-Bott formalism that we developed in [HN3], our work [HN2, HN4] establishes the invariance of cylindrical contact homology under choices of \(J\) and \(\lambda\).

**Theorem 2.1.** [HN16, HN2, HN4] Let \(\lambda\) be a nondegenerate, dynamically convex contact form on a closed three-manifold \(Y\). Suppose further that:

(*) A contractible Reeb orbit \(\gamma\) has \(\text{CZ}(\gamma) = 3\) only if \(\gamma\) is embedded\(^1\).

Then for generic \(\lambda\)-compatible almost complex structures \(J\) on \(\mathbb{R} \times Y\), \((\mathbb{C}^\infty_{EGH}(Y, \lambda, J), \partial_{EGH})\), is a well-defined chain complex and its homology is invariant under choices of dynamically convex \(\lambda\) defining the contact structure \(\xi\) and generic \(J\),

\[
\text{CH}^\infty_{EGH}(Y, \xi; \mathbb{Q}) := H_*(\mathbb{C}^\infty_{EGH}(Y, \lambda, J), \partial_{EGH}).
\]

Under the assumptions of Theorem 2.1 we also define the following related contact homology theories. Further details are provided in Section 4.1.

\(^1\)Work in progress of D. Cristofaro-Gardiner, M. Hutchings, and B. Zhang will allow this assumption to be dropped.
Theorem 2.2. [HN2, HN4] Under the assumptions of Theorem 2.1,

(i) There is a well-defined nonequivariant contact homology defined with coefficients in \( \mathbb{Z} \) which is a contact invariant and denoted by \( \text{NCH}_* \).

(ii) There is a well-defined equivariant contact homology defined with coefficients in \( \mathbb{Z} \) which is a contact invariant and denoted by \( \text{CH}^{S^1}_* \).

(iii) There is a canonical isomorphism, \( \text{CH}^{S^1}_* \simeq \text{CH}^{EGH}_* \otimes \mathbb{Q} \).

As a result of Theorem 2.2(iii) we call \( \text{CH}^{S^1}_* \) an integral lift of cylindrical contact homology. Our constructions also produce a definition of local contact homology in any dimension, a theory which when well-defined yields significant dynamical results [GG09, GGM15, GGM2, HM15]. Long term dynamical applications include existence results for homoclinic orbits in Hamiltonian systems [HW90] and the study of periodic orbits of iterated disk maps [BrHof12]. The analytic techniques we developed also have broader applications. For example, recent work of Cristofaro-Gardiner, Hind, and McDuff [CGHM] made use of the analytic techniques we pioneered in [HN16] to construct a sophisticated obstruction theory for certain higher dimensional symplectic embedding problems.

Our work is connected to several other contact and symplectic invariants. In particular, the nonequivariant and integral theories are expected to be isomorphic\(^2\) to positive symplectic homology and positive \( S^1 \)-equivariant symplectic homology, respectively. Symplectic homology is a Floer type invariant of symplectic manifolds with contact type boundary which detects the symplectic structure of the interior and the Reeb dynamics of the boundary [CFHW, Se08, Vi99]. The positive symplectic homology complex is a quotient complex of the total complex by the complex generated by critical points in the interior of the symplectic manifold.

Our work should allow for further progress on the classification of Legendrian knots up to Legendrian isotopy in closed dynamically convex contact 3-manifolds. Legendrian knots are smooth knots whose tangent vectors live in the contact structure, see Figure 3, and are said to be Legendrian isotopic when they are isotopic through a family of Legendrian knots. The classification of Legendrian knots is an interesting problem because Legendrian knots are surgery loci for constructions of new contact manifolds and have close ties with smooth knots via geometric and quantum knot invariants. Connections with our work to symplectic homology [BO17] and Legendrian contact homology [BEE12, EES07, EENS13] are included in Section 4.2.

There are the following alternate approaches to defining contact homology invariants.

Remark 2.3. (On related approaches)

(Alternatives to \( \text{CH}^{EGH}_* \)) Bao and Honda [BaHon1] give a definition of a contact invariant akin to cylindrical contact homology for contact manifolds which admit no contractible Reeb orbits in dimension 3, defined with coefficients in \( \mathbb{Q} \). It remains to show this definition agrees with the “classical” one from [EGH00]. Gutt [Gu] has shown that the positive equivariant portion of symplectic homology is also an invariant for some contact manifolds, see also Remark 4.4.

(Kuranishi) Pardon’s approach [Pa] to defining full contact homology via virtual fundamental cycles is only expected to yield a definition of cylindrical contact homology in the absence of contractible Reeb orbits. Bao and Honda [BaHon2] give a definition of the full contact homology differential graded algebra for any closed contact manifold in any dimension. These approaches make use

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\(^2\)See Section 4.2 for a more precise explanation.
of Kuranishi structures to construct contact and symplectic invariants and while they hold more generally, they are also more abstract and hence more difficult to work with in computations and applications; see also [FO99], [FO31]-[FO34], [IP], [Joy], [MW1]-[MW17b], [Pa16], [TZ1], [TZ2].

(Polyfolds) Hofer, Wysocki, and Zehnder have developed the abstract analytic framework [HWZI]-[HWZII], [HWZ-gw], collectively known as polyfolds, to systematically resolve issues of regularizing moduli spaces. Contact homology awaits foundations via polyfolds and the use of abstract perturbations can make computations difficult. Hofer and I plan to investigate the connections between the polyfold approach and the one taken by Hutchings and myself.

3 Computations of contact invariants and their applications

In [Ne] I proved a correspondence between the cylindrical contact homology of certain 3-dimensional circle bundles over closed, oriented surfaces and the Morse homology of the base. These circle bundles are prequantization bundles [BW58], to which is associated a canonical contact form $\lambda_0$. A specific example of a prequantization bundle is the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ from Figure 1.

**Definition 3.1** (Prequantization). Let $(\Sigma^{2n-2}, \omega)$ be a closed symplectic manifold such that the cohomology class $-\omega/(2\pi) \in H^2(\Sigma; \mathbb{R})$ is the image of an integral class $e \in H^2(\Sigma; \mathbb{Z})$. The principal $S^1$ bundle $\pi : Y^{2n-1} \to \Sigma$ with first Chern class $e$ is the **prequantization bundle**. The prequantization bundle $Y$ admits a contact form $\lambda_0$ which is the real-valued connection 1-form $\lambda$ on $Y$ whose curvature is $\omega$.

The form $\lambda_0$ is degenerate, meaning that the closed Reeb orbits are non-isolated. The Reeb orbits of $R_{\lambda_0}$ are in fact the fibers of the $S^1$ fibration. We can directly perturb the critical manifolds realized as Reeb orbits with a Morse-Smale function $H$ on the base $B$ that is invariant under the $S^1$-action of the bundle $S^1 \hookrightarrow Y \xrightarrow{\pi} B$. The perturbation,

$$\lambda_\epsilon = (1 + \epsilon \pi^* H)\lambda_0,$$

yields the Reeb dynamics,

$$R_\epsilon = \frac{1}{(1 + \epsilon \pi^* H)} R + \frac{\epsilon}{(1 + \epsilon \pi^* H)^2} \tilde{X}_H,$$

where $X_H$ is a Hamiltonian vector field on $S^2$ and $\tilde{X}_H$ its horizontal lift.

The only fibers that persist as Reeb orbits of $R_\epsilon$ are those over the critical points of $H$. Additional Reeb orbits of $R_\epsilon$ must come from horizontal lifts of closed orbits of $X_H$. Since $\epsilon H$ and $\epsilon dH$ are small, these Reeb orbits all have action $A(\gamma) := \int_{\gamma} \lambda_\epsilon$ greater than $\sim \epsilon$. Thus, up to large action, the only closed Reeb orbits left intact by this perturbation must occur as a multiple cover of a simple Reeb orbit lying over a critical point of $H$. In particular, the only orbits which generate the chain complex up to large action must project to critical points on the base.

In [Ne] I proved that the pseudoholomorphic cylinders counted by the differential $\partial^{\text{EGH}}$ correspond to the count of Morse-Smale trajectories of $H$ in the base. More precisely, we have the following result.

**Theorem 3.2.** [Ne] Fix $L > 0$ and let $J$ be a generic $\lambda_\epsilon$ compatible almost complex structure on a prequantization bundle $(Y, \tilde{\xi})$ over an integral closed symplectic surface $(\Sigma^2, \omega)$. If $H$ is a Morse-Smale function on $(\Sigma, \omega)$ which is $C^2$ close to 1 then there exists $\epsilon > 0$ such that all orbits $\gamma$ with $A(\gamma) < L$ project to critical points of $H$. Then with respect to each free homotopy class, $CH^{\text{EGH},L}_*(Y, \lambda_\epsilon, J; Q)$ is the homology of the chain complex generated by copies of $H^{\text{Morse}}_*(\Sigma, H; Q)$ with $\partial_\epsilon = \partial^{\text{Morse}}_H$. 

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Figure 4: $-\nabla H$ for $H = z$ with a fiber over $S^2$, respectively $S^2/Z_2$; note $X_H = J\nabla H$.
Remark 3.3 (Applicability to higher dimensional prequantization bundles). Recent work by Wendl [We], establishes transversality for certain multiply covered curves in higher dimensions. It is thus conceivable that Theorem 3.2 could be generalized to apply to higher dimensional prequantization bundles over closed monotone symplectic manifolds.

There is a proportionality between the action of these orbits which project to critical points of \( H \) and their index. This filtration on both the action and index leads to a formal version of filtered contact homology, allowing one to recover cylindrical contact homology in a well-defined way [Ne, §4]. In particular, the use of direct limits in conjunction with the above geometric perturbation allows us avoid the analytic difficulties of directly degenerating moduli spaces of pseudoholomorphic cylinders. The methods of Theorem 3.2 will permit direct computations of nonequivariant contact homology as defined in [HN2, HN4].

We illustrate how Theorem 3.2 can be used to compute \( \text{CH}^{EGH}_* (S^3, \xi_{std}) \) by taking \( H \) to be the height function on \( S^2 \), see also Figure 4. The simple Reeb orbit projecting to the north pole has index 2 and the one projecting to the south pole has index 0. The index increases by 4 after each iteration of the Reeb orbit.\(^3\)

We thus recover the well known computation of cylindrical contact homology for the sphere \( (S^3, \xi_{std}) \).

Example 3.4 ([Ne13]). The cylindrical contact homology for the sphere \( (S^3, \xi_{std}) \) is given by

\[
\text{CH}^{EGH}_* (S^3, \xi_{std}) = \begin{cases} Q^n & * \geq 2, \text{ even}; \\ 0 & * \text{ else}. \end{cases}
\]

If we consider the lens space \( (L(n+1, n), \xi_{std}) \) with its contact structure induced by the one on \( S^3 \), then the free homotopy classes of the orbits of \( R_\epsilon \) over critical points of \( H \) are in 1-1 correspondence with the conjugacy classes of \( L(n+1, n) \). While (*) from Theorem 2.1 is not satisfied, the methods of [Ne15, Ne] still allow us to conclude that we get a well-defined chain complex which is invariant under the choice of almost complex form and a large class of contact forms.

Theorem 3.5 ([Ne]). The cylindrical contact homology for the sphere \( (L(n+1, n), \xi_{std}) \) is given by

\[
\text{CH}^{EGH}_* ((L(n+1, n), \xi_{std}) = \begin{cases} Q^{n+1} & * \geq 2, \text{ even}; \\ Q^n & * = 0; \\ 0 & * \text{ else}. \end{cases}
\]

This computation agrees with what is obtained in [AHNS17] for the link of the \( A_n \) singularity, which is to be expected given that these two manifolds are contactomorphic.

These computational methods have numerous applications to singularity theory and dynamics, which we explain further in Sections 3.1 and 3.2 respectively.

3.1 Topological applications and McKay correspondence

In future work, I plan to generalize Theorem 3.2 so that one can work with prequantization bundles over symplectic orbifolds. In particular, I expect that the contact homology differential agrees with the Morse orbifold differential. This would allow us to compute cylindrical contact homology of Seifert fiber spaces and for three dimensional links of many weighted homogeneous polynomials. Such links are examples of prequantization bundles over symplectic orbifolds. When the defining polynomial is homogeneous the link can be realized as a prequantization bundle over a symplectic manifold.

This generalization would yield a Floer theoretic interpretation of the McKay correspondence [IM96] in terms of the Reeb dynamics of the links of the simple singularities. The simple singularities can be characterized as the absolutely isolated double point quotient singularity of \( C^2/\Gamma \), where \( \Gamma \) is a finite subgroup of \( \text{SL}(2; \mathbb{C}) \) or in terms of an isolated singularity of a hypersurface in \( \mathbb{C}^3 \) cut out by certain weighted

\[^3\text{The grading is } |\gamma^k_p| = CZ(\gamma^k_p) - 1 = 4k - 2 + \text{index}_p(H) \text{ where } p \in \text{Crit}(H), \text{ and } \gamma^k_p \text{ is the } k\text{-fold iterate of a simple Reeb orbit } \gamma \text{ over } p.\]
homogenous polynomials. This agrees with work in progress by McLean and Ritter [McRi] which establishes a relationship between the cohomological McKay correspondence and symplectic homology. The Floer theoretic McKay correspondence conjecture is stated below.

**Conjecture 1.** The cylindrical contact homology of the link of a simple singularity is a free $\mathbb{Q}[u]$ module of rank equal to the number of conjugacy classes of the respective finite subgroup $\Gamma$ of $\text{SL}(2; \mathbb{C})$.

My approach to prove this above conjecture relies on the fact that the link of the simple singularities is (contact) diffeomorphic to $S^3/\Gamma$, so that one could appeal to a generalized version of Theorem 3.2 as follows. First, one must pick $H$ in (2) so that $X_H$ is invariant under the image of $\Gamma$ in $\text{SO}(3)$. In the case of the $E_6$-singularity $\Gamma = \mathbb{T}^+$ is the binary tetrahedral group and taking $H = \text{xyz}$ yields the desired $\mathbb{T}$-invariant vector field $X_H$ on $S^3$, as in Figure 5. Then after quotienting out the entire bundle by the action of $\Gamma$, the free homotopy classes of the orbits of $R_\epsilon$ over critical points of $H$ are in 1-1 correspondence with the conjugacy classes of $\Gamma$.

As a result, the rank of $CH^*_E \mathbb{G}_{\mathbb{H}}$ is governed by the number of these conjugacy classes, producing the desired Floer theoretic McKay correspondence. An interesting by product of this approach is that the presentation of the spherical manifolds $S^3/\Gamma$ as Seifert fiber spaces is realized by the perturbed Reeb dynamics. We obtain $S^1$-bundles over the 2-sphere with the expected number of exceptional fibers in 1-1 correspondence with the generators of the chain complex. Moreover, the weight of an exceptional fibers corresponds to the number of free homotopy classes that can be realized as iterates of the Reeb orbit in that fiber. Finally, these methods also suggest that one can generalize Ritter’s work [Ri14] in regards to Floer theory for negative line bundles via Gromov-Witten theory of orbifolds.

### 3.2 Applications to dynamics

Many interesting dynamical applications have been explored under the assumption that cylindrical or local contact homology would one day become a well-defined contact invariant and that computational methods as in Theorem 3.2 could be made rigorous. Detailed below are several results which I expect can be rigorously achieved through a combination of computational results and the foundational work with Hutchings.

Ginzburg, Gürel, and Macarini explain in [GGM2, §6] how one could use cylindrical contact homology in conjunction with Morse-Bott methods to refine [GGM2, Theorem 2.1] in regards to lower bounds of geometrically distinct contractible (non-hyperbolic) periodic Reeb orbits of prequantization bundles. Another application is a refinement of the Conley Conjecture [GGM15, Theorem 2.1], which under certain assumptions guarantee that for every sufficiently large prime $k$, the Reeb flow has a simple closed orbit in the $k$-th iterate of the free homotopy class of the fiber. I expect that one can adapt the proofs of Theorems 2.1, 2.2, and 3.2 to obtain these extensions for prequantization bundles $(Y^3, \xi)$ over closed oriented surfaces $(\Sigma^2, \omega)$.

In [HM15] Hryniewicz and Macarini give several applications of local contact homology which are conditional on completion of foundational work. Preliminary discussions with these authors and Hutchings indicate that Theorems 2.1 and 2.2 should be sufficient to provide the necessary foundations in dimension 3. It is possible that partial results can be made rigorous in higher dimensions by way of [HN2]. This would then permit us to relate properties of the contact homology such as homological unboundedness and sufficiently large growth rate to properties of Reeb orbits of an arbitrary contact form. One of the more interesting applications is the following generalization of a result by Gromoll-Meyer [GM69].
Conjecture 2 (Generalization of Gromoll-Meyer). Let $M$ be a closed oriented manifold and assume that the rank of $H_2(LM/S^1, M; \mathbb{Q})$ is asymptotically unbounded. Then every hypersurface in $T^*M$ which is fiberwise starshaped with respect to the zero section has infinitely many, geometrically distinct, periodic orbits. In particular, the result holds if $M$ is simply connected and its cohomology algebra over $\mathbb{Q}$ is not generated by a single class.

The last dynamical application is to establish growth rates of the number of periodic Reeb orbits of prequantization bundles over closed oriented surfaces. Theorems 2.1, 2.2, and Theorem 3.2 in conjunction with Vaugon’s work [Va15] are expected to produce an expression of the growth rate in terms of the Euler characteristic of the base and the Euler number of the fibration.

3.3 Undergraduate research supervision

Many aspects of computing Floer theoretic invariants can be made appropriate research projects for graduate students and talented undergraduates. The first REU project I ran at Columbia University in 2014-2015 resulted in a direct verification of the Floer theoretic McKay correspondence for the link of the $A_n$-singularity [AHNS17]. The $A_n$ singularity is one of the simple singularities and its link is given by $S^3 \cap f^{-1}(0)$ where $f(z_0, z_1, z_2) = z_0^{n+1} + z_0^2 + z_2^2$.

Theorem 3.6. [AHNS17] The positive $S^1$-equivariant symplectic homology of the $A_n$-link is a free $\mathbb{Q}[u]$ module of rank equal to the number of conjugacy classes of the finite cyclic subgroup $A_n$ of $SL(2; \mathbb{C})$.

In work from 2015-2016 with Christianson [CN18], a former REU mentee, we found new obstructions to symplectic embeddings of the four-dimensional polydisk,

$$ P(a, 1) = \{ (z_1, z_2) \in \mathbb{C}^2 \mid \pi|z_1|^2 \leq a, \pi|z_2|^2 \leq 1 \}, $$

into the ball,

$$ B(c) = \{ (z_1, z_2) \in \mathbb{C}^2 \mid \pi|z_1|^2 + \pi|z_2|^2 \leq c \}, $$

extending work done by Hind-Lisi [HL15] and Hutchings [Hu16]. We made use of Hutchings’ refinement of embedded contact homology from [Hu16], which established a necessary condition for one convex toric domain, such as an ellipsoid or polydisk, to symplectically embed into another.

Theorem 3.7 ([CN18]). Let $2 \leq a < \frac{\sqrt{7}-1}{\sqrt{7}-2} = 2.54858...$. If the four dimensional polydisk $P(a, 1)$ symplectically embeds into the four dimensional ball $B(c) = E(c, c)$ then $c \geq 2 + \frac{a}{2}$.

Schlenk’s folding construction permits us to conclude our bound on $c$ is optimal, [Sc15, Prop. 4.3.9]. Additionally, we proved that if certain symplectic embeddings of four dimensional convex toric domains exist then a modified version of the combinatorial criterion from [Hu16] must hold, thereby reducing the computational complexity of the original criterion from $O(2^n)$ to $O(n^2)$.

4 Technical aspects of contact homology and related invariants

This section provides further details on Theorems 2.1 and 2.2, in regards to the aforementioned class of dynamically convex class of contact forms.

Definition 4.1. (cf. [HWZ99]) Let $\lambda$ be a nondegenerate contact form on a closed three-manifold $Y$. We say that $\lambda$ is dynamically convex if either:

- $\lambda$ has no contractible Reeb orbits, or
- $c_1(\xi)|_{\pi_2(Y)} = 0$, and every contractible Reeb orbit $\gamma$ has CZ($\gamma$) $\geq 3$.

Generic convex, compact, star-shaped hypersurfaces $Y$ in $\mathbb{R}^4$ admit a dynamically convex contact form via the restriction of $\lambda = \frac{1}{2} \sum_{k=1}^{\infty} (x_k y_k - y_k x_k)$ to $Y$.

Theorem 2.1 required the following additional assumption, which we briefly explain.
Remark 4.2 (On the hypotheses of Theorem 2.1 and related future work).

(a) The hypothesis (*) automatically holds when \( \pi_1(Y) \) contains no torsion:

(*) A contractible Reeb orbit \( \gamma \) has \( CZ(\gamma) = 3 \) only if \( \gamma \) is embedded,

(b) In general, the hypothesis (*) can be removed from Theorem 2.1 assuming a certain technical conjecture on the asymptotics of holomorphic curves.

(c) We expect that the hypothesis of dynamical convexity can be further weakened.

4.1 Discussion of Theorem 2.2

Our work [Ne13, Ne15, HN16] demonstrates that generic \( S^1 \)-independent almost complex structures yield sufficient transversality to ensure that \( \partial^{EGH} \) is well-defined and that \( (\partial^{EGH})^2 = 0 \) for large classes of contact manifolds in dimension 3. However, \( S^1 \)-independent almost complex structures do not give sufficient transversality to count index zero cylinders in cobordisms, which is necessary to define the chain maps and chain homotopy equations. This is because the covers of index zero cylinders can live in a moduli space of negative virtual dimension. In order to prove topological invariance of \( CH^*_S(\gamma, \lambda, J) \) in [HN2, HN4] we make use of \( S^1 \)-dependent almost complex structures.

Breaking the \( S^1 \) symmetry invalidates certain properties needed to prove \( (\partial^{EGH})^2 = 0 \) and the chain map and chain homotopy equations. As a result, the use of \( S^1 \)-dependent almost complex structures leads to a Morse-Bott version of the chain complex. The homology of this chain complex is not the desired cylindrical contact homology but rather a nonequivariant version of it. This nonequivariant version is denoted by \( NCH_\ast \) and the content of Theorem 2.2(i). Moreover, the presence of contractible Reeb orbits necessitates the use of obstruction bundle gluing [HT07, HT09], producing a correction term in the expression of the nonequivariant differential [HN4]. The desired cylindrical contact homology \( CH^*_S \) can be regarded as an “\( S^1 \)-equivariant” version of non-equivariant contact homology, and recovering this [HN2, HN4] requires additional family Floer theoretic constructions in the spirit of [BO17, Hu08, SeSm10, Vi99].

Our constructions of nonequivariant contact homology \( NCH_\ast \) and its \( S^1 \)-equivariant version \( CH^*_S \) rely on extracting homological invariants from “Morse-Bott” data in which the “critical set” is a union of manifolds, and the moduli spaces of “flow lines” have evaluation maps taking values in the critical set. This requires a mix of analytic arguments and formal arguments. In [HN3] we isolated the formal arguments which turn analytic data into invariants when the critical set is a union of circles. We expect this formalism to have further applications in wider realm of Floer theory.

Theorem 4.3. [HN3] There is an almost category of “Morse-Bott systems” to which one can associate a “cascade homology” functor as well as a “current homology” functor. Both of these homological theories are homotopy invariant and cascade homology is naturally isomorphic to current homology.

The outcome of these constructions is our integral lift of contact homology, denoted by \( CH^*_S \), and the content of Theorem 2.2(ii)-(iii). We have examples which show that this equivariant version integral lift of contact homology also contains interesting torsion information pertaining to the qualitative behavior of the Reeb dynamics[HN2, §5]. The entirety of Theorem 2.2 allows us to conclude that \( CH^*_S \) is a contact invariant.

4.2 Connections with symplectic homology, linearized and Legendrian contact homology

Linearized contact homology is a variant of a more general contact homology theory that typically depends on a symplectic filling of a contact manifold and requires abstract perturbations or Kuranishi structures to be rigorously defined. The linearized theory is identical to cylindrical contact homology in the dynamically convex case in dimension three. Recent work of Bourgeois and Oancea [BO17] suggests that an isomorphism between the positive part of \( S^1 \)-equivariant symplectic homology and linearized contact
homology should exist in general. Their argument requires the assumption that an almost complex structure $J$ can be chosen so that all relevant moduli spaces of $J$-holomorphic curves are cut out transversely, a condition which is not guaranteed by dynamical convexity.\(^4\)

Our work indicates that the construction of a geometric isomorphism must involve an obstruction bundle contribution term in the presence of contractible orbits. We plan to pursue this avenue of study in the coming year, and expect an affirmative answer to the following conjecture.

**Conjecture 3.** Under the assumptions of Theorem 2.1 there exist natural isomorphisms,
\[
\begin{align*}
NCH^*_+(Y,\xi) &\simeq SH^*_+(\mathbb{R} \times Y,\xi;\mathbb{Z}); \\
CH^{S^1}_+(Y,\xi) &\simeq SH^{+,S^1}_+(\mathbb{R} \times Y,\xi;\mathbb{Z}).
\end{align*}
\]

**Remark 4.4.** More precisely, $NCH^*_+(Y,\xi)$ should be isomorphic to the positive symplectic homology $SH^*_+$ of a filling of $Y$. In fact, for a contact manifold with no contractible Reeb orbits, or more generally for a contact manifold of dimension $2n - 1$ in which every contractible Reeb orbit satisfies $CZ(\gamma) > 4 - n$, one can define positive symplectic homology directly in terms of the symplectization, without reference to a filling [BO17, Section 4.1.2(2)]; this is satisfied by the dynamically convex assumption in dimension 3. Positive symplectic homology (and its equivariant version) are also shown to be invariants of certain closed contact manifolds in [Gu, Theorems 1.2, 1.3].

Finally, we expect that our work can be incorporated into the framework of Legendrian contact homology [BEE12, CELN, EES07, EENS13]. This relative version of contact homology is currently defined for Legendrians in 1-jet spaces [EES07] and for Legendrian links in $#^k(S^1 \times S^2)$ [EN15]. We anticipate that our results will permit this theory to be geometrically defined for Legendrian knots in closed dynamically convex contact 3-manifolds and that the proof of the following conjecture is within reach of our methods.

**Conjecture 4.** The contact homology of Legendrian knots of dynamically convex contact 3-manifolds is well-defined: The stable tame isomorphism class of the DGA associated to a Legendrian knot $K$ is independent of the choice of compatible almost complex structure $J$ and is invariant under Legendrian isotopies of $K$.

There is also the following hope that one can relate the Legendrian DGA to other combinatorial knot invariants as in [ENS, EENS13]. Such a knot invariant in combination with a combinatorial expression would lead to progress on the following question.

**Question 5.** Do there exist infinite families of Legendrian knots in dynamically convex contact 3-manifolds, which have the same classical invariants but are pairwise non-Legendrian isotopic?

**References**


[BaHon2] E. Bao and K. Honda, Semi-global Kuranishi charts and the definition of contact homology, arxiv:1512.00580


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\(^4\)This assumption is only true for contact manifolds arising as unit cotangent bundles $DT^*L$ with dim $L \geq 5$ or those Riemannian manifolds $L$ which admit no contractible closed geodesics.


Research Statement


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